A Monte Carlo Solution for Stochastic Programming Problems with Recourse

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Abstract
This paper describes and evaluates numerical models for the solution of stochastic programming problems with recourse. However, the emphasis is given to the mathematical characterization of the joint distribution of the random parameters associated to the data of the problem. In this respect, two different approaches are used. When the joint probability distribution is characterized by one or two random factors, lattice techniques can be used to substitute continuous distributions by discrete ones with good accuracy. Thought, in general, the Monte Carlo approach is the only alternative to deal with more complex distributions. The details for the inclusion of this approach in the solution methodologies developed with the lattice approach are the main subject of this study. It is concluded that with the Monte Carlo approach the number of factors of uncertainty of the problem doesn’t influence significantly the efforts to obtain an approximation of the solution.

Keywords: Stochastic Programming, Recourse, Monte Carlo.

1. Introduction
Stochastic linear programming problems are often defined to incorporate in a linear programming model the uncertainty of the data of the problem. If the uncertainty is related with the change over time of the problem parameters it is assumed, normally, that the decision maker has the possibility to take corrective actions after the realization of the random events that influence the system he wants to control. In this case, the problem involves decisions that have to be made in different time stages, and the objective is, roughly, to prepare the system so that the adverse impact of the random events is minimized. To perform this action, the decision maker needs information about the set of decisions he may need to take after the realization of the random events.

Since there are decisions that need to be made prior to the data observation, the aim of the model is to optimize decisions that are made in two-stages. Consequently, the mathematical formulation of these problems involves two-linked models: the first model defining the decisions that have to be taken before the realization of some random parameters (first stage decisions), and the second model that evaluates the decisions to be taken after the realization of the random parameters (second stage decisions). The second-stage decisions are influenced by the first stage ones and can often be seen as corrective actions to be taken in order to minimize the consequences of the system equilibrium deviations.

The objective of the problem is to determine the optimal first-stage variables that minimize a cost function that involves both the costs associated to the first-stage decisions and the expected costs associated to all possible realizations of the second stage variables. Therefore, it has to be obtained considering a variety of future alternative realizations of the random parameters, requiring usually the characterization of a multidimensional probability distribution.

The solution of stochastic linear programming (SLP) problems is more often than not difficult to obtain due, essentially, to two main reasons. The first one is the size of the problem originated by the random data, which grows exponentially with the number of outcomes considered. The second one is related with handling general probability distributions. Therefore, the first step is often to construct discrete distributions, close enough to the original one. The initial problem is then replaced by a computationally tractable approximation. This approach raises issues on the quality of the computed solutions and on the computational effort to achieve a given level of optimality with respect to the original problem.

In this study we consider random factors with a joint Gaussian distribution. To evaluate the obstacles due to the size of the problem originated by the random data, we consider both the cases in which the parameters are perfectly correlated and have arbitrary correlations. Additionally, for validation purposes, two different approaches are used to obtain the discrete distributions of the parameters involved in the data of the problem: one is based in the construction of trinomial trees (lattice technique) and the other is based in the Monte Carlo simulation. To deal with the general probability distribution we assume that all random parameters follow generalized Winner processes.

In this study we show that by using the Monte Carlo method to characterize the joint probability distribution of the random properties, two-stage stochastic linear programming problems, can be solve efficiently even if several...
number random factors are considered.
The solution procedure is iterative and makes use of an auxiliary problem to generate tentative values of the first stage variables consistent with the first stage restrictions. The auxiliary problem is defined using the underlying linear programming (ULP) problem and partitions of the values of the random parameters. The advantage of this approach, over defining directly the tentative values of the first stage variables, is that the random parameters are only bounded by their physical domain, and thus they are not subject to complex restrictions when they are selected as inputs of the auxiliary problem.

This article is organized as follows. The next section presents the mathematical formulation of typical two-stage stochastic linear programming problems. It also includes a subsection dedicated to the mathematical characterization of the joint probability distribution giving special emphasis to the use of the Monte Carlo approach. The use of lattice techniques in this type of problems is analyzed in detail in [1]. The next section includes also a subsection related with the iterative searching technique of the optimal first-stage decision variables. This method is also described in [1] but here the details necessary for the utilization of the Monte Carlo representation of the joint distribution are also analyzed. The third section is dedicated to the presentation and interpretation of the results obtained with a test case available in the literature (e.g. [2]). The last section summarizes the main conclusions of this work.

