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MINIMIZING SEDIMENTATION IN CANALS: A GRADIENT-FREE APPROACH

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Abstract. *This work deals with the mathematical modelling and control of the processes related to the sedimentation of suspended particles in large streams. To analyze this environmental problem, we propose two alternative mathematical models (1D and 2D, respectively) coupling the system for shallow water hydrodynamics with the sediment transport equations. Our main goal is related to establishing the optimal management of a canal (for instance, from a wastewater treatment plant) to avoid the settling of suspended particles and their unwanted effects: channel malfunction, undesired growth of vegetation, etc. So, we formulate the problem as an optimal control problem of partial differential equations, where we consider a set of design variables (the shape of the channel section and the water inflow entering the canal) in order to control the velocity of water and, therefore, the settling of particles in suspension. To compute a minimal value of the sedimentation, in this work we propose the use of a direct search algorithm: a modification of the classical Nelder-Mead method. In this first approach to the problem from the viewpoint of environmental control, in addition to a mathematically well-posed formulation of the problem, we present theoretical results and numerical examples for a simple realistic case (using MIKE21 software package).*

1 INTRODUCTION

The geological work in streams is based on three closely related activities: erosion, transport and deposition. The current erosion is caused by the progressive removal of mineral matter from the bottom and from the banks of the channel. Transport is the movement of the eroded particles by dragging along the bottom, suspended in water or in solution. Finally, sedimentation is the progressive accumulation of the particles carried on the bed of the river or canal, or in the bottom of a water body at rest into which opens a watercourse. Naturally, the erosion cannot take place without transportation, and suspended particles are finally deposited. Therefore, erosion, transport and settling can be viewed simply as three phases of a single activity.

Sediment is solid material, accumulated on the Earth surface, derived from the actions of different external processes (for instance, wind, temperature variations, meteorological precipitations, movement of surface water or groundwater, displacement of water masses in marine or lacustrine environments, chemical actions, actions of living organisms, etc.) [7]. On the other hand, we should also note that the sedimentation processes can be considered beneficial in some cases (just think of the wastewater treatment where the water passes through a settling device in order to deposit fine solids for later disposal) or harmful in other circumstances (for example, when considering the reduction in useful volume of reservoirs, or decreasing the capability of an irrigation or drainage channel, as in our case).

Natural sediments are constituted by a wide variety of particles differing in size, shape and density. From the point of view of the resistance to be dragged there are two main classes, clearly differentiated:

- non-cohesive or frictional sediment is formed by coarse or loose particles, such as sand and gravel. Gravity predominates over any other force, so all the particles have a similar behavior: Weight is the main force resisting drag and lift forces.

- cohesive sediment is formed by very fine particles, consisting of clay minerals, which are held together by the cohesive force, which opposes to that individual particles are separated from the whole. That bond strength is considerably greater than the weight of each grain, and resists the drag and lift forces. These aqueous sediments in which very fine settled solids concentrate are usually knows as sludge. Beads of these sediments are not presented separately (as the sand, for example) but as aggregates or clusters of particles called flocs, which are usually composed of a large quantity of solid particles, and therefore have completely different shapes and densities to those of individual particles [11, 12].

A large number of quality problems in surface waters (estuaries, reservoirs, rivers, canals, etc.) are related to the behavior and characteristics of these sediments, responsible, among others, of the loss of capacity in reservoirs, of the formation of deltas, or of the instability in canals for surface drainage.

2 MATHEMATICAL FORMULATION AND ANALYSIS OF THE PROBLEM

A wastewater treatment plant aims to achieve from wastewater, by different physical, chemical or biotechnological processes, an effluent water with better quality features, based on certain standard parameters. Inside a treatment plant (consider, for example, in a cooling system corresponding to a thermoelectric plant) water transfers are originated from different containers through channels. In these channels typically occurs naturally deposition of numerous particles in suspension, which causes a change in the geometry of the canal bottom, with the consequent appearance of accumulated sludge and algae or vegetation growth. All this will trigger a malfunction of the treatment processes, which results in the electric plant. Our goal is then directed to the study of sedimentation in the bed of a channel (although its validity can be directly extended to the case of a river) to optimally design the geometry of the canal section, so as to avoid the problems mentioned above.

