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ESTIMATION OF TWO-DIMENSIONAL THERMAL CONTACT CONDUCTANCES USING THE RECIPROCITY FUNCTIONAL APPROACH

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Abstract. *In this paper, a method based on the reciprocity functional approach is used to estimate a space-dependent thermal contact resistance between two three-dimensional bodies without using intrusive measurements. A three-dimensional steady state heat conduction problem is considered, where different two-dimensional thermal contact conductance profiles are tested. The method consists of two stages: initially, two auxiliary problems that do not depend on the thermal contact conductance are solved and, after this step, it is possible to obtain both the temperature jump and the heat flux at the inaccessible interface, through the reciprocity functional, which takes into account only measurements taken at an external boundary.*

1 INTRODUCTION

In the study of heat conduction between two or more solid surfaces, one must consider the heat transfer at the contact interface. This heat transfer is strongly dependent on the roughness of the contacting surfaces, which generate gaps where the contact is not perfect. Therefore, the junction of two or more solids will present valleys resulting from imperfections of these materials, as can be seen in figure (1). These gaps are filled by an environment fluid where heat transfer occurs mainly by conduction.

When the thermal conductivity of this fluid is lower than the ones for the contacting solids, these valleys will function as a thermal insulation, causing a temperature drop at the interface. This resistance to heat transfer is known as *Thermal Contact Resistance* [1][2]. Another quantity of interest is the thermal contact conductance, defined as the reciprocal of the thermal contact resistance.

The thermal contact conductance is used in different areas of the knowledge. Its detection is important in various real problems such as nuclear reactors [3], biomedicine [4], and others. There are different studies regarding the estimation of the thermal contact

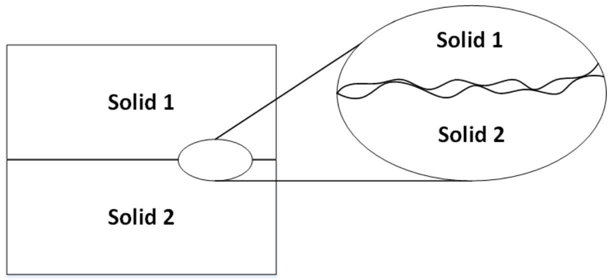

Figure 1: Contact between two materials.

resistance. Gill *et al.* [5] estimated the spatial variation of the thermal contact resistance between two materials and showed the need for regularization since the results were sensitive to noise in measurements. Furthermore, the temperature measurements used for this analysis were taken near the interface in an intrusive way. Milosević *et al.* [3] estimated a constant thermal contact resistance between two solids together with other parameters of the mathematical model using the laser flash method with the Gauss method, i.e. using non-intrusive measurements. Also, an analysis of the sensitivity coefficients was performed and three combinations of materials were tested.

In addition to the difficulties on estimating the thermal contact resistance, many works in this area required temperature measurements close to the contacting interface, i.e. intrusive measurements, which is avoided in the proposed methodology. In this work, we intend to estimate a two-dimensional thermal contact conductance in a three-dimensional body by means of the reciprocity functional approach [6]. For this task, two auxiliary problems are used and, subsequently, through the developed methodology, we can obtain different thermal contact conductance profiles. This technique is considered fast, since the auxiliary problems are not dependent on the thermal contact conductance and can be solved only once. Thus, it is possible to estimate different thermal contact conductance profiles simply calculating different integrals, as will be shown.

Two important works of the authors Stéphane Andrieux and Amel Ben Abda [6][7] show the concept and use of Reciprocity Functional. From these, other studies based on the Reciprocity Functional began to emerge in different areas. Delbary *et al.*[8], developed a qualitative method for breast cancer detection by combining the reciprocity functional method with the linear sampling method. Colaço and Alves [9] estimated spatial variation of the thermal contact conductance by using a reciprocity functional approach with the method of fundamental solutions and non-intrusive temperature measurements. Shifrin and Shushpannikov [10] developed a method for identifying small defects in an anisotropic elastic body based on the reciprocity functional. Other studies, regarding the estimation of the thermal contact conductance through reciprocity functional, using non-intrusive measures, can be found in [9][11][12].