2. Mathematical Modeling

The early formulations of the stochastic problem emerged with the purpose of extending linear programming methods (e.g. [3], [4] and [5]). Since then, the consideration of uncertainty in linear programming problems has been discussed with increasing frequency. Generally, the mathematical formulation of two-stage SLP problems can be represented by the following two linked problems:

\[
\begin{align*}
\text{(1)} & \quad \min \{ c^T x + E_{\omega_1} \left[ Q(x, \omega) \right] \mid Ax \leq b, x \in R^n_R \} \\
\text{(2)} & \quad \min \{ q^T y \mid T_k x + Wy \geq h, y \in R^n_R \}
\end{align*}
\]

The objective of the problem is to minimize a function that represents an expected cost with two components: the first corresponds to the cost associated to first-stage decisions, which are represented by the vector \(x\), and the second corresponds to the expected cost associated to second-stage decisions, which are represented by the vector \(y\). The costs associated to these variables are represented by \(c\) and \(q\), respectively. In the second-stage constraints, \(W\) is known as the recourse matrix and \(T\) as the technology matrix. The matrix \(A\) is associated to the first-stage constraints.

The uncertainty of the data is represented mathematically in the second-stage problem by the random variable \(\omega \in \Omega\), which is defined on the probability space \((\Omega, \mathcal{F}, P)\), where \(\mathcal{F}\) is a complete and right continuous filtration and \(P\) is a probability measure on \(\Omega\). The function \(Q(x, \omega)\) is known as the second-stage value function, and the operator \(E_{\omega}\) is the expectation under the probability measure \(P\).

Within the framework of discrete distributions, the realizations of the random parameters are specified in the form of scenarios, containing each of them a complete description of \(q, T\) and \(h\) (typically, the components of \(W\) are considered deterministic). Though, in this case, it is possible to formulate a deterministic equivalent linear programming (DELP) problem [6], which can be represented as:

\[
\begin{align*}
\min \quad & c^T x + \sum_{k=1}^{n_R} p_k q_k^T y_k \\
\text{s.t.} \quad & Ax \leq b \\
& T_k x + Wy_k \geq h, \quad k = 1, \ldots, n_R \\
& x \in R^n_R, \quad y_k \in R^n_R, \quad k = 1, \ldots, n_R
\end{align*}
\]

where \(n_R\) is the number of joint realizations and \(p_k\) is the probability of realization of scenario \(k\).

2.1. The probability distribution

To specify the scenarios corresponding to the realization of the random parameters, it is assumed that the random parameters are stochastic time dependent variables following generalized Wiener processes, which are represented mathematically by the following expression (e.g. [7])

\[
df = \mu dt + \sigma dz
\]

where \(\mu\) is the expected rate of change of \(f\), \(\sigma^2\) is its variance per unit of time and \(dz\) is a Wiener process:

\[\text{where } \mu \text{ is the expected rate of change of } f, \sigma^2 \text{ is its variance per unit of time and } dz \text{ is a Wiener process:} \]
where \( \varepsilon \) is a stochastic variable with a normalized Gaussian distribution and \( d\tau \) is an infinitesimal time interval.

To reproduce the local statistical properties of the random parameters two different methods are considered. If the number of correlated factors is one or two, the temporal evolution of the parameters is simulated using trinomial trees, a technique that was developed to evaluate financial derivative securities (e.g. [7]). This technique can be adapted easily to simulate at least two driving random factors with Gaussian or lognormal distributions (e.g. [1]). If more than two parameters are considered only the Monte Carlo approach is adopted. In this case, considering the case of Gaussian distributions, the temporal evolution of the random data can be simulated using the following expression:

\[
dz = \varepsilon \sqrt{d\tau}
\]  

(5)

Where, taking into consideration \( n \) correlated random parameters, the set of the stochastic variables \( \varepsilon_i, i = 1,..,n \), can be defined using the following equations:

\[
\varepsilon_i = \sum_{j=1}^{n} \alpha_j c_j, \quad i = 1,..,n
\]  