The mathematical model proposed here to model sedimentation in a channel system couples partial differential equations for the one-dimensional version of the hydrodynamics of the shallow waters (formulated in terms of the wet area and the averaged flow) with the equations for sediment transport (formulated in terms of the concentration of suspended particles and the sedimented area). Specifically, the following system is proposed for a channel of length L over a period of time T:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = \sum_j q_j\, \delta(x - p_j) \quad \text{in } (0, L) \times (0, T) \tag{1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial}{\partial x}(b + z + H) = -\frac{gP}{C^2}\frac{Q}{A^2}\,|Q| + \sum_j q_j\, V_j\, \cos(\beta_j)\, \delta(x - p_j) \quad \text{in } (0, L) \times (0, T) \tag{2}$$

$$\frac{\partial}{\partial t}(Ac) + \frac{\partial}{\partial x}(Qc) - \frac{\partial}{\partial x}\left(kA\frac{\partial c}{\partial x}\right) - \phi = \sum_j m_j\, \delta(x - p_j) \tag{3}$$

$$\rho_s(1 - \eta)\frac{\partial A_s}{\partial t} + \phi = 0 \quad \text{in } (0, L) \times (0, T) \tag{4}$$

with boundary conditions:

$$A(L, t) = A_L(t), \quad Q(0, t) = Q_0(t) \quad \text{in } (0, T) \tag{5}$$

$$c(0, t) = c_0(t), \quad k\,\frac{\partial c}{\partial x}(L, t) = c_L(t), \quad A_s(0, t) = A_{s0}(t) \quad \text{in } (0, T) \tag{6}$$

and initial conditions:

$$A(x,0) = A^0(x),\ Q(x,0) = Q^0(x),\ c(x,0) = c^0(x),\ A_s(x,0) = A_s^0(x) \ \text{ in } (0,L) \tag{7}$$

where $A(x,t)$ is the wet area; $Q(x,t)$ is the flux of water ($Q = A\,u$, with $u(x,t)$ the averaged velocity of water); $c(x,t)$ is the averaged concentration of sediment transported in suspension; $A_s(x,t)$ is the sedimented area; p_j are the points of inflow water to the canal with water flux $q_j(t)$, velocity $V_j(t)$, angle β_j, and sediment flux $m_j(t)$; $b(x)$ represents the geometry of the (fixed) bottom of canal; $z(x,t)$ is the height of settled sediment on the channel bottom (if we assumed known the shape of the section: rectangular, circular, trapezoidal... there exists a bijective mapping between height and area: $z = B(A_s) \Leftrightarrow A_s = S(z)$); $H(x,t)$ is the height of water (in a similar way to previous case, $H = B(A+A_s) - B(A_s) \Leftrightarrow A = S(z+H) - S(z)$); P represents the wetted perimeter; g is the gravity acceleration; C is the Chézy friction coefficient; k is the diffusion coefficient; ϕ measures the interchange of sediment with the bottom (balancing eroded vs. deposited): a possible expression for ϕ can be $\phi = Q(c^* - c)/L_A$, with c^* the transport capacity of sediment, and L_A the adaptation length; and $\eta \in [0,1]$ corresponds to the porosity of bottom.