The proposed technique follows a line of study in inverse problems [13][14] that is of interest in several current topics, mainly in industrial processes (non-destrutive testing)

and biomedicine (tumors and fractures detection). The development of non-intrusive techniques allows for the identification of properties, failures and internal heterogeneity in materials in a non-destructively manner, becoming thus an extremely versatile tool for tackling issues of interest.
The physical model presented in this work involves the steady-state heat transfer process between two three-dimensional bodies in contact. The goal is to identify the spatial variation of the thermal contact conductance, based on temperature measurements taken at the top surface of the test body in a non-intrusive way, for example using infrared measurements.

2 MATHEMATICAL FORMULATION

This work considers a three-dimensional steady-state heat conduction problem, with a two-dimensional thermal contact conductance. Two bodies are in non-perfect contact, where the first domain, Ω_1, has thermal conductivity κ_1 and the second domain, Ω_2, has thermal conductivity κ_2 with a contact surface Γ between them. Therefore, it is considered one domain Ω, divided in three parts $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$.
The lateral surfaces of both domains (Γ_1 and Γ_2), are assumed to be thermally insulated. On the upper surface Γ_0, a prescribed heat flux q is imposed and the lower surface Γ_∞ is subjected to a prescribed temperature. Figure (2) shows the geometry of the problem.

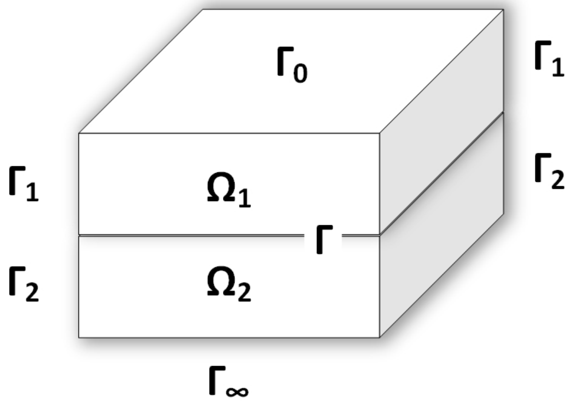

Figure 2: Geometry of the problem.

The mathematical formulation of this three-dimensional heat transfer problem with constant thermal conductivities (κ_1 and κ_2), can be written as follows:

$$\nabla^2 T_1 = 0 \quad \text{in} \quad \Omega_1 \tag{1}$$

$$\frac{\partial T_1}{\partial n} = 0 \quad \text{on} \quad \Gamma_1 \tag{2}$$

$$-k_1\frac{\partial T_1}{\partial n} = h(T_1 - T_2) \quad \text{on} \quad \Gamma \tag{3}$$

$$-k_1\frac{\partial T_1}{\partial n} = q \quad \text{on} \quad \Gamma_0 \tag{4}$$

$$\nabla^2 T_2 = 0 \quad \text{in} \quad \Omega_2 \tag{5}$$

$$\frac{\partial T_2}{\partial n} = 0 \quad \text{on} \quad \Gamma_2 \tag{6}$$

$$T_2 = 0 \quad \text{on} \quad \Gamma_\infty \tag{7}$$

$$k_2\frac{\partial T_2}{\partial n} = -k_1\frac{\partial T_1}{\partial n} \quad \text{on} \quad \Gamma \tag{8}$$

To estimate the thermal contact conductance on the surface Γ, we will use the methodology developed in [9, 11, 12] based on the reciprocity functional approach [6] where the solution of two auxiliary problems are required. The first auxiliary problem will determine the temperature jump and the second auxiliary problem will determine the heat flux, both on the contact surface Γ. By solving these two problems it is possible to obtain the thermal contact resistance by dividing these two quantities, as can be seen in equation (9).

