Congresso de Métodos Numéricos em Engenharia 2015
Lisboa, 29 de Junho a 2 de Julho, 2015

HIGH-ORDER ALGORITHM FOR NON-LINEAR DYNAMICS

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Keywords: Hermitian algorithm, non-linear dynamic, numerical instability, single-step algorithm

Abstract *The objective of this paper is presenting an efficient step-by-step direct integration algorithm for the structural dynamic equation. The proposed algorithm is formulated in terms of two Hermitian finite difference operators of fifth-order local truncation error and it is unconditionally stable with no numerical damping presenting a fourth-order truncation error for period dispersion (global error). In addition, although it is in competition with higher-order algorithms presented in the literature, the computational effort is similar to that of the classical second-order Newmark method. Classical single degree of freedom as Duffing and Van Der Pol equations are considered as illustrative applications.*

1. INTRODUCTION

Numerical single step integration methods for the solution of equilibrium equations in dynamic analysis are based in general on the use of Hermitian finite-difference operators [1] as the well-known Newmark family that have been employed to solve linear and nonlinear problems [2]. The Newmark family are characterized by a linear combination of the function to be integrated and its derivatives whose accuracy depends on the combination parameters.

As the developed algorithm takes into account the repeated differentiation of the governing equation, additional nonlinear terms are required to solve nonlinear problems. Thus, it is interesting to consider, for example, the classic iterative procedures presented by Argyris and Mlejek [3]. Although the presence of these additional nonlinear terms increases the number of operations in the iterative operations and introduces some numerical noise in comparison to the Padè-P_{22} algorithm family [4], the reduction obtained in the matrix factorization and higher orders of the relative radii errors are interesting attributes of the proposed algorithm.

2. HERMITIAN OPERATORS

The step-by-step integration algorithm to be considered in this paper takes into account the following Hermitian operators [1] [3]

$$\begin{aligned} &Ay_i - By_{i+1} + C\Delta t\dot{y}_i + D\Delta t\dot{y}_{i+1} + E\Delta t^2\ddot{y}_i - F\Delta t^2\ddot{y}_{i+1} + G\Delta t^3\dddot{y}_i - H\Delta t^3\dddot{y}_{i+1} = 0 \\ &A_1y_i - B_1y_{i+1} + C_1\dot{y}_i - D_1\dot{y}_{i+1} + E_1\Delta t^2\ddot{y}_i + F_1\Delta t^2\ddot{y}_{i+1} + G_1\Delta t^3\dddot{y}_i - H_1\Delta t^3\dddot{y}_{i+1} = 0 \end{aligned} \quad (1)$$

where Δt is the time step, i and i+1 indicate the step, y is the function to be integrated, $\dot{y}$, $\ddot{y}$ and $\dddot{y}$ are derivatives of the function with respect to time; A, B… C_1, D_1..are combination non-dimension parameters that define the order of accuracy (local truncation error). As for Newmark method, it is necessary to consider two independent Hermitian operators. Table 1 presents the combination parameters for the some popular algorithms of Newmark family.

	A	B	C	D	E	F	G	H
Laier [1]	12	-12	6	6	1	-1	0	0
Newmark	0	0	-1	1	-1/2	-1/2	0	0
Hoff [5]	0	0	1	-1	1/2	-1/2	0	0
Bazzi [6]	-2	2	-1	-1	0	0	0	0
Argyris [3]	1	-1	1	0	21/60	9/60	3/60	-2/60
	A_1	B_1	C_1	D_1	E_1	F_1	G_1	H_1
Laier [1]	0	0	12	-12	6	6	1	-1
Newmark	1	-1	-1	0	-1/4	-1/4	0	0
Hoff [5]	1	-1	1	0	1/4	1/4	0	0
Bazzi [6]	-4	4	-4	0	-1	-1	0	0
Argyris [3]	0	0	1	-1	6/12	6/12	1/12	-1/12

