Optimal Punch Shape Under Probabilistic Data Concerning External Loading

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Abstract
Shape optimization problem for rigid punch interacted with elastic medium is investigated. Additional forces applied to the elastic medium are supposed to be random in considered problem formulation. Probabilistic approach is applied as for formulation as for solution of the optimization problem. As a result the optimal designs of the punch with circular base are obtained and presented.

**Keywords:** structural optimization, contact interaction, probabilistic approach.

The problem of contact interaction of rigid punch and elastic medium occupied the half-space \(z \geq 0\) in rectangular coordinate system \(xyz\) is considered. The boundary of elastic half-space \(\Omega (z = 0)\) contains the domain of contact \(\Omega_f\) representing the base of the punch, the region \(\Omega_0\), which is free of loading, and the regions \(\Omega^i_q\) \((i = 1, 2, ..., N)\) of application of external forces, i.e.

\[
\Omega = \Omega_f + \Omega_0 + \Omega_q, \quad \Omega_q = \bigcup_{i=1}^{N} \Omega^i_q
\]  

where

\[
\Omega_f \cap \Omega_0 = 0, \quad \Omega_f \cap \Omega^i_q = 0, \quad \Omega^i_q \cap \Omega_0 = 0
\]  

The surface of the punch penetrated into elastic medium without friction is given by the equation

\[
z = f(x, y), \quad (x, y) \in \Omega_f \quad \text{and} \quad z = 0, \quad (x, y) \in \partial \Omega_f, \quad \text{where} \quad f(x, y) \text{ is positive continuous and smooth. In what follows this function will be considered as unknown design variable. External forces } q^i = q^i_x, q^i_y, q^i_z \text{ applied to the domain } \Omega^i_q \text{ are considered as functions depending on two random variables} \xi, \eta, \text{ i.e.}
\]

\[
q^i_j = q^i_j(x, y, \xi, \eta), \quad (x, y) \in \Omega^i_q
\]

Random variables \(\xi, \eta\) have the following joint probability densities \(g^i(\xi, \eta)\) and the corresponding joint probability distribution functions \(F^i(\xi, \eta)\), where

\[
g^i(\xi, \eta) = \frac{\partial^2 F^i}{\partial \xi \partial \eta}
\]  

Boundary conditions at the \(\Omega\) have the form

\[
\sigma_{xz} = \sigma_{yz} = 0, \quad w = f(x, y), \quad (x, y) \in \Omega_f
\]

\[
\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0, \quad (x, y) \in \Omega_0
\]

\[
\sigma_{xz} = q^i_x(x, y, \xi, \eta), \quad \sigma_{yz} = q^i_y(x, y, \xi, \eta), \quad \sigma_{zz} = q^i_z(x, y, \xi, \eta), \quad (x, y) \in \Omega^i_q
\]

Here \(\sigma_{xz}, \sigma_{yz}, \sigma_{zz}\) - components of the stress tensor and \(w\) - projection of the displacement vector on the \(z\)-axis. The contact pressure distribution \(p(x, y, \xi, \eta) = -\sigma_{xz}(x, y, \xi, \eta)\) defined at the bottom of the punch \((x, y) \in \Omega_f\) can be used for determination of the resulting force \(P\) applied to the punch and the total applied moments \(M_x, M_y\) (with respect to the axes \(x\) and \(y\)) by means of the following expressions

\[
P(\xi, \eta) = \int_{\Omega_f} p(x, y, \xi, \eta) d\Omega_f,
\]

\[
M_x(\xi, \eta) = \int_{\Omega_f} yp(x, y, \xi, \eta) d\Omega_f, \quad M_y(\xi, \eta) = \int_{\Omega_f} xp(x, y, \xi, \eta) d\Omega_f
\]
Evaluation of the reaction of elastic medium is performed with the help of the Betti’s reciprocity theorem \([1]\). We consider two punches with the same contact region \(\Omega_f\). The first is standard rigid punch with the plane bottom having the shape described by a linear function
\[
z = f^0(x, y) = \alpha + \beta x + \gamma y, \quad (x, y) \in \Omega_f
\]
with given coefficients \(\alpha, \beta, \gamma\). This standard punch is penetrated into elastic half-space in the case, when
\[
g^0_i(x, y, \xi, \eta) = g^0_i(x, y, \xi, \eta) = g^0_i(x, y, \xi, \eta) = 0, \quad (x, y) \in \Omega_f
\]
for \(i = 1, 2, ..., N\). Contact pressure \(p^0((x, y) \in \Omega_f)\) and displacements \(w^0, v^0, w^0 ((x, y) \in \Omega_f, i = 1, 2, ..., N)\) along the axes \(x, y, z\) corresponding to the first standard punch penetration without friction are supposed to be known. These values can be represented in the form \([2]\)