$$R = \frac{\Delta T}{q} \tag{9}$$

The thermal contact conductance can be defined as

$$h = \frac{1}{R} \tag{10}$$

Auxiliary Problem 1

Considering a auxiliary problem for harmonic test functions $F_1 \in C^2(\Omega_1)$ and $F_2 \in C^2(\Omega_2)$, the first auxiliary problem is defined by equations (11-18) [9][11][12].

$$\nabla^2 F_{1,j} = 0 \quad \text{in} \quad \Omega_1 \tag{11}$$

$$\frac{\partial F_{1,j}}{\partial n} = 0 \quad \text{on} \quad \Gamma_1 \tag{12}$$

$$F_{1,j} = F_{2,j} \quad \text{on} \quad \Gamma \tag{13}$$

$$F_{1,j} = \psi_j \quad \text{on} \quad \Gamma_0 \tag{14}$$

$$\nabla^2 F_{2,j} = 0 \quad \text{in} \quad \Omega_2 \tag{15}$$

$$\frac{\partial F_{2,j}}{\partial n} = 0 \quad \text{on} \quad \Gamma_2 \tag{16}$$

$$F_{2,j} = 0 \quad \text{on} \quad \Gamma_\infty \tag{17}$$

$$k_2 \frac{\partial F_{2,j}}{\partial n} = -k_1 \frac{\partial F_{1,j}}{\partial n} \quad \text{on} \quad \Gamma \tag{18}$$

The function ψ_j appearing in the equation (14) is a $L^2(\Gamma)$ orthonormal basis. In this work, we used a combination of sine and cosine functions. From this auxiliary problem, described by equations (11-18), we may develop the reciprocity functional methodology [9].
Considering the identity (19) for the domain Ω_1,

$$0 = \int_{\Omega_1} [F_{1,j}(\nabla^2 T_1) - T_1(\nabla^2 F_{1,j})] \, d\Omega_1 \tag{19}$$

and using the Green's second identity, we may obtain,

$$\int_{\Omega_1} [F_{1,j}(\nabla^2 T_1) - T_1(\nabla^2 F_{1,j})] \, d\Omega_1 = \int_{\partial\Omega_1} \left[F_{1,j} \frac{\partial T_1}{\partial n} - T_1 \frac{\partial F_{1,j}}{\partial n} \right] d(\partial\Omega_1) \tag{20}$$

Then,

$$0 = \int_{\Gamma_0 \cup \Gamma 1 \cup \Gamma} \left[F_{1,j} \frac{\partial T_1}{\partial n} - T_1 \frac{\partial F_{1,j}}{\partial n} \right] d(\partial\Omega_1) \tag{21}$$

Using the boundary conditions (2) and (12), the following expression is obtained

$$0 = \int_{\Gamma_0 \cup \Gamma} \left[F_{1,j} \frac{\partial T_1}{\partial n} - T_1 \frac{\partial F_{1,j}}{\partial n} \right] d(\partial\Omega_1) \tag{22}$$

Using now the boundary condition (4) and the fact that some temperature measurements Y are available on the boundary Γ_0, we have

$$0 = \int_{\Gamma_0} \left[F_{1,j} \left(-\frac{q}{\kappa_1} \right) - Y \frac{\partial F_{1,j}}{\partial n} \right] d\Gamma_0 + \int_{\Gamma} \left[F_{1,j} \frac{\partial T_1}{\partial n} - T_1 \frac{\partial F_{1,j}}{\partial n} \right] d\Gamma \tag{23}$$

Using another identity, now for the domain Ω_2, we can write

$$0 = \int_{\Omega_2} [F_{2,j}(\nabla^2 T_2) - T_2(\nabla^2 F_{2,j})] \, d\Omega_2 \tag{24}$$

Using again the Green's second identity, and equations (6),(7),(16) and (17) we can obtain

$$0 = \int_{\Gamma} \left[F_{2,j} \frac{\partial T_2}{\partial n} - T_2 \frac{\partial F_{2,j}}{\partial n} \right] d\Gamma \tag{25}$$