Table 1 Combination Parameters

3. NONLINEAR DYNAMIC EQUILIBRIUM EQUATION

The non-linear dynamic equilibrium can be expressed in matrix notation by [4]

$$M\ddot{y} + C^*\dot{y} + Ky + N(y,\dot{y}) - f(t) = 0 \quad (2)$$

where y, y^{l} and y^{ll} are respectively the displacement, velocity and acceleration experienced by the mass M; C^* and K are respectively the damping and the stiffness parameters. The function $f(t)$ represents a given applied force and any non-linear terms are represented by the function $N(y, y^I)$. As the Hermitian operators (1) involves the third derivative of the displacement one has to be consider the first derivative of eq. (2), that is

$$M\dddot{y} + C^{*}\ddot{y} + K\dot{y} + N_{y}(y,\dot{y})\dot{y} + N_{\dot{y}}(y,\dot{y})\ddot{y} - \dot{f}\left(t\right) = 0 \tag{3}$$

The classical Duffing equation and its derivative can be expressed by:

$$\begin{aligned} &M\ddot{y} + C^{*}\dot{y} + Ky + \mu y^{3} - f\left(t\right) = 0 \\ &M\dddot{y} + C^{*}\ddot{y} + K\dot{y} + 3\mu y^{2}\dot{y} - \dot{f}\left(t\right) = 0 \end{aligned} \tag{4}$$

where μ is a constant parameter of the nonlinear stiffness; similarly for Van Der Pol equation and its derivative can be written as:

$$\begin{aligned} &M\ddot{y} + \delta\dot{y} + Ky + \delta\dot{y}y^{2} - f\left(t\right) = 0 \\ &M\dddot{y} - \delta\ddot{y} + K\dot{y} + \delta\ddot{y}y^{2} - 2\delta\dot{y}^{2}y - \dot{f}\left(t\right) = 0 \end{aligned} \tag{5}$$

where δ is a constant parameter of the linear and nonlinear dumping.

4. NUMERICAL INTEGRATION

Taking into account equations (2) and (3) the second and third derivatives of the function to be integrated can be expressed in terms of the function and its first derivative at the step i and i+1 that substituting in equation (1) yields:

$$\begin{aligned} &y_{i}\left(A - E\Delta t^{2}K + GCK\Delta t^{3}\right) + y_{i+1}\left(B - FK\Delta t^{2} + HCK\Delta t^{3}\right) + \\ &\dot{y}_{i}\left(C^{*}\Delta t - EC\Delta t^{2} + G\left(C^{2} + K\right)\Delta t^{3} - 3G\mu\Delta t^{3}y_{i}^{2}\right) + \\ &\dot{y}_{i+1}\left(D\Delta t - FC\Delta t^{2} + H\left(C^{2} + K\right)\Delta t^{3} - 3H\mu\Delta t^{3}y_{i}^{2}\right) + \\ &y_{i}^{3}\left(-E\mu\Delta t^{2} + GC\mu\Delta t^{3}\right) + y_{i+1}^{3}\left(-F\mu\Delta t^{2} + HC\mu\Delta t^{3}\right) + \\ &f_{i}\left(E\Delta t^{2} - GC\Delta t^{3}\right) + f_{i+1}\left(F\Delta t^{2} - HC\Delta t^{3}\right) + \dot{f}_{i}\left(G\Delta t^{3}\right) + \dot{f}_{i+1}\left(H\Delta t^{3}\right) = F\left(y_{i+1}, \dot{y}_{i+1}\right) = 0 \end{aligned} \tag{6}$$

and

$$
\begin{aligned}
& y_i\left(A_1 - E_1\Delta t^2 K + G_1 C_1 K\Delta t^3\right) + y_{i+1}\left(B_1 - F_1 K\Delta t^2 + H_1 C_1 K\Delta t^3\right) + \\
& \dot{y}_i\left(C^*\Delta t - E_1 C_1\Delta t^2 + G_1\left(C_1^2 + K_1\right)\Delta t^3 - 3G_1\mu\Delta t^3 y_i^2\right) + \\
a\, & \dot{y}_{i+1}\left(D_1\Delta t - F_1 C_1\Delta t^2 + H_1\left(C_1^2 + K_1\right)\Delta t^3 - 3H_1\mu\Delta t^3 y_i^2\right) + \\
& y_i^3\left(-E_1\mu\Delta t^2 + G_1 C_1\mu\Delta t^3\right) + y_{i+1}^3\left(-F_1\mu\Delta t^2 + H_1 C_1\mu\Delta t^3\right) + \\
& f_i\left(E_1\Delta t^2 - G_1 C_1\Delta t^3\right) + f_{i+1}\left(F_1\Delta t^2 - H_1 C_1\Delta t^3\right) + \dot{f}_i\left(G_1\Delta t^3\right) + \dot{f}_{i+1}\left(H_1\Delta t^3\right) = 0
\end{aligned}
\qquad (6)
$$