(6)

\[
\sum_{j=1}^{n} \alpha_j^2 = 1, \quad i = 1,..,n
\]  

(7)

\[
\rho_{ik} = \sum_{j=1}^{n} \alpha_j \alpha_{kj}, \quad i = 1,..,n, \quad k = i+1,..,n
\]  

(8)

In these equations \( \rho_{ik} \) coincide with the correlation coefficient between \( \varepsilon_i \) and \( \varepsilon_k \) and \( c_j, j = 1,..,n \), represent a set of independent stochastic variables (or factors of uncertainty) with normalized Gaussian distributions. Equations (8) and (9) can be rewritten to define a system of recurrence relations for the evaluation of the weights \( \alpha_{ij} \) of the equation (7):

\[
\alpha_i = \left(1 - \sum_{j=1}^{n} \alpha_j^2\right)^{1/2}
\]  

(9)

\[
\rho_{ik} = \frac{\sum_{j=1}^{n} \alpha_j \alpha_{kj}}{\alpha_i}, \quad k > i
\]  

(10)

Thought, and without loss of generality, we assume in this study that all the correlation coefficients are evaluated adopting the first parameter of the random data as reference and that they verify the following additional condition:

\[
\rho_{ij} = \rho_{ii}\rho_{ij}
\]  

(11)

Under these conditions, the following relations are obtained:

\[
\begin{align*}
\sum_{j=1}^{n} \alpha_j &= 0 \\
\alpha_{ii} &= \frac{\rho_{ii}}{1 - \rho_{ii}^2}, \quad i = 1,..,n \\
\alpha_{ij} &= -\rho_{ij} \quad j \neq i
\end{align*}
\]  

(12)

2.2. The Solution algorithm

To obtain the solutions of the problems considered in this study, two different approaches are used. If the probability distribution is characterized by a relative small number of realization points an approximation of the exact solution can be obtained solving directly the DELP problem using the SIMPLEX algorithm. But, if the representation of the distribution function requires a large number of realization points an iterative procedure is necessary. When the number of scenarios increase, the number of variables that have to be considered in problem (3) increases considerably, making the application of the standard SIMPLEX method difficult with great needs of computational memory and increasingly significantly the number of iterations and the computational time to achieve the optimal solution.

The iterative solution involves the determination of a sequence of values of the first-stage variables that lead to
consecutive decreasing values of the first-stage objective function. This process is explained next, illustrating the steps involved in a single iteration. The first step is to guess values of the first-stage decisions, say $x_t$, which are obtained using the underlying linear programming (ULP) problem:

$$\begin{align*}
\text{Min } & \bar{c}^T x_t + q_t^T \bar{y}_t \\
\text{s.t. } & Ax_t \leq b \\
& T_t x_t + W \bar{y}_t \geq h_t \\
& x_t \in R^n, \quad \bar{y}_t \in R^n
\end{align*}$$

Being this LP problem assembled, each iteration, with different tentative values of the random parameters ($T_t$, $h_t$, and $q_t$). These values are selected simply from their partitions, since, at this step, the objective is to generate only values of $x \in R^n$ consistent with the restriction $Ax \leq b$. Considering the DELP problem, this course of action can be interpreted as searching for some active restrictions that lead to the optimal solution of $x$.

The set of all-possible second-stage decisions associated to $x_t$ is evaluated afterwards from the second-stage problem, considering the joint realization of the random parameters. Accordingly, with $x_t$, the following $n_R$ second-stage problems are solved:

$$\begin{align*}
\text{Min } & c_{2k} = q_k^T y_{t_k} \\
\text{s.t. } & W y_{t_k} \geq (h_t - T_t x_t) \\
& y_{t_k} \in R^{n_k}
\end{align*}$$

The value of the objective function, and of the expected values of the second-stage decisions, associated to $T_t$, $h_t$ and $q_t$, are then evaluated from:

$$c_i = c_i + \bar{c}_i = c^T x_i + \sum_{t=1}^{n_R} p_t c_{2k}$$

$$\bar{y}_i = \sum_{t=1}^{n_R} p_t y_{t_k}$$

The starting values of $T_t$, $h_t$ and $q_t$, are defined using the expected values of the random parameters. In subsequent iterations, independent elementary increases and decreases of each parameter are tested. The optimal generating values of these parameters are updated after each modification that leads to an improvement in the objective function.