As κ_1 and κ_2 are constants, summing equations (23) and (25) we obtain

$$0 = \int_{\Gamma_0} \kappa_1 \left[F_{1,j} \left(-\frac{q}{\kappa_1} \right) - Y \frac{\partial F_{1,j}}{\partial n} \right] d\Gamma_0 + \int_{\Gamma} \kappa_1 \left[F_{1,j} \frac{\partial T_1}{\partial n} - T_1 \frac{\partial F_{1,j}}{\partial n} \right] d\Gamma +$$

$$+ \int_{\Gamma} \kappa_2 \left[F_{2,j} \frac{\partial T_2}{\partial n} - T_2 \frac{\partial F_{2,j}}{\partial n} \right] d\Gamma \tag{26}$$

or

$$\int_{\Gamma_0} \kappa_1 \left[F_{1,j} \left(-\frac{q}{\kappa_1} \right) - Y \frac{\partial F_{1,j}}{\partial n} \right] d\Gamma_0 = \int_{\Gamma} \left[-\kappa_2 F_{2,j} \frac{\partial T_2}{\partial n} - \kappa_1 F_{1,j} \frac{\partial T_1}{\partial n} \right] d\Gamma +$$

$$+ \int_{\Gamma} \left[\kappa_2 T_2 \frac{\partial F_{2,j}}{\partial n} + \kappa_1 T_1 \frac{\partial F_{1,j}}{\partial n} \right] d\Gamma \tag{27}$$

Using equations (8), (13) and (18), we can obtain

$$\int_{\Gamma_0} \kappa_1 \left[F_{1,j} \left(-\frac{q}{\kappa_1} \right) - Y \frac{\partial F_{1,j}}{\partial n} \right] d\Gamma_0 = \int_{\Gamma} \kappa_1 \frac{\partial F_{1,j}}{\partial n} [T_1 - T_2] \, d\Gamma \tag{28}$$

Then, through of the methodology developed by Andrieux and Abda [6], we can define the Reciprocity Functional in terms of the function F_1, given as

$$\Re(F_{1,j}) = \int_{\Gamma_0} \kappa_1 \left[F_{1,j} \left(-\frac{q}{\kappa_1} \right) - Y \frac{\partial F_{1,j}}{\partial n} \right] d\Gamma_0 \tag{29}$$

Using equations (28) and (29), we obtain

$$\Re(F_{1,j})\kappa_1 = \left\langle T_1 - T_2, \kappa_1 \frac{\partial F_{1,j}}{\partial n} \right\rangle_{L^2(\Gamma)} \tag{30}$$

We can now define

$$k_1 \frac{\partial F_{1,j}}{\partial n} = \beta_j \tag{31}$$

and write equation (30) as follows

$$\Re(F_{1,j})\kappa_1 = \langle T_1 - T_2, \beta_j \rangle_{L^2(\Gamma)} \tag{32}$$

Considering that the temperature jump in the interface can be written as [12]

$$[T_1 - T_2]_\Gamma = \sum_i \alpha_i \beta_i \tag{33}$$

and using equations (32) and (33) we can write [12]

$$\Re(F_{1,j})\kappa_1 = \langle \beta_i, \beta_j \rangle_{L^2(\Gamma)} \alpha_j \tag{34}$$

where the unknown coefficients (α_j) can be obtained from the solution of equation (34) and thereby, through of the equation (33), we can obtain the temperature jump in the interface $[T_1 - T_2]_\Gamma$.

Auxiliary Problem 2

Considering now another auxiliary problem for harmonic test functions $G_1 \in C^2(\Omega_1)$, the second auxiliary problem is defined by equations (35-38) [9][11][12].

$$\nabla^2 G_{1,j} = 0 \quad \text{in} \quad \Omega_1 \tag{35}$$

$$G_{1,j} = \psi_j \quad \text{on} \quad \Gamma_0 \tag{36}$$

$$\frac{\partial G_{1,j}}{\partial n} = 0 \quad \text{on} \quad \Gamma_1 \tag{37}$$

$$\frac{\partial G_{1,j}}{\partial n} = 0 \quad \text{on} \quad \Gamma \tag{38}$$