for Duffing equation and

$$
\begin{aligned}
& y_i\left(A - E\Delta t^2 K + GCK\Delta t^3\right) + y_{i+1}\left(B - FK\Delta t^2\right) + \\
& \dot{y}_i\left(\delta E\Delta t^2\left(1 - y_i^2\right) + H\delta\left(1 - y_i^2\right)^2\Delta t^3\right) + \dot{y}_{i+1}\left(\delta F\Delta t^2\left(1 - y_{i+1}^2\right) + G\delta^2\left(1 - y_i^2\right)^2\Delta t^3\right) \\
& y_i\delta\left(1 - y_i^2\right)HK\Delta t^3 - \delta y_{i+1}\left(1 - y_{i+1}^2\right)G\Delta t^3 - 2\delta H\dot{y}_i^2 y_i\Delta t^3 - 2\delta G\dot{y}_{i+1}^2 y_{i+1}\Delta t^3 \\
& f_i\left(E\Delta t^2 - H\left(1 - y_i^2\right)\Delta t^3\right) + f_{i+1}\left(F\Delta t^2 - G\delta\Delta t^3\left(1 - y_{i+1}^2\right)\right) \\
& +\dot{f}_i\left(-H\Delta t^3\right) + \dot{f}_{i+1}\left(-G\Delta t^3\right) = F\left(y_{i+1}, \dot{y}_{i+1}\right) = 0
\end{aligned}
\qquad (7)
$$

and

$$
\begin{aligned}
& y_i\left(A - E\Delta t^2 K + GCK\Delta t^3\right) + y_{i+1}\left(B - FK\Delta t^2\right) + \\
& \dot{y}_i\left(\delta E_1\Delta t^2\left(1 - y_i^2\right) + H_1\delta\left(1 - y_i^2\right)^2\Delta t^3\right) + \dot{y}_{i+1}\left(F_1\delta\Delta t^2\left(1 - y_{i+1}^2\right) + G_1\delta^2\left(1 - y_i^2\right)^2\Delta t^3\right) \\
& y_i\delta\left(1 - y_i^2\right)H_1 K_1\Delta t^3 - \delta y_{i+1}\left(1 - y_{i+1}^2\right)G_1\Delta t^3 - 2H_1\delta\dot{y}_i^2 y_i\Delta t^3 - 2\delta G_1\dot{y}_{i+1}^2 y_{i+1}\Delta t^3 \\
& f_i\left(E_1\Delta t^2 - H_1\left(1 - y_i^2\right)\Delta t^3\right) + f_{i+1}\left(F_1\Delta t^2 - G_1\delta\Delta t^3\left(1 - y_{i+1}^2\right)\right) \\
& +\dot{f}_i\left(-H_1\Delta t^3\right) + \dot{f}_{i+1}\left(-G_1\Delta t^3\right) = G\left(y_{i+1}, \dot{y}_{i+1}\right) = 0
\end{aligned}
\qquad (7)
$$

for Van der Pol equation.

To solve iteratively nonlinear equations (6) and (7) at each step one has to take into account the two dimensional Newton Formula:

$$\begin{Bmatrix} y \\ \dot{y} \end{Bmatrix}_{j+1} = \begin{Bmatrix} y \\ \dot{y} \end{Bmatrix}_{j} + \begin{bmatrix} F_y & F_{\dot{y}} \\ G_y & G_{\dot{y}} \end{bmatrix}^{-1} \begin{Bmatrix} F\left(y_j, \dot{y}_j\right) \\ G\left(y_j, \dot{y}_j\right) \end{Bmatrix} \tag{8}$$