The implementation of this approach is straightforward if the joint probability distribution can be defined with a small number of realization points. In this case all scenarios can be available at the starting of the iteration procedure. Thought, if the number of independent random parameters is large (i.e. greater than 2) the simultaneous definition of all scenarios becomes impractical, since its number is of the order $n_f^R$, where $n_f$ is the number of points necessary to represent the distribution of a single parameter and $n$ is the number of random parameters. Nevertheless, for the convergence of the algorithm it is required that the joint distribution is the same for all iterations. This may seem an obstacle in the case of large number of random parameters and for the utilization of the Monte Carlo approach. However, we have concluded that in the Monte Carlo approach the set of independent random factors $c_j$, $j=1,...,n$, can be generated from a single stochastic vector, $\bar{c}_i = \{c_j : k=1,...,n_R + n\}$, of independent random values with a standardized Gaussian distribution. The number of components of this vector is $n_R + n$, where $n_R$ represents the number of Monte Carlo simulations (i.e. the number of scenarios) and $n$ is the number of random parameters. In this way, at scenario $k$, the independent random factors $c_{jk}$, $j=1,...,n$, can be defined from the relation: $c_{jk} = \bar{c}_i(k + j - 1)$.

3. Results and discussion

To illustrate the application of the model described in the previous section, we consider a simple test case that we have used in previous studies (e.g. [1] and [8]). The problem is adapted from a farming example that is presented in [2]. With an additional crop, to test the model with a multidimensional distribution, the problem reads as follows. A farmer has 500 acres of land, which he uses to raise (1) barley, (2) wheat, (3) corn, and (4) sugar beets. During
the winter, he must decide how much land to devote to each crop. The planting costs are €200, €150, €230 and €260 per acre of land devoted to barley, wheat, corn and sugar beets, respectively. He knows that at least 150 ton of barley, 200 ton of wheat and 240 ton of corn are needed for internal consumption. These amounts can be raised on the farm or bought from a wholesaler. Any production in excess of the feeding requirements will be sold. Selling prices are €200, €170, and €150 per ton of barley, wheat and corn, respectively, and purchase prices are 40% higher. Sugar beets are sold at €36/ton but, due to quota limitations, he is not allowed to sell more than 6000 ton. Based on historical data, the farmer knows that the mean yields on his land are 2.6, 2.5, 3.0, and 20.0 ton/acre, and the variances are 6/25, 1/6, 6/25 and 32/3 ton^2/acre^2, for barley, wheat, corn and sugar beets, respectively.

Under these conditions, the optimization problem that must be solved to determine the optimal allocation of land is represented as follows:

\[
\begin{align*}
\text{Min } \bar{c} &= \sum_{i=1}^{n} c_i x_i + E_{\omega} \left[ c_2 \left( x_1, \ldots, x_n, \omega \right) \right] \\
\text{s.t. } \sum_{i=1}^{n} x_i &\leq b \\
x_i &\geq 0, \quad i = 1, \ldots, n_c
\end{align*}
\]

With:

\[
c_2 = \text{Min} \left( \sum_{i=1}^{n_c} q_i^+ y_i^+ - q_i^- y_i^- \right) \\
\text{s.t. } t_i x_i + y_i^+ - y_i^- &\geq h_i, \quad i = 1, \ldots, n_c \\
y_i^- &\leq u_i \\
y_i^+, y_i^- &\geq 0, \quad i = 1, \ldots, n_c
\]