The function ψ_j again, appearing in equation (36), is a $L^2(\Gamma)$ orthonormal basis. Using the same process described in the auxiliary problem 1, we obtain

$$\int_{\Gamma_0} \kappa_1 \left[G_{1,j} \left(-\frac{q}{\kappa_1} \right) - Y \frac{\partial G_{1,j}}{\partial n} \right] d\Gamma_0 = \int_\Gamma -\kappa_1 G_{1,j} \frac{\partial T_1}{\partial n} \, d\Gamma \tag{39}$$

Defining the Reciprocity Functional in terms of the function G_1 as

$$\Re(G_{1,j}) = \int_{\Gamma_0} \left[G_{1,j} \left(-\frac{q}{\kappa_1} \right) - Y \frac{\partial G_{1,j}}{\partial n} \right] d\Gamma_0 \tag{40}$$

and using equations (39) and (40), results in

$$\Re(G_{1,j})\kappa_1 = - \left\langle G_{1,j}, \kappa_1 \frac{\partial T_1}{\partial n} \right\rangle_{L^2(\Gamma)} \tag{41}$$

Now, defining

$$G_{1,j} = \phi_j \tag{42}$$

it is possible to write equation (41) as follows,

$$\Re(G_{1,j})\kappa_1 = - \left\langle \phi_j, \kappa_1 \frac{\partial T_1}{\partial n} \right\rangle_{L^2(\Gamma)} \tag{43}$$

Considering that the heat flux at the inaccessible interface Γ can be written as [12]

$$\left[\kappa_1 \frac{\partial T_1}{\partial n} \right]_{\Gamma} = \sum_i \gamma_i \phi_i \tag{44}$$

and using the equations (43) and (44), we can write [12]

$$\Re(G_{1,j})\kappa_1 = \langle \phi_i, \phi_j \rangle_{L^2(\Gamma)} \gamma_j \tag{45}$$

where the unknown coefficients (γ_j) can be obtained through solution of equation (45) and, thereby, through equation (44), we can obtain the heat flux at the inaccessible interface Γ.

Thermal contact conductance

After solving the two auxiliary problems defined previously, it is possible to obtain the thermal contact conductance at the inaccessible interface Γ as [12]

$$h = \frac{\sum_i \gamma_i \phi_i}{\sum_i \alpha_i \beta_i} \tag{46}$$

The auxiliary problems and the calculation of the reciprocity functional only depends on the geometry and the thermal conductivity of the direct problem, i.e. they do not depend on the direct problem itself.
Another important fact is that, because the thermal contact conductance is two-dimensional, we must use the function $\psi_j(x, y)$ as an orthonormal basis in the x and y direction. Therefore, the number of functions used is the product of the number of functions in the x and y directions.

3 RESULTS

In this paper we considered a three-dimensional heat conduction problem with a two-dimensional thermal conductance profile. As previously mentioned, the lateral surfaces of both domains (Γ_1 and Γ_2) were assumed thermally insulated. On the upper surface Γ_0, a prescribed heat flux of $-10\mathrm{W/m^2}$ was imposed and the lower surface Γ_∞ was subjected to a prescribed temperature of 0^oC, as shown in figure (2). The geometry had 0.04 m of width, 0.04 m of length and 0.01 m of height for each domain. The thermal conductivity was considered equal for both domains (Ω_1 and Ω_2) with a value of 54 W$(m/^oC)$. A grid convergence analysis was performed and a grid with $120 \times 120 \times 30$ points was used in each domain.
Measurements with and without noise, taken on the upper surface Γ_0, were considered to verify the stability of the solution. Measurements with experimental noise were modeled according to

$$Y = T + \varepsilon\sigma \tag{47}$$

where ε is a random variable with a Gaussian distribution and unitary standard deviation, and σ is the standard deviation of the measurements. To generate a Gaussian random number with zero mean and unit variance, the following Box-Muller transformation was used

$$\varepsilon = \sqrt{-2\ln(u_1)}\cos(2\pi u_2) \tag{48}$$

where u_1 and u_2 are two uniformly distributed random numbers.
The finite difference method was used to solve the auxiliary problems 1 and 2, with the same mesh used in the direct problem. As these problems are not dependent on thermal contact conductance, they can be solved only once. Changing only the calculation of Functional Reciprocity (equations 29 and 40), and the calculation of the resulting systems (equations 34 and 45) different thermal contact conductance profiles can be obtained in short computational time.
As stated before, we used a orthonormal basis of sine and cosine functions. We can vary the number of functions and thereby the number of times that the auxiliary problems are be solved, since these problems depend on ψ_j.