where for Duffing equation one has

$$\begin{aligned}
F_y &= \frac{F\left(y_j, \dot{y}_j\right)}{y_j} = B - FK\Delta t^2 + HCK\Delta t^3 + 3y_{i+1}^2\left(H\Delta t^3 C\mu - F\mu\Delta t^2\right) \\
&-6\dot{y}_{i+1}H\Delta t^3\mu y_{i+1} \\
F_{\dot{y}} &= \frac{F\left(y_j, \dot{y}_j\right)}{\dot{y}_j} = D\Delta t - FC\Delta t^2 + H\Delta t^3\left(C^2 - K\right) - 3H\mu\Delta t^3 y_{i+1}^2 \\
G_y &= \frac{G\left(y_j, \dot{y}_j\right)}{y_j} = B_1 - F_1K\Delta t^2 + H_1C_1K_1\Delta t^3 + 3y_{i+1}^2\left(H_1\Delta t^3 C_1\mu - F_1\mu\Delta t^2\right) \\
&-6\dot{y}_{i+1}H_1\Delta t^3\mu y_{i+1} \\
G_{\dot{y}} &= \frac{G\left(y_j, \dot{y}_j\right)}{\dot{y}_j} = D_1\Delta t - F_1C_1\Delta t^2 + H_1\Delta t^3\left(C_1^2 - K_1\right) - 3H\mu\Delta t^3 y_{i+1}^2
\end{aligned} \tag{9}$$

in which j indicate the step of the iteration process and for Van der Pol equation one has

$$\begin{aligned}
F_y &= B + F\Delta t^2\left(-4\delta y_{i+1}\dot{y}_{i+1} - 1\right) + \\
&H\Delta t^3\left(-4\delta^2 y_{i+1}\dot{y}_{i+1}\left(1 - y_{i+1}^2\right) - \delta\left(1 - y_{i+1}^2\right) + 2\delta y_{i+1}^2 + 2\delta\dot{y}_{i+1} - 1\right) \\
F_{\dot{y}} &= D\Delta t - F\Delta t^2\delta\left(1 - y_{i+1}^2\right) + H\Delta t^3\left(\delta^2\left(1 - y_{i+1}^2\right)^2 + 2\delta y_{i+1}\right) \\
G_y &= B_1 + F_1\Delta t^2\left(-4\delta y_{i+1}\dot{y}_{i+1} - 1\right) + \\
&H_1\Delta t^3\left(-4\delta^2 y_{i+1}\dot{y}_{i+1}\left(1 - y_{i+1}^2\right) - \delta\left(1 - y_{i+1}^2\right) + 2\delta y_{i+1}^2 + 2\delta\dot{y}_{i+1} - 1\right) \\
G_{\dot{y}} &= D_1\Delta t - F_1\Delta t^2\delta\left(1 - y_{i+1}^2\right) + H_1\Delta t^3\left(\delta^2\left(1 - y_{i+1}^2\right)^2 + 2\delta y_{i+1}\right)
\end{aligned} \tag{10}$$

5. NUMERICAL APPLICATION

Let us consider the following Dufing equation:

$$\ddot{y} + 0.4\dot{y} + y + 0.5y^3 = 0.5\cos(0.5t) \qquad (11)$$

in which M=1, C^* =0.4, K=1, F=0.5, $\omega = 0.5$ and $\mu = 0.5$ (see Eq. 4).

Table 4 compare the first peak results for three practical time-steps. The numerical results indicate that the cubic Hermitian algorithm presents results similar to the Newmark´s method.

Δt	LAIER[1]	ARGYRES[3]	NEWMARK
0.2s	0.5017	0.4840	0.4816
0.1s	0.5217	0.5117	0.5111
0.05s	0.5302	0.5253	0.5251

TABLE 4- Duffing first peak

Now, considering the van der Pol equation:

$$\ddot{y} - 1.5\dot{y}(1 - y^2) + y = 0 \qquad (9)$$

the result are shown in Table 5. The first peak is presented for three time-steps. For the van der Pol equation the cubic Hermitian and the proposed methods present quite similar results.

Δt	LAIER[1]	ARGYRES[3]	NEWMARK
0.2s	-0.3193	-0.3192	-0.3130
0.1s	-0.3193	-0.3192	-0.3177
0.05s	-0.3193	-0.3193	-0.3189

TABLE 5 – van der Pol results

6. CONCLUSIONS

This paper presents a very efficient unconditional stable step-by-step Hermitian algorithm to solve nonlinear dynamic equation. As numerical results are shown it is in competition with the high-order cubic algorithm presented in the literature. Although attained high order accuracy, the proposed algorithm presents similar matrix factorization as Newmark´s method [1].

AKNOWLEDGEMENTS

The author wishes to thanks to FAPESP (São Paulo State Research Foundation) and CNPq (National Council for Technological and Scientific Development) for their financial support.

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