\[
p^0 = \alpha p^0_0(x, y) + \beta p^0_0(x, y) + \gamma p^0_0(x, y), \quad (x, y) \in \Omega_f
\]
and

\[
w^0 = \alpha w^0_0(x, y) + \beta w^0_0(x, y) + \gamma w^0_0(x, y),
\]

\[
v^0 = \alpha v^0_0(x, y) + \beta v^0_0(x, y) + \gamma v^0_0(x, y),
\]
where \((x, y) \in \Omega_q, i = 1, 2, ..., N\). The functions \(p^0_i(x, y), p^0_i(x, y), p^0_i(x, y) ((x, y) \in \Omega_f)\) and \(w^0_i(x, y), w^0_i(x, y) ((x, y) \in \Omega_q)\) do not depend on the constants \(\alpha, \beta, \gamma\).

The second considered punch has unknown (optimized) shape described by the shape function \(f = f(x, y)\). The second punch with desired shape is penetrated to elastic half-space in accordance with the boundary conditions (5)-(7) in general case, when \(q^0_j \neq 0 (j = x, y, z)\). To write out the reciprocity relation consider two different systems of values. The first system \(w^0 = f^0, p^0, ((x, y) \in \Omega_f), v^0, e^0, w^0 ((x, y) \in \Omega_q, i = 1, 2, ..., N)\) corresponds to the contact problem with a standard punch, and the second system \(w = f, p, ((x, y) \in \Omega_f)\) and \(q^0_i, q^0_i, q^0_i ((x, y) \in \Omega_q, i = 1, 2, ..., N)\) is realized in the contact problem with optimized punch. In accordance with the Betti’s reciprocity theorem \([1]\) we have

\[
\int_{\Omega_f} p^0 d\Omega_f = \int_{\Omega_f} f^0 d\Omega_f + \sum_{i=1}^{N} \int_{\Omega_q} \left[ w^0 q^0_i + v^0 q^0_i + w^0 q^0_i \right] d\Omega_q
\]

If we substitute the expressions (9), (11), (12) for \(f^0, p^0, w^0, v^0, w^0\) into the relation (13) and perform elementary transformations, we obtain

\[
\alpha \left( \int_{\Omega_f} p d\Omega_f - \int_{\Omega_f} p^0 d\Omega_f \right) + \beta \left( \int_{\Omega_f} w d\Omega_f - \int_{\Omega_f} w^0 d\Omega_f \right) + \gamma \left( \int_{\Omega_f} v d\Omega_f - \int_{\Omega_f} v^0 d\Omega_f \right) = 0
\]

Then consider the values \(\alpha, \beta, \gamma\) as an arbitrary constants. To satisfy the equation (14) for any \(\alpha, \beta, \gamma\) it is necessary to equate the expressions in the round brackets to zero. As a result we obtain the following presentations for the total force \(P\) and the moments \(M_j\) applied to the optimized punch

\[
P = P_f(f) - P_q(q)
\]

\[
M_j = M^f_j(f) - M^q_j(q), \quad j = x, y
\]

where \(P_f(f), M^f_j(f)\) are the linear functionals given by the expressions

\[
P_f(f) = \int_{\Omega_f} p^0 d\Omega_f,
\]

\[
M^f_j(f) = \int_{\Omega_f} p^0 d\Omega_f,
\]

and the linear functionals \(P_q(q), M^q_j(q)\) depending on random forces \(q^0_i(x, y, \xi, \eta), q^0_i(x, y, \xi, \eta), q^0_i(x, y, \xi, \eta)\) are expressed with the help of the following formulas

\[
P_q(q) = \sum_{i=1}^{N} \int_{\Omega_q} \left[ w^0 q^0_i + v^0 q^0_i + w^0 q^0_i \right] d\Omega_q
\]

\[
M^q_j(q) = \sum_{i=1}^{N} \int_{\Omega_q} \left[ w^0 q^0_i + v^0 q^0_i + w^0 q^0_i \right] d\Omega_q
\]

\[
M^q_j(q) = \sum_{i=1}^{N} \int_{\Omega_q} \left[ w^0 q^0_i + v^0 q^0_i + w^0 q^0_i \right] d\Omega_q
\]
Consider the case when the random external pointed load \( q_j = q(x, y, \xi, \eta), j = x, y, z \) are applied to the only domain \( \Omega_q = \{ x_1 \leq x \leq x_2, \ y_1 \leq y \leq y_2 \} \) and these loads are statistically independent and uniformly distributed, i.e.

