A Method to Improve the Calculation of the Bicriteria Pareto Frontier

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Abstract
If the optimization design problem considers multiple conflicting objectives, Pareto-optimality results in a number of trade-off optimal solutions shaping the Pareto frontier. Each of these solutions to the multicriteria problem represents a point in the boundary of the feasible objective space, such that the improvement in one of the objectives results in the worsening of at least one of the other objectives. In calculating these points it is important to get their uniform distribution and not include non-Pareto or local Pareto solutions.

The goal of this paper is to study methods of calculating the Pareto frontier and to purpose improvements towards obtaining a well-distributed and well-representing set of points for a bicriteria problem. In order to get a better distribution of points on the Pareto frontier it is used a strategy following closer the main shape of that frontier. To get only the global Pareto points, this strategy sweeps the criterion space just once, getting automatically rid off the non-Pareto and local Pareto points, without any further filtering.

Keywords: Optimal design, Vector Optimization, Pareto optimality.

1. Introduction
Most real-world engineering design problems have more than one criterion or objective. The multiobjective or vector optimal design problem is that of optimizing simultaneously several conflicting objectives. Pareto optimality is the concept used to characterize the optimality of multicriteria optimization problems [1,2]. A (global) Pareto solution is one such that the improvement in one of the objectives results in the worsening of at least one of the other objectives. The set of Pareto points in the criterion space forms the Pareto frontier (Figs. 1, 2). This frontier is always on the boundary of the feasible criterion region. Other points on this boundary (but not on the Pareto frontier) are local-Pareto and non-Pareto points (Figs. 1, 2). A local-Pareto point is a solution that obeys to the Pareto concept only in its neighborhood. Any point is better than another on the Pareto frontier (non-dominated points). All the other points are worse than at least one point of this frontier (dominated points). There are infinitely many Pareto solutions for a multicriteria optimization problem. A point in the criterion space corresponding to the minimum of one objective independently of the other objectives is called an anchor point. The point in the criterion space corresponding to the minimum of all objectives independently obtained is called the utopia point.

Mathematical programming techniques to find the Pareto solutions transform the vector optimization problem into a series of scalar optimization problems. A simple and common approach for this scalarization is to optimize a weighted sum of the objectives. Each set of weights generates a solution point [3]. However, this approach does not get an even distribution of points on the Pareto curve and it obtains points from all parts of the Pareto curve only if this curve is convex [4].

Another strategy is the Normal Boundary Intersection (NBI) method developed by [5]. This method aims to get the Pareto solutions by starting from normal directions to the plane containing all the anchor points (utopia plane or...
ideal plane) and intersecting the boundary of the feasible domain. This method generates even spread solutions and also generates all available Pareto points, but may also generate dominated solutions. Since the criteria are generally noncommensurable and their values may differ greatly, normalization has been proposed for the criterion space, the normalized values of all criteria ranging from zero to one.

In [6] is developed the Normalized Normal Constraint (NNC) method. This method works in a normalized criterion space and introduces, for each scalar optimization problem, an inequality constraint that splits the whole normalized feasible criterion space, and then its boundary, in two regions: a feasible region and an unfeasible one. For a bi-objective problem, each scalar optimization problem considers a point on the ideal line and the introduced constraint sets to be no greater than zero the scalar product of the vector directed downwards along the ideal line and the vector connecting a feasible point to the point on the ideal line (Fig. 3). Then, the scalar optimization minimizes the normalized objective 2 subject to the constraint above. The series of scalar problems initializes at the anchor point (1,0), i.e., the algorithm advances from right to left. Since this strategy does not guarantee to generate only global Pareto points, a post-filtering algorithm is proposed in [6] to keep the solutions off local-Pareto and non-Pareto points.