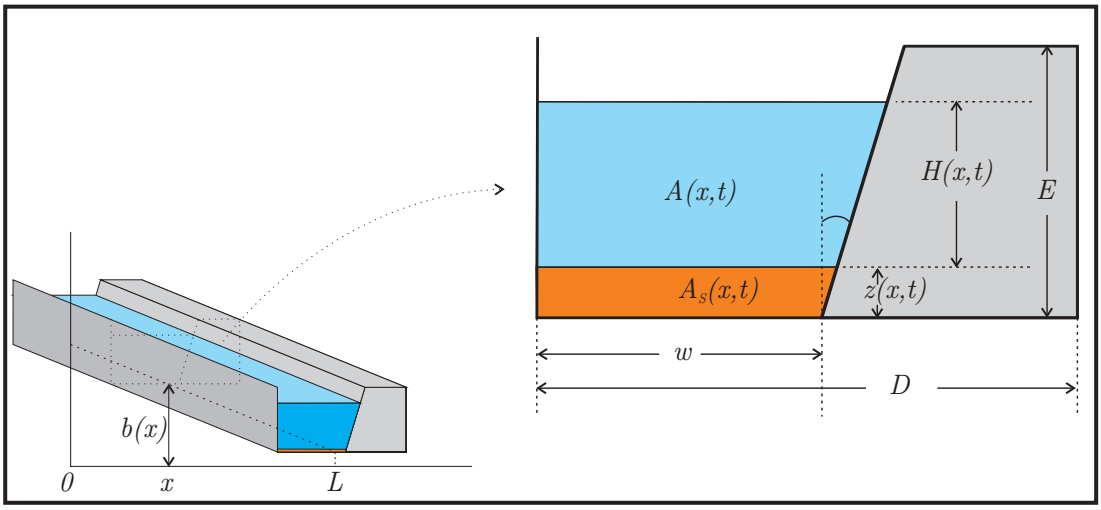

Figure 1: Schematic section of the channel

In our case, we start from a rectangular channel whose initial dimensions are a width D and a depth E. A modification of the shape of its section will be obtained by filling a lateral side (for example, with concrete) so that the new modified width of the basis of the channel is w, and that one of the side walls presents an inclination angle α with relation to the vertical (see Fig. 1). By technical reasons, several bound constraints will be imposed for these design variables (controls entering the state system through the domain on which the partial differential equations are posed):

$$\frac{D}{4} = \underline{w} \leq w \leq \overline{w} = D - E, \qquad 0 = \underline{\alpha} \leq \alpha \leq \overline{\alpha} = \frac{\pi}{4}. \tag{8}$$

Then, for our particular design variables (w, α), above referenced geometric functions S and B take the form:

$$S(h) = wh + \frac{\tan(\alpha)}{2} h^2, \qquad B(a) = \frac{\sqrt{w^2 + 2\tan(\alpha)a} - w}{\tan(\alpha)}. \tag{9}$$

Thus, once replaced the previous expressions and transformed the concentration equation to a non-conservative formulation, the state system can be rewritten as:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = \sum_j q_j\, \delta(x - p_j) \quad \text{in } (0, L) \times (0, T) \tag{10}$$

$$\begin{aligned} &\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + g\, \frac{1}{\sqrt{w^2 + 2\tan(\alpha)(A + A_s)}}\, \frac{\partial}{\partial x}\left(\frac{A^2}{2}\right) \\ &\quad + g\left(b' + \frac{1}{\sqrt{w^2 + 2\tan(\alpha)(A + A_s)}}\, \frac{\partial A_s}{\partial x}\right) A \\ &\quad = -\frac{gP}{C^2}\, \frac{Q}{A^2}\, |Q| + \sum_j q_j\, V_j\, \cos(\beta_j)\, \delta(x - p_j) \quad \text{in } (0, L) \times (0, T) \end{aligned} \tag{11}$$

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Qc}{A}\right) - \frac{\partial}{\partial x}\left(k \frac{\partial c}{\partial x}\right) - \frac{1}{L_A}\, \frac{Q}{A}(c^* - c) = \frac{1}{A} \sum_j m_j\, \delta(x - p_j) \tag{12}$$

$$\frac{\partial A_s}{\partial t} = -\frac{1}{\rho_s(1 - \eta)}\, \frac{1}{L_A}\, Q(c^* - c) \quad \text{in } (0, L) \times (0, T) \tag{13}$$

As cost function to be optimized we can take a combination of several factors including, necessarily, the minimization of the thickness of the settled sludge layer in the bottom of the channel. One of the simplest examples of cost functional to be minimized is of the form:

$$J(w, \alpha) = \frac{1}{2} \int_0^T \int_0^L A_s^2\, dx\, dt \tag{14}$$

In the particular case of a rectangular channel (i.e., with $\alpha = 0$ fixed), only the channel width w will be optimized in the space of admissible controls given by the real interval

$U_{ad} = [\underline{w}, \overline{w}]$, that is, we look for the value w, satisfying the bound constraints, that minimizes the functional $J(w)$, given by expression (14).