Where \( n_c \) represents the number of crops. The first stage variables, \( x_i, i = 1, \ldots, n_c \), represent the number of acres assigned to each crop. The constants \( c_i, i = 1, \ldots, n_c \), represent the planting costs per unit of area associated to each crop. The objective function of the first-stage problem includes the planting costs and the net expected costs associated to the purchasing and selling of all crops. The first stage restriction states that the number of acres assigned can not exceed the number of acres available (\( b \)). The second-stage variables, \( y_i^+ \) and \( y_i^- \), \( i = 1, \ldots, n_c \), represent the number of tons of crop \( i \) that must be acquired to meet the internal requirements (\( h_i \)) or that can be sold if the internal requirements are exceeded, respectively. The parameters \( q_i^+ \) and \( q_i^- \), \( i = 1, \ldots, n_c \), represent the purchasing and selling prices, respectively. The second stage objective function represents the net cost associated to the purchasing and selling of the planted products. The parameters \( t_i, i = 1, \ldots, n_c \), represent the rates of production of each crop. They are the random parameters of the data considered in this problem. The first group of restrictions state, for each crop, that the production and the quantity purchased can’t be lower than the internal needs and the quantity put up for sale. An additional restriction states that the quantity put up for sale of the last crop (sugar beets) should not exceed the allowed quota (\( u_i \)).

To evaluate the models described in the previous section, we consider 3 different hypotheses related with the correlation among the production rates of the crops.

In the first hypothesis we assume that they are perfectly correlated. In this case, a single random factor can be used to reproduce all random parameters (i.e., the production rates). Under this hypothesis the solution can be obtained using 3 alternative methods: (M1) using the DELP problem and the trinomial technique for the distribution representation, (M2) using the iterative procedure based in the ULP problem and the trinomial technique for the distribution representation, and (M3) using the iterative procedure based in the ULP problem and the Monte Carlo method to describe the probabilistic properties of the random parameters. However, in any case, it is expected that method M3 lead to similar results of the others only when the number of scenarios in the other alternatives is sufficiently high so that the discrete distribution constitute a good approximation of the continuous one.

In the second hypothesis we consider the case in which the first 3 production rates are perfectly correlated and imperfectly correlated with the forth rate. In this case, all the three methods still can be applied, but, as problem (3) illustrates, the number of variables and the number of restrictions that must be considered in the DELP problem becomes unaffordable.

In the third hypothesis, we consider the case in which all rates can have arbitrary correlations, including the case of
independency. In this case only the solution methodology M3 is used. The next two tables show the properties of the scenarios defined under the hypothesis of perfectly correlation, using bi-dimensional trinomial trees, and temporal evolutions of the Winner processes of 1, 2, 3 and 4 time steps. Table 1 presents the probabilities of each scenario for each number of time steps considered, and table 2 shows the correspondent values of the variable $\varepsilon$, with mean 0 and variance 1. From these values the correspondent values of each rate of production are obtained from the expression: $t_i(k) = E(t_i) + S(t_i)\varepsilon(k)$, where $E(t_i)$ represents the expected value of $t_i$ and $S(t_i)$ its standard deviation.

Table 1: Probabilities of realization considering trinomial evolutions in 1, 2, 3 and 4 time steps

<table>
<thead>
<tr>
<th>$n_g$</th>
<th>$p(k)$ ($k = 1,...,n_g$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.3333 0.3333 0.3333</td>
</tr>
<tr>
<td>5</td>
<td>0.1111 0.2222 0.3333 0.2222 0.1111</td>
</tr>
<tr>
<td>7</td>
<td>0.0370 0.1111 0.2222 0.2593 0.2222 0.1111 0.0370</td>
</tr>
<tr>
<td>9</td>
<td>0.0123 0.0494 0.1235 0.1975 0.2346 0.1975 0.1235 0.0494 0.0123</td>
</tr>
</tbody>
</table>

Table 2: Realization values of the random parameter $\varepsilon$, with a standardized Gaussian distribution

<table>
<thead>
<tr>
<th>$n_g$</th>
<th>$\varepsilon(k)$ ($k = 1,...,n_g$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-1.2247 0.0000 1.2247</td>
</tr>
<tr>
<td>5</td>
<td>-1.7321 -0.8660 0.8660 1.7321</td>
</tr>
<tr>
<td>7</td>
<td>-2.1213 -1.4142 -0.7071 0.0000 0.7071 1.4142 2.1213</td>
</tr>
<tr>
<td>9</td>
<td>-2.4495 -1.8371 -1.2247 -0.6124 0.0000 0.6124 1.2247 1.8371 2.4495</td>
</tr>
</tbody>
</table>

The next two tables show the optimal solutions for the distributions defined using the information of tables 1 and 2, obtained with the solution methodologies identified previously as M1 and M2.