Figure 3: NNC method

In [7] a modification is made to the NNC strategy for bicriteria optimization problems such that no post-filtering is needed. Firstly, each scalar optimization problem described above is added a new constraint stating that the current value of the normalized criterion 1 should be no greater than its value obtained for the preceding scalar problem; secondly, a second series of scalar problems is initialized at the anchor point (0,1), then advances from left to right but now minimizing the normalized objective 2 and imposing the new constraint to the normalized objective 2. After solving this bi-directional search of scalar problems, the solutions in both directions are compared point by point and excluded those that are not coincident. The coincident ones are (global) Pareto solutions.

The present work proposes a different strategy in order to improve the efficiency and the distribution of the solutions relatively to the methods just described. The proposed strategy stems basically in substituting the scalar product constraint by a constraint formulated throughout the ray with origin at the point (1,1) of the normalized criterion space (Fig. 4). This fashion, no post-filtering is necessary and only is enough the search from right to left of the scalar optimization problems. Besides that, since the rays originated at the point (1,1) go along better with the global curvature of the Pareto frontier than the normal directions to the ideal line, then we get a better distribution of solutions along that frontier.

2. The Bicriteria Optimization Problem

The bicriteria optimization problem is formulated as

\[
\min_b \{ \Psi_b = (\Psi_{b_1}, \Psi_{b_2}) \} \\
\text{s.t.} \\
\Psi_j \leq 0; \quad j = 1, 2, ..., m' \\
\Psi_j = 0; \quad j = m'+1, ..., m \\
b_{j-1} \leq b_j \leq b_{j+1}; \quad i = 1, 2, ..., n
\]

where \( \Psi_b \) is the vector objective of components \( \Psi_{b_1}(b) \) and \( \Psi_{b_2}(b) \). \( \Psi_j(b) \) are the \( m \) constraints, and \( b \) is the
design vector with \( n \) components \( b_i \) which lower and upper bounds are respectively \( b_{il} \) and \( b_{iu} \). By using the Pareto concept, this problem has infinitely many Pareto solutions, the Pareto points. Each point in the design space corresponds to a point in the criterion space. The Pareto solutions form the Pareto frontier and are on the boundary of the feasible objective space.

Since the values of the criteria components may differ greatly, they are normalized as

\[
\Psi_k = \frac{\Psi_k - \Psi_{0k}}{\Psi_{nk} - \Psi_{0k}} \quad (k = 1, 2)
\]

where \( \Psi_{0k} \) is the solution of the scalar problem

\[
\begin{align*}
\min_{\Psi_k} & \Psi_k \\
\text{s.t.} & \Psi_j \leq 0; \quad j = 1, 2, \ldots, m' \\
& \Psi_j = 0; \quad j = m'+1, \ldots, m \\
& b_{il} \leq b_i \leq b_{iu}; \quad i = 1, 2, \ldots, n
\end{align*}
\]

(3)

corresponding to the minimization of \( \Psi_{0k} \) independently of other criteria, and \( \Psi_{nk} \) is the maximum value of \( \Psi_k \) when all the objectives are independently minimized. This way, the values of the normalized criteria components range from 0 to 1. The problem of Eqs. (3) is solved for \( k = 1 \) and \( k = 2 \) to get the anchor points of the normalized criterion space \((0,1)\) and \((1,0)\) respectively (Figs. 3, 4). The utopia point in this space is the point \((0,0)\).

3. The Normalized Normal Constraint Method

By this method [6], the utopia or ideal line in the normalized criteria space (Fig. 3) is divided in \( s \) segments, this way setting a grid of evenly distributed points \( \Psi^s = (\Psi_{01}, \Psi_{02}) \) on this line. The segment size is equal to \( \sqrt{2}/s \).

Define also a left-to-right-direction from the anchor point \((0,1)\) to the anchor point \((1,0)\) as \( \vec{N} = (1,0) - (0,1) \).