For this particular case, we define the adjoint system as:

$$
\begin{aligned}
&-\frac{\partial r}{\partial t} + \left(\frac{Q^2}{A^2} - g\,\frac{1}{w}\,A\right)\frac{\partial p}{\partial x} + g\left(b' + \frac{1}{w}\,\frac{\partial A_s}{\partial x} - \frac{2P}{C^2}\frac{Q}{A^3}|Q|\right)p + \frac{Qc}{A^2}\,\frac{\partial s}{\partial x} \\
&\qquad + \left(\frac{1}{L_A}\frac{Q}{A^2}(c^* - c) + \frac{1}{A^2}\sum_j m_j\,\delta(x - p_j)\right)s = 0 \quad \text{in } (0,L)\times(0,T) \qquad (15)\\
&-\frac{\partial p}{\partial t} - \frac{\partial r}{\partial x} - 2\frac{Q}{A}\,\frac{\partial p}{\partial x} - \frac{c}{A}\,\frac{\partial s}{\partial x} + \frac{2gP}{C^2}\frac{|Q|}{A^2}p \\
&\qquad + \frac{1}{\rho_s(1-\eta)}\frac{(c^* - c)}{L_A}v - \frac{1}{L_A}\frac{(c^* - c)}{A}s = 0 \quad \text{in } (0,L)\times(0,T) \qquad (16)\\
&-\frac{\partial s}{\partial t} - \frac{Q}{A}\,\frac{\partial s}{\partial x} - \frac{\partial}{\partial x}\left(k\frac{\partial s}{\partial x}\right) + \frac{1}{L_A}\frac{Q}{A}s - \frac{1}{\rho_S(1-\eta)}\frac{Q}{L_A}v = 0 \qquad (17)\\
&-\frac{\partial v}{\partial t} - \frac{g}{w}A\,\frac{\partial p}{\partial x} - \frac{g}{w}\frac{\partial A}{\partial x}p = A_s \quad \text{in } (0,L)\times(0,T) \qquad (18)
\end{aligned}
$$

with final conditions:

$$r(x,T) = p(x,T) = s(x,T) = v(x,T) = 0 \quad \text{in } (0,L), \qquad (19)$$

and boundary conditions:

$$
\begin{aligned}
&p(0,t) = p(L,t) = s(0,t) = 0 \qquad \text{in } (0,T) \qquad (20)\\
&\left(r + \frac{c}{A}\,s\right)(L,t) = \left(k\,\frac{\partial s}{\partial x} + \frac{Q}{A}\,s\right)(L,t) = 0 \qquad \text{in } (0,T) \qquad (21)
\end{aligned}
$$

Thus, we can derive the following formal first order optimality condition characterizing the optimal solution of the control problem:

Theorem 1. *Let $w \in [\underline{w}, \overline{w}]$ be the optimal solution of the control problem corresponding to minimizing the functional $J(w)$ in the admissible interval $U_{ad} = [\underline{w}, \overline{w}]$. Then, there exist (A, Q, c, A_s), solutions of the state system, and (r, p, s, v), solutions of the adjoint system, such that the following optimality condition is verified:*

$$(\tilde{w} - w)\int_0^T\int_0^L \frac{\partial}{\partial x}(A + A_s)A\,p\,dx\,dt \geq 0, \quad \forall \tilde{w} \in [\underline{w}, \overline{w}]. \qquad (22)$$

Proof. After a sequence of tedious (but elemental) computations, and taking into account the definition of the adjoint system, we can demonstrate that:

$$DJ(w)\cdot\tilde{w} = \frac{g}{w^2}\left(\int_0^T\int_0^L \frac{\partial}{\partial x}(A+A_s)A\,p\,dx\,dt\right)\tilde{w}. \tag{23}$$

Then, the formal condition characterizing the constrained minimum of J, i.e.,

$$DJ(w)\cdot(\tilde{w}-w)\geq 0, \quad \forall\tilde{w}\in[\underline{w},\overline{w}] \tag{24}$$

is equivalent to the optimality condition (22), which concludes the proof.