Table 3: Optimal solutions obtained with M1 for the distributions defined with tables 1 and 2

<table>
<thead>
<tr>
<th>$n_g$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$E(c)$</th>
<th>$Var(c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>75.000</td>
<td>95.000</td>
<td>80.000</td>
<td>250.000</td>
<td>-81538.333</td>
<td>2.56792E+09</td>
</tr>
<tr>
<td>5</td>
<td>85.642</td>
<td>80.000</td>
<td>71.528</td>
<td>262.830</td>
<td>-80389.139</td>
<td>2.33312E+09</td>
</tr>
<tr>
<td>7</td>
<td>78.650</td>
<td>90.444</td>
<td>71.719</td>
<td>259.188</td>
<td>-80038.754</td>
<td>2.27243E+09</td>
</tr>
<tr>
<td>9</td>
<td>88.384</td>
<td>88.889</td>
<td>72.727</td>
<td>250.000</td>
<td>-80327.747</td>
<td>2.44515E+09</td>
</tr>
</tbody>
</table>

Table 4: Optimal solutions obtained with M2 for the distributions defined with tables 1 and 2

<table>
<thead>
<tr>
<th>$n_g$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$E(c)$</th>
<th>$Var(c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>75.000</td>
<td>95.000</td>
<td>80.000</td>
<td>250.000</td>
<td>-81538.333</td>
<td>2.56792E+09</td>
</tr>
<tr>
<td>5</td>
<td>85.669</td>
<td>80.000</td>
<td>71.749</td>
<td>262.582</td>
<td>-80381.788</td>
<td>2.33437E+09</td>
</tr>
<tr>
<td>7</td>
<td>78.653</td>
<td>84.141</td>
<td>71.719</td>
<td>265.487</td>
<td>-80029.694</td>
<td>2.27243E+09</td>
</tr>
<tr>
<td>9</td>
<td>88.384</td>
<td>88.889</td>
<td>72.727</td>
<td>250.000</td>
<td>-80327.747</td>
<td>2.44515E+09</td>
</tr>
</tbody>
</table>

Table 5 presents the optimal solutions, for a case of perfectly correlation, obtained with the methods M1 and M2 with a distribution of 125 scenarios, defined with the lattice technique, and for the method M3 considering simulations.

Table 5: Optimal solutions for the case of perfectly correlation: in M1 and M2 a distribution with 125 scenarios, defined with the lattice technique, was used; in M3 simulations were considered

<table>
<thead>
<tr>
<th>M</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$E(c)$</th>
<th>$Var(c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>83.617</td>
<td>84.797</td>
<td>75.112</td>
<td>256.475</td>
<td>-80208.688</td>
<td>2.32949E+09</td>
</tr>
<tr>
<td>2</td>
<td>83.616</td>
<td>84.768</td>
<td>75.206</td>
<td>256.410</td>
<td>-80208.152</td>
<td>2.33003E+09</td>
</tr>
<tr>
<td>3</td>
<td>83.934</td>
<td>84.923</td>
<td>75.195</td>
<td>255.949</td>
<td>-80327.747</td>
<td>2.44515E+09</td>
</tr>
</tbody>
</table>

The results of Tables 3 and 4 show, firstly, that the solution methods M1 and M2 lead to similar results. Since the method M1 solves the exact problem (for the given distribution), we can conclude that the iterative solution used in M2 leads to similar results.

The results of table 5 for M1 and M2 can be seen as an approximation of the exact solution, since the large number
of scenarios used provides a good approximation of the shape of the exact distribution. Comparing these results with those of Tables 3 and 4 we can conclude that the precision of the representation of the distribution has a considerable influence in the optimal solution.

The results presented in Table 5 for M3 shown that this method provides also a good approximation of the exact solution.

The next table presents solutions for the situation in which randomness can be described by two random factors. These results were obtained considering \( \rho_1 = \rho_3 = 1 \) and \( \rho_4 = -0.5 \). In this case the bi-dimensional distributions in M1 and M2 were obtained with a three-dimensional lattice. The number of steps considered was 10 and the number of scenarios obtained, in the end of the evolution, was 441. In the solution of M3, \( 5 \times 10^5 \) simulations were considered. The results of table 6 show again that the three methods lead to similar solutions. The different processes adopted for the characterization of the joint distributions support the adequacy of the Monte Carlo approach to deal with more complex situations.