First we present the solution of the direct problem by the finite difference method using the boundary conditions and heat flux mentioned above. Figures (3) and (4) show the temperature jump and the heat flux, respectively, on the inaccessible interface Γ.

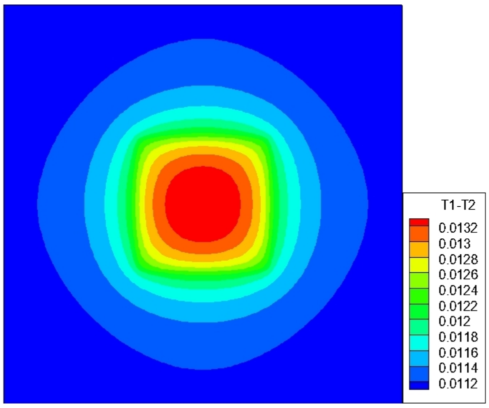

Figure 3: Temperature jump on Γ.

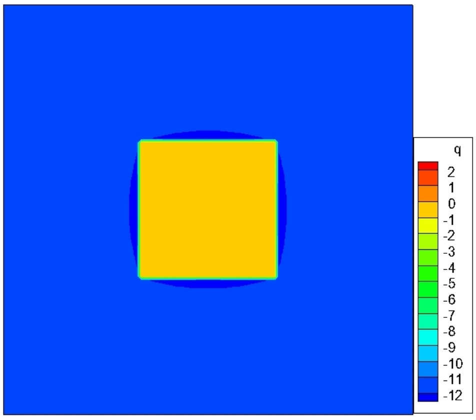

Figure 4: Heat flux on Γ.

Figure (5) shows the profile of thermal contact conductance used in this test problem. As stated previously, we are using orthonormal functions in the x and y directions, so the number of functions required in each direction for a good estimate should be tested. For this reason, we varied the number of functions in x and y from 1 to 30 and calculated the error between the exact and estimated temperature jump, using equation (49). Figure (6) shows the temperature jump error graph, in logarithmic scale.

$$\text{error} = \frac{\sum (T_1 - T_2)_{exact} - (T_1 - T_2)_{estimated}}{\sum (T_1 - T_2)_{exact}} \tag{49}$$

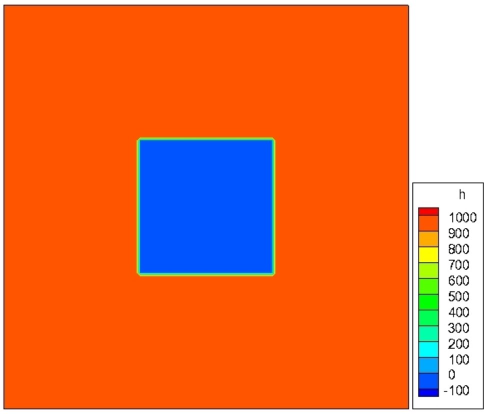

Figure 5: Thermal contact conductance on Γ.

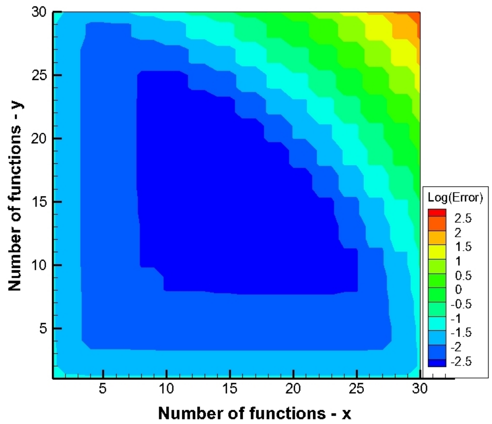

Figure 6: Temperature jump error graph.