Remark. In the objective functional to be minimized we can also include other aspects, such as the cost of the filling material added to modify the shape of the channel. In this case, the functional to minimize would take the form:

$$J(w,\alpha) = \frac{1}{2}\int_0^T\int_0^L A_s^2\,dx\,dt + \beta\{E(D-w) - \frac{E^2}{2}\tan(\alpha)\} \tag{25}$$

where the last term measures the area of the modified section and β is a weight parameter (relative to the length of the canal and the economic cost of the material). For the particular case of a rectangular canal ($\alpha = 0$), this functional would read:

$$J(w) = \frac{1}{2}\int_0^T\int_0^L A_s^2\,dx\,dt + \beta E(D-w) \tag{26}$$

Thus, a new optimality condition - similar to that one in Theorem 1 - could be obtained, only bearing in mind that

$$DJ(w)\cdot\tilde{w} = \left(\frac{g}{w^2}\int_0^T\int_0^L \frac{\partial}{\partial x}(A+A_s)Ap\,dx\,dt - \beta E\right)\tilde{w}. \tag{27}$$

3 NUMERICAL EXPERIENCES

As follows from the previous section, the calculation of the optimal shape of the channel section passes through the resolution of the state system (and the adjoint system when necessary). To solve them we combine the method of characteristics for the time discretization with a finite element method for the space discretization.

The method of characteristics [10] is based on the use of the total derivative with respect to u and t, that is,

$$\frac{Dy}{Dt}(x,t) = \frac{\partial}{\partial\tau}\Big[y(X(x,t;\tau),\tau)\Big]\Big|_{\tau=t} = \frac{\partial y}{\partial t}(x,t) + u\cdot\nabla y \tag{28}$$

where the characteristic function $\tau \to X(x,t;\tau)$ represent s the trajectory of the particle of fluid that occupied the position x at time t. Then, for the time discretization, we take $N \in \mathbb{N}$, $\Delta t = T/N$, and we define discretized time steps $t^n = n\Delta t$. This leads us to a time semi-discretization of the system.

For the space discretization we set a space of Lagrange $\mathcal{P}_1$ finite elements [5] that, by a variational formulation of the semi-discretized equations and the use of a nodal basis, transforms the fully discretized problem into an algebraic linear system, which can be solved by any standard method.

Using these achieved approximations of the state variables, we are able to compute the value of the discretized cost functional, which can be minimized by a wide range of numerical algorithms (both with and without derivatives) [2]. The simplest alternative (avoiding the computation of the gradient and, consequently, the resolution of the adjoint system) is the utilization of any direct search method. In order to do this, the bound constrained problem must be previously transformed into an unconstrained optimization problem using, for instance, a penalty function: we introduce a function g collecting all the constraints corresponding to the characterization of the admissible set U_{ad} (g is chosen in such a way that $(w,\alpha) \in U_{ad} \Leftrightarrow g(w,\alpha) \leq 0$), and we define the penalty function:

$$\Phi(w,\alpha) = \gamma\, J(w,\alpha) + \max\{g(w,\alpha), 0\}, \tag{29}$$

where the parameter $\gamma > 0$ determines the relative contribution of J and the penalty term. So defined, Φ is an exact penalty function (cf. Han [6]) in the sense that, for γ small enough, the solutions of our original bound constrained problem are equivalent to the minima of function Φ.