### Table 6: Optimal solutions for the case of \( \rho_2 = \rho_3 = 1 \) and \( \rho_4 = -0.5 \): in M1 and M2 a distribution with 441 scenarios, defined with the lattice technique, was used; in M3, \( 5 \times 10^5 \) simulations were considered.

<table>
<thead>
<tr>
<th>M</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( E(c) )</th>
<th>( Var(c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>84.323</td>
<td>84.636</td>
<td>75.241</td>
<td>255.800</td>
<td>-80205.995</td>
<td>6.21977E+08</td>
</tr>
<tr>
<td>2</td>
<td>84.186</td>
<td>84.656</td>
<td>75.294</td>
<td>255.864</td>
<td>-80205.620</td>
<td>6.21730E+08</td>
</tr>
<tr>
<td>3</td>
<td>83.883</td>
<td>84.923</td>
<td>75.195</td>
<td>256.000</td>
<td>-80195.324</td>
<td>6.22542E+08</td>
</tr>
</tbody>
</table>

The next table presents complete solutions for two situations in which the joint distributions are multidimensional. In this case only the method based in the Monte Carlo approach can be used without significant additional effort. Comparatively to the other solutions obtained with this approach, the memory requirements are almost the same and the CPU time is also of the same order (about 30 seconds).

### Table 7: Optimal solutions for the cases of independency and imperfect correlation of all rates, obtained with the methodology M3, considering \( 5 \times 10^5 \) simulations.

<table>
<thead>
<tr>
<th>( \rho_{12} = 0.8, \rho_{13} = 0.6, \rho_{14} = -0.5 )</th>
<th>( \rho_2 = \rho_3 = \rho_4 = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{(opt)} )</td>
<td>2.4444</td>
</tr>
<tr>
<td>( x_i )</td>
<td>83.81</td>
</tr>
<tr>
<td>( y_i^c )</td>
<td>68.72</td>
</tr>
<tr>
<td>( y_i^c )</td>
<td>0.81</td>
</tr>
<tr>
<td>( \overline{c}, Var(c) )</td>
<td>-80209.9</td>
</tr>
</tbody>
</table>

These results show that in any scenario related with the correlation of the rates of production, on average, the farmer has excess production of crops 1, 2 and 4, and scarcity of only crop 3, consequently the average net cost is negative (i.e., it is profit). The most sensitive parameter to the correlation structure is the variance of the net cost. The greater variance is obtained in the case where the rates can fluctuate freely. The case in which \( \rho_1 = \rho_3 = 1 \) and \( \rho_4 = -0.5 \) is similar to the case in which \( \rho_2 = 0.8, \rho_3 = 0.6 \) and \( \rho_4 = -0.5 \), but the variance is lower in this second case. This lower value of the variance is consistent to the fact that the third crop is on average scarce. Consequently some degree of freedom from the other two crops has the effect of providing some compensation when its production rate is lower.

### Conclusions

Recently developed new solution methodologies, for stochastic programming problems with recourse are presented and analysed in this article. The emphasis of this study rests on the mathematical characterization of the multi-dimensional joint distributions, which are associated to the randomness of the data of the problems. The general solution approach proposed in this study makes use of an iterative procedure for searching the optimal first-stage restrictions, which is developed using the underlying deterministic LP problem. Additionally, this algorithm is combined with the Monte Carlo approach for representation of the joint probability distribution.

To access the ability of this approach to obtain good approximations of the problem’s solutions, two additional methodologies were used. The first makes use of the deterministic equivalent linear programming problem, to obtain the exact solutions of problems in which the continuous joint distribution can be substituted by a discrete one with good accuracy. The second uses an iterative approach combined with a technique appropriated for the
definition of the discrete distributions. By obtaining the solution of problems in which the three methods could be used, we were able to verify that the methodology base in the Monte Carlo approach can be used to deal adequately with these problems and presents the flexibility do deal with more complex problems in which the randomness of the that is described by a multi-dimensional distribution.

References