The series of scalar optimization problems \((p = 1, 2, \ldots, s-1)\) initializes now at the point \( \Psi^0 = (\Psi_{01}^0, \Psi_{02}^0) \) nearest the anchor point \((1,0)\) and finishes at the point \( \Psi^s = (\Psi_{01}^s, \Psi_{02}^s) \) nearest the anchor point \((0,1)\). The scalar problem corresponding to the general point \( \Psi^p \) is stated as

\[
\begin{align*}
\min_{\Psi_k} & \Psi_k \\
\text{s.t.} & \Psi_j \leq 0; \quad j = 1, 2, \ldots, m' \\
& \Psi_j = 0; \quad j = m'+1, \ldots, m \\
& \vec{N} \cdot (\Psi^0 - \Psi^p) \leq 0 \\
& b_{il} \leq b_i \leq b_{iu}; \quad i = 1, 2, \ldots, n
\end{align*}
\]

(4)

where \((\Psi_{01}^0, \Psi_{02}^0)\) represents the points of the feasible criteria space and \( \cdot \) stands for scalar product. One should note that the scalar product constraint obliges the minimization problem to search in a region where the vectors \( \vec{N} \) and \( \Psi^0 - \Psi^p \) are in opposition. When the scalar product constraint is active, then the solution is found on the intersection of the normal to the ideal line at the point \( \Psi^0 \) and the boundary of the normalized feasible criteria space.

This method obtains Pareto solutions evenly distributed but generates other than global Pareto points when the boundary of the feasible criteria domain is not convex. So it needs a post-filtering process in order to get rid off these non required points as mentioned in [6].

4. A Different Proposed Method: the Normalized Angular Constraint Method

This article proposes a different strategy that looks to eliminate the drawbacks of the NNC method and gets a better distribution of the solutions. Instead using the scalar product constraint of the problem formulated in the Eqs. (4), we split the criterion space in two regions, for each scalar optimization problem, by using the ray with origin at the point \((1,1)\) that forms the angle \( \beta^p \) given as

\[
\tan \beta^p = \Psi_{01}^p / \Psi_{02}^p
\]

(5)

where the points \( \Psi^p \) on the ideal line were by this method obtained by dividing the objective space in angular
segments (Fig. 4). If the number of segments is \( s \), then the segment size will be equal to \( (\pi/2)/s \). Hence, the criterion space is split by the line

\[
\Psi_{01} + \tan \beta_p (1 - \Psi_{01}) - 1 = 0
\]

and the feasible region will be the region above this line. Since this line goes better with the global curvature of the feasible space boundary, we may obtain a better distribution of solutions. The scalar problem may be formulated as

\[
\min_b \Psi_{02}
\]

s.t.

\[
\Psi_j \leq 0; \quad j = 1, 2, \ldots, m'
\]

\[
\Psi_j = 0; \quad j = m' + 1, \ldots, m
\]

\[
\Psi_{02} + (1 - \Psi_{01}) \tan \beta_p - 1 \geq 0
\]

\[
\Psi_{01} \leq \Psi_{01}^{p-1}; \quad p = 1, 2, \ldots, s-1
\]

\[
b_j \leq b_j \leq b_n; \quad i = 1, 2, \ldots, n
\]

The series of scalar optimization problems \( (p = 1, 2, \ldots, s-1) \) starts, as for the NNC method, at the point \( \Psi_{01}^k \) nearest the anchor point \( (1, 0) \), where now \( \beta_p = \pi/2 - (\pi/2)/s = (\pi/2) (1 - 1/s) \).

Points in non-convexities as from A to C in the Fig. 5 are automatically eliminated if we use the angular constraint. Points in non-convexities as from D to F in the Fig. 7 are automatically filtered by the constraint \( \Psi_{01}^p \leq \Psi_{01}^{p-1} \).

5. Examples

Consider two very simple examples. The first one is an example solved in [6,7] and presents a non-convexity on a part of the feasible space boundary where the relationship \( \Psi_{01}/\Psi_{01} \) is smaller; the other one is presented here to show the case where the non-convexity is where the relationship \( \Psi_{01}/\Psi_{01} \) is larger. Both are going to be solved by using the method introduced in this paper. Mathematical programming has been used.