\[
g = g(\xi, \eta) = g(\xi)g(\eta),
\]

\[
g_\xi(\xi) = \begin{cases} 
0, & \xi < x_1 \\
1/(x_2 - x_1), & x_1 < \xi < x_2 \\
0, & \xi > x_2
\end{cases}, \quad g_\eta(\eta) = \begin{cases} 
0, & \eta < y_1 \\
1/(y_2 - y_1), & y_1 < \eta < y_2 \\
0, & \eta > y_2
\end{cases}, \quad F_\xi(\xi) = \begin{cases} 
0, & \xi < x_1 \\
(\xi - x_1)/(x_2 - x_1), & x_1 < \xi < x_2 \\
1, & \xi > x_2
\end{cases}, \quad F_\eta(\eta) = \begin{cases} 
0, & \eta < y_1 \\
(\eta - y_1)/(y_2 - y_1), & y_1 < \eta < y_2 \\
1, & \eta > y_2
\end{cases}
\]

where \( x_1, x_2, y_1, y_2 \) - given values \((x_1 < x_2, y_1 < y_2)\).

Mathematical expectations of the total reaction force \( \hat{P} \) and moments \( \hat{M}_x, \hat{M}_y \) evaluated with the help of the formulas

\[
\hat{P} = \mathcal{E}(P) = P_f(f) - \mathcal{E}(P_0(q)), \quad \hat{M}_j = \mathcal{E}(M_j) = M_j^f(f) - \mathcal{E}(M_j^0(q)), \quad j = x, y
\]

Consider shape optimization problem for rigid punch-shell that consists of minimization of mass functional

\[
J = \rho \int_{\Omega_f} \sqrt{1 + (\nabla f)^2} d\Omega_f \approx \rho S_{f} + \frac{\rho}{2} \int_{\Omega_f} (\nabla f)^2 d\Omega_f \to \min_f
\]

under the following constraints

\[
\hat{P} = P^*, \quad \hat{M}_x = M_x^*, \quad \hat{M}_y = M_y^*
\]

imposed on mathematical expectations of the total reaction force and moments. Here \( S_f = \text{meas} \Omega_f, \rho = h \rho_0 \) \( (h \) - thickness of the considered rigid shell, \( \rho_0 \) - density of the material of the shell-punch) and \( h, \rho, P^*, M_x^*, M_y^* \) - given problem parameters. In what follows we suppose that the domain \( \Omega_f \) is a circular with given radius \( a \) \((\Omega_f: x^2 + y^2 \leq a^2)\). Let us now construct the Lagrange augmented functional corresponding to optimization problem (21), (22)

\[
J^L = \int_{\Omega_f} \left[ \frac{\rho}{2}(\nabla f)^2 - \lambda_\alpha p_\alpha^0 f - \lambda_\beta p_\beta^0 f - \lambda_\gamma p_\gamma^0 f \right] d\Omega_f
\]

where \( p_\alpha^0(r), p_\beta^0(r, \theta), p_\gamma^0(r, \theta) \) are represented as

\[
p_\alpha^0(r) = E/\left[ \pi(1 - \nu^2)\sqrt{a^2 - r^2} \right], \quad r = \sqrt{x^2 + y^2}
\]

\[
p_\beta^0(r, \theta) = 2E r \cos \theta/\left[ \pi(1 - \nu^2)\sqrt{a^2 - r^2} \right], \quad r_\gamma^0(r, \theta) = 2E r \sin \theta/\left[ \pi(1 - \nu^2)\sqrt{a^2 - r^2} \right]
\]

and Lagrange multipliers \( \lambda_\alpha, \lambda_\beta, \lambda_\gamma \) are determined with the help of conditions (22). The radius \( r \) and the angle \( \theta \) belong to the domain \( \Omega_f = \{ 0 \leq r \leq a, 0 \leq \theta \leq 2\pi \} \). Necessary optimality condition and boundary condition for desired function \( f \) constitute the following boundary-value problem

\[
\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2} = -\lambda_\alpha p_\alpha^0(r) - \lambda_\beta p_\beta^0(r, \theta) - \lambda_\gamma p_\gamma^0(r, \theta)
\]

\[
(f(r, \theta))_{r=a} = 0, \quad 0 \leq \theta \leq 2\pi
\]

It is convenient to represent the desired bounded solution of the Poisson equation (25) with the Dirichlet condition (26) in the following form

\[
f(r, \theta) = f_\alpha(r) + f_\beta(r, \theta) + f_\gamma(r, \theta)
\]

\[
f_\alpha(r) = \lambda_\alpha \chi_\alpha(r), \quad f_\beta(r, \theta) = \lambda_\beta \chi_\beta(r, \theta), \quad f_\gamma(r, \theta) = \lambda_\gamma \chi_\gamma(r, \theta)
\]