In order to minimize the function Φ, in this work we propose the use of Nelder-Mead algorithm [9], given the essentially geometric nature of the problem. This algorithm is a gradient-free method, based on the mere comparison of values of the minimizing function, that constructs a sequence of simplices as an approximation to the optimal point, and has been successfully used by the authors in other environmental problems [1, 3, 4]. Moreover, the method presents good convergence properties in a low dimension case [8], as is ours.

We show the numerical results for a sloped canal of $L = 100\,m$ length (whose initial configuration is shown in Fig. 2) and a time interval of $T = 3600000\,s$ (about 42 days). We consider, for the design variables, the bounds $\underline{w} = 0.5$, $\overline{w} = 4, \underline{\alpha} = 0$, $\overline{\alpha} = \frac{\pi}{2}$, and a material cost parameter $\beta = 0$. Then, starting from a random configuration, we arrive - after 43 iterations - to the optimal section given by $w = 0.5$, $\alpha = 0$ (that is, a rectangular section as narrow as possible, which is the expected solution, since we are not considering the economic cost of the material filling). Fig. 2 shows the heights of water and sediment for the optimized canal at initial and final times.

At present time we are working in other numerical examples with non-zero parameter β (where we reasonably hope to arrive to optimal solutions not corresponding to lower

bound values), and also in an alternative 2D formulation of the problem (with the help of the commercial software MIKE 21).

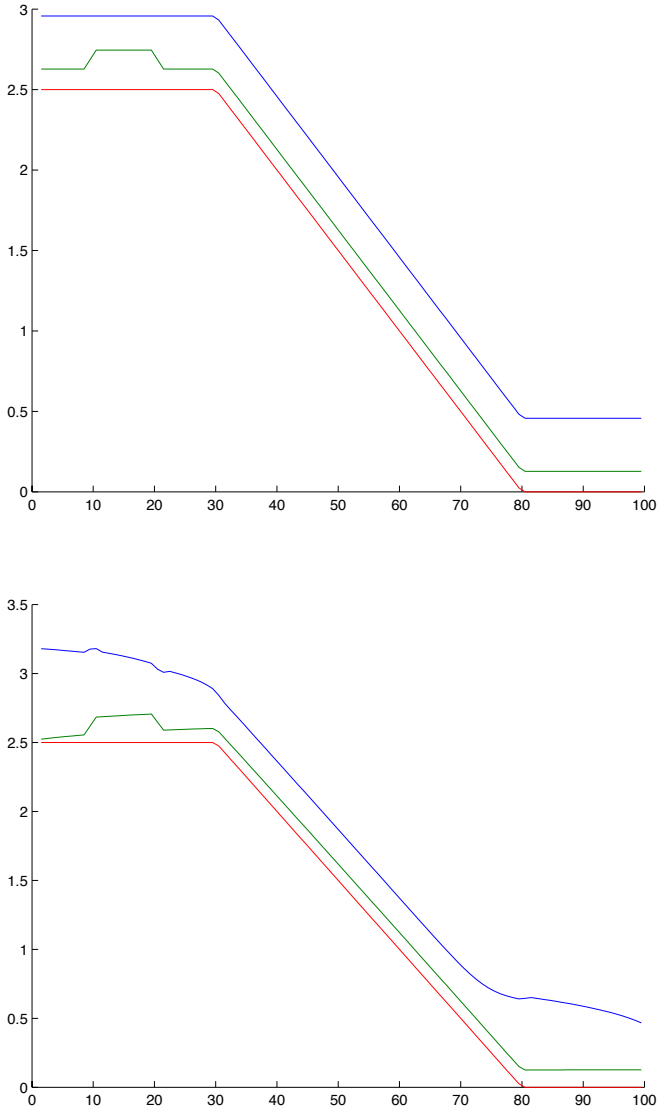

Figure 2: Profiles of the optimized canal for initial time $T = 0$ (up) and $T = 3600000$ (down). The lower (red) lines correspond to the bottom of the channel, the upper (blue) ones to the height of water, and the intermediate (green) ones to the height of sediment

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