5.1. Example 1

Consider the following optimization problem:

\[
\min_{b_1, b_2} \{ \Psi_0 = (\Psi_{01} = b_1, \Psi_{02} = b_2) \}
\]

s.t.

\[
5e^{-b_1} + 2e^{-0.5(b_1 - b_2)} - b_2 \leq 0
\]

\[
0 \leq b_1, b_2 \leq 5
\]

The normalized criteria of this problem, by using the Eq. (2), may be written as

\[
\Psi_{01} = 0.20018b_1 - 0.00090368, \quad \Psi_{02} = 0.21296b_1 - 0.064818
\]

The boundary of the feasible criterion area of this problem is shown in the Fig.5. The boundary zones between the points \( (0,1) \) and \( C \) and between \( A \) and \( (1,0) \) are the global Pareto frontier. The points on the segment CB are non-Pareto points and the points on the segment BA are local Pareto points.

![Figure 5](image1.png)

![Figure 6](image2.png)
The results obtained by the angular constraint method with a grid of rays of 1.5° angular size (60 segments), are shown in the Fig. 6. We may observe the algorithm has filtered the points between A and C, and has kept only the global Pareto points. The Pareto frontier goes from the lower anchor point to the point 14 at 69°, then for a constraint at β = 67.5° the point 15 jumps to 45.283°. Then goes on from 45° until the 1.5° and the anchor point (0,1). So, they have been avoided 15 minimizations in 59. The Pareto points are well-distributed due to the near perpendicularity between the rays and the Pareto frontier. The distance between the point (1,1) and this frontier varies between 0.98305 for point 1 and 0.74567 for the point 14, and between 0.97578 for the point 30 and 0.99529 for the point 59.

By using the NNC method, the local-Pareto points of the zone AB were also calculated, needing a post filtering process to vanish them.

5.2. Example 2
Consider now the optimization problem

\[
\begin{align*}
\min \{ & \Psi \equiv \left( \Psi_{01} = h_1, \Psi_{02} = b_2 \right) \} \\
\text{s.t.,} \\
5e^{5h} + 2e^{-0.5(b-b)} - b_l & \leq 0 \\
0 & \leq h, b \leq 5
\end{align*}
\]

where, relatively to the first example were exchanged the roles of \(h_1\) and \(b_2\) in the constraint.

The normalized criteria of this problem, by using the Eq. (2), may be written as

\[
\begin{align*}
\Psi_{01} &= 0.21296h_1 - 0.064818, \\
\Psi_{02} &= 0.20018b_2 - 0.00090367
\end{align*}
\]

The boundary of the feasible criterion area of this problem is shown in the Fig. 7. The boundary zones between the points (0,1) and D and between F and (1,0) are the global Pareto frontier. The points on the segment DE are non-Pareto points and the points on the segment EF are local Pareto points. The results obtained by the proposed method with the same grid as in the Example 1, are shown in the Fig. 8. Again we may observe the algorithm has filtered the points between D and F, keeping only the global Pareto points. The Pareto frontier goes from the lower anchor point to the point 30 at 45°, then for a constraint at β = 43.5° the point 31 jumps to 21.523°. Then goes on from 21° until the 1.5° and the anchor point (0,1). So, they have been avoided 15 minimizations in 59.

By using the NNC method without post-filtering, the non-Pareto points (zone DE) and the local Pareto points (zone EF) were also calculated.

6. Concluding Remarks
This article proposes a new strategy to calculate the global Pareto solutions for a biobjective optimization problem. The method can calculate the scalar minimization sub-problems in only one-direction series and filtering automatically all the local Pareto and non-Pareto solutions, this way increasing the efficiency relatively to other strategies as, namely, the Normalized Normal Constraint method. Also, the method presented may obtain a better distribution of the Pareto points, since it is based on the division of the criterion space by rays originated at the right upper corner and directed nearly perpendicular to the Pareto frontier.

References
York, 1986.