The same error calculation was made for the heat flux. Figure (7) shows the heat flux error graph, in logarithmic scale.

We can see that the best estimates for both temperature jump and for heat flux are located in the dark blue region of the figures (6) and (7), respectively. From these figures, it is possible to see that errors decrease when more functions are used, up to a certain limit. Starting from this limit, if more functions are used, errors raise very rapidly, due to the ill-conditioned character of this inverse problem. Therefore a deep investigation of the relationship between the accuracy of the solution, and the number of functions used is needed.

For this estimate, we choose 15 functions in x and y, both for the temperature jump and the heat flux estimation. Figures (8), (9) and (10), shows the estimated temperature jump, the heat flux and the thermal contact conductance, respectively.

As we can see figures (8), (9) and (10) are in good agreement with (3), (4) and (5), respectively. Thus, the proposed methodology is able to estimate the thermal contact

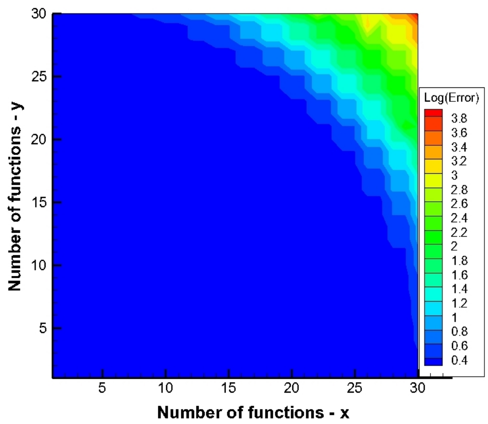

Figure 7: Heat flux error graph.

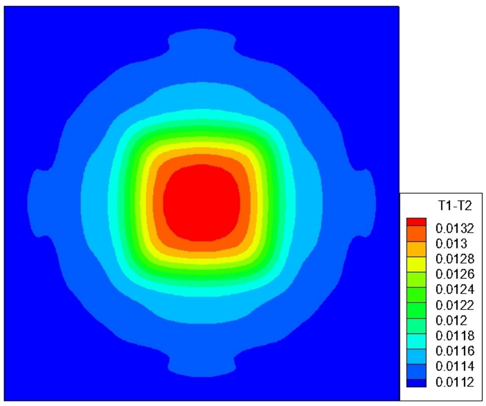

Figure 8: Estimated temperature jump.

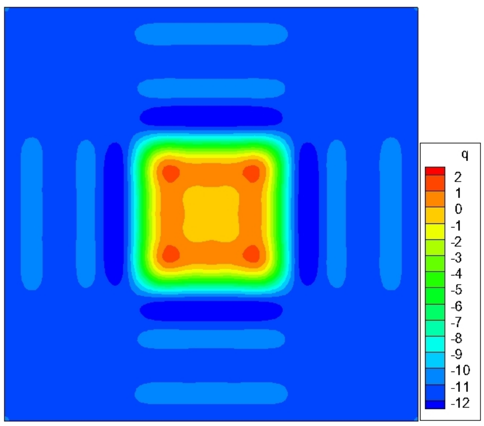

Figure 9: Estimated heat flux.

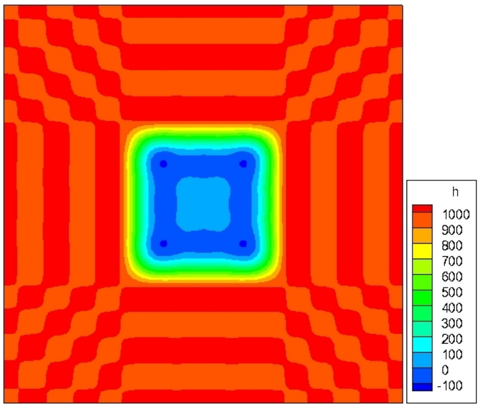

Figure 10: Estimated thermal contact conductance.

conductance without intrusive measures.