Shape functions \( \chi_i \) \((i = \alpha, \beta, \gamma)\) must satisfy the following boundary-value problems

\[
\Delta \chi_i = -\frac{1}{\rho} p_i^0, \quad i = \alpha, \beta, \gamma
\]

\[
(\chi_i)_{r=a} = 0, \quad (\chi_i)_{r=0} < \infty
\]
and Lagrange multipliers $\lambda_i$ ($i = \alpha, \beta, \gamma$) are determined with the help of the isoperimetric conditions (22). Taking into account that in the case $i = \alpha$ the function $p_{\alpha}^0 = p_{\alpha}(r)$ is axisymmetric with respect to $z$-axis we solve the boundary-value problem (28) and find corresponding symmetric shape function

$$
\chi_{\alpha}(r) = E \left[ \sqrt{a^2 - r^2} - a \ln \left( \frac{a + \sqrt{a^2 - r^2}}{a} \right) \right] / (\rho \pi (1 - \nu^2))
$$

(29)

Figure 1: Axisymmetric shape function $z = \chi_{\alpha}\rho\pi(1 - \nu^2)/E$

Optimal axisymmetric shape function (29) corresponds to the case where only constraint on the external load is taken into account. This case is illustrated in Fig. 1 where the value $a = 1$ and the value $\chi_{\alpha}\rho\pi(1 - \nu^2)/E$ is shown along the axis $oz$.

In the case when $i = \beta, \gamma$ the corresponding solutions of the boundary-value problem (28) can be also found in analytical form. To find shape function we rewrite the equation (28) as

$$
\frac{\partial^2 \chi_{\beta}}{\partial r^2} + \frac{1}{r} \frac{\partial \chi_{\beta}}{\partial r} + \frac{1}{r^2} \chi_{\beta} = -\frac{2}{\rho} \rho_{\alpha}^0(r) \cos \theta
$$

(30)

and represent the solution of the equation (30) as

$$
\chi_{\beta}(r, \theta) = (\cos \theta)\chi_0^0(r)
$$

(31)

Substituting (31) into (30) and performing elementary transformations we will have

$$
\frac{d^2 \chi_0^0}{dr^2} + \frac{1}{r} \frac{d \chi_0^0}{dr} - \frac{1}{r^2} \chi_0^0 = \frac{2K}{\rho} \frac{d}{dr} \left( \sqrt{a^2 - r^2} \right)
$$

(32)

where $K = E/\pi(1 - \nu^2)$. Performing two times integration of the equation (32) we obtain

$$
r\chi_0^0(r) = -\frac{2K}{3\rho} (a^2 - r^2)^{3/2} + \frac{C_2}{2} r^2 + D
$$

(33)

Here $C$ and $D$ are the arbitrary constants of integration. To find these constants we use the boundary condition boundedness condition (28). We have $C = -4K a / 3\rho$, $D = 2Ka^3 / 3\rho$. As a result we find $\chi_{\beta}(r, \theta)$ and by a similar manner $\chi_{\gamma}(r, \theta)$ as

$$
\begin{align*}
\chi_{\beta}(r, \theta) &= \chi_0^0(r) \cos \theta = 2E \cos \theta \left\{ a(a^2 - r^2) - (a^2 - r^2)^{3/2} \right\} (3\pi(1 - \nu^2)\rho r) \\
\chi_{\gamma}(r, \theta) &= \chi_0^0(r) \sin \theta = 2E \sin \theta \left\{ a(a^2 - r^2) - (a^2 - r^2)^{3/2} \right\} (3\pi(1 - \nu^2)\rho r)
\end{align*}
$$

(34)
Figure 2: Shape function \( z = \chi_\beta 3\pi \rho (1 - \nu^2)/(2E) \)

Figure 3: Shape function \( z = \chi_\gamma 3\pi \rho (1 - \nu^2)/(2E) \)
Fig. 2 corresponds to the case when we take into account the constraint on the moment $M_y$. Here the value $\chi_3 3\pi \rho (1 - \nu^2)/(2E)$ is shown along the axis $oz$. The value $\chi_3 3\pi \rho (1 - \nu^2)/(2E)$ represented in Fig. 3 describes the shape function under the constraint on $M_x$. It is seen, in particularly, that the functions shown in Fig. 2 and Fig. 3 are not symmetrical. Both in Fig. 2 and Fig. 3 the value $a = 1$.

Note that the application of the reciprocity theorem gave us the possibility to represent the original constraints on $E(P)$ and $E(M_j)$ in the form (21), (22). As it is seen from (20), (22) each of the expressions for $\hat{P}$, $\hat{M}_x$, $\hat{M}_y$ can be divided to two parts. The first part is a linear functional of the unknown optimized shape $f$ and the second part represents an integral depending on the joint probability density $g(\chi, \eta)$ and the solution of the contact problem for standard punch with plane bottom. If this solution is known then the second part of the constraint can be independently estimated. Thus, with the application of Betti’s theorem the original probabilistic optimization problem is reduced to estimation of probabilistic parts of the constraints and solution of deterministic optimization problem.

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References