The next step included the analysis of the influence of the measurement errors in the estimate. Results are presented for a standard deviation σ equal to 0.1% of $|Y_{max}|$. As in the previous case, 15 functions were used in the x direction and 15 functions in the y direction. Figures (11), (12) and (13) show the estimated temperature jump, the heat flux and the thermal contact conductance, respectively.

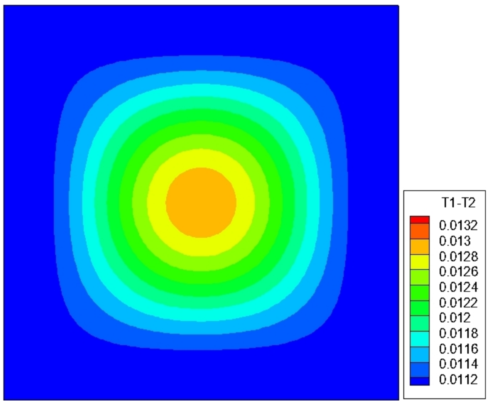

Figure 11: Estimated temperature jump.

As we can see, the estimates are worse than the ones obtained by using noiseless measurements, but are still in good agreement with the exact results shown in figures (3), (4) and (5). A further analysis, however, is necessary, in order to find the optimum number of basis functions, as it was done for the case with errorless measurements.

The IMSL subroutine DLSARG was used to solve the linear system given by Eqs. (34) and (45) for the case were noiseless measurements were used. When measurement noises were included, a SVD solver was used, where the minimum singular value allowed was 1.5×10^6 for the temperature jump and 5×10^{-4} for heat flux. A further analysis of these

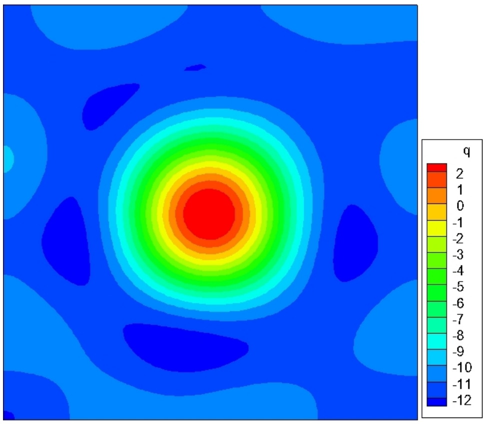

Figure 12: Estimated heat flux.

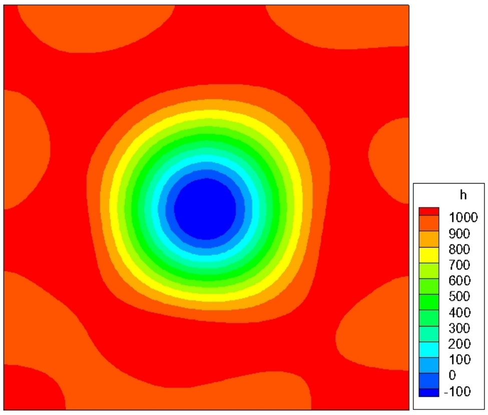

Figure 13: Estimated thermal contact conductance.

values is also necessary.

Overall, the estimates were very good, and the estimated functions could be recovered in a short computational time, showing that the method might be a good alternative to some traditional techniques.

4 CONCLUSIONS

In this paper, we used the reciprocity functional approach to estimate an unknown two-dimensional space-dependent thermal contact conductance in a three-dimensional body. After solving two auxiliary problems and calculating different integrals, it was possible to estimate the temperature jump and the heat flux and, consequently, the thermal contact conductance at an inaccessible boundary, only using non-intrusive data. Results show that the estimates are in good agreement with the exact values. However, test cases where measurement noises were included show that a further analysis of the ortonormal functions is required. The methodology is able to quickly identify contact failures between materials with good results.

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