

Optimal Design of Reinforced Concrete T-Beams under Ultimate Loads

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Abstract

The present paper deals with the structural design optimization of reinforced concrete T-beams under ultimate design loads. An analytical approach of the problem based on a minimum cost design criterion and a reduced number of design variables, is developed. It is shown, among other things, that the problem formulation can be cast into a non linear mathematical programming format.

Typical examples are presented to illustrate the applicability of the formulation in accordance with the current French design code BAEL-99. The results are then confronted to design solutions of reinforced concrete T-beams derived from current design practice. The optimal solutions show clearly that significant savings can be made in the predicted amounts and consequently in the absolute costs of the construction materials to be used.

Keywords: Non-linear Optimization, Reinforced Concrete, T-beams, Optimal Design, Ultimate Loads.

1. Introduction

Reinforced Concrete T-beams are frequently used in industrial construction, especially in building floors, retaining walls, bridge decks, and more generally in all reinforced concrete construction projects where an appropriate portion of the slab is associated to the resisting section of the supporting beam.

For large scale use of such T-shaped beams, as may be the case for precast reinforced concrete component production, a cost effective design approach can be developed by using non-linear programming techniques. The optimal designed beams can be suitably manufactured in a prefabrication factory and then used for the specific purpose. This may lead to significant savings in the costly construction materials for both the superstructure and the foundation elements.

The total cost to be minimized is essentially divided into the construction material costs of concrete, steel and formwork. From an economical perspective, it is also desirable to consider in the design process optimization of the critical sections, the nonlinear ultimate behaviour of the concrete and reinforcing steel in accordance with the current design codes [1].

Within this framework the present study deals with the cost effective design of reinforced concrete T-shaped beams under ultimate loads. Consideration of serviceability conditions at working loads will be addressed elsewhere as it requires further attention in terms of restrictions on bending moment capacity, stress limitation in the concrete and in steel as a function of cracking conditions, as well as limits on deflections. Such restrictions will have direct consequences on the boundaries of the design space and the feasible design solutions of the optimization problem.

The art of cost effective design consists of a procedure in which a structural optimization model is first formulated and then solved by using a suitable mathematical programming algorithm [2]. The structural optimization model consists of an objective function and a set of constraints. The latter generally include i) limits of search for the decision variables; ii) restrictions on structural behaviour; and iii) various stress and strain conditions and their limits. Ideally, the final design must ensure to include compatibility between the geometrical dimensions of the optimized T-cross section and the ultimate loading condition including self weight of the T-beam. Very little attention has been given to this aspect by some of the earlier investigations.

Another relevant aspect in the optimal design is the use of a particular suitable optimization algorithm. Various mathematical programming algorithms have been used in structural design optimization [3]. This study demonstrates the formulation and the solution of the non linear minimum cost design problem of reinforced concrete T-beams under ultimate loads by utilizing a suitable mathematical programming technique.

2. Problem formulation

Consider the T-cross section shown in Figure 1 and let C_0 be the objective function representing the cost of reinforced concrete and reinforcing steel to be used. This function can be defined as:

$$C_0 = L [C_c (b_0 d + (b-b_0) h_0) + C_s A_s]$$

Where:

- b : effective width of compressive flange
- b_0 : web width
- h : total depth
- d : effective depth
- h_0 : flange depth
- L : length of beam
- A_s : area of reinforcing steel
- C_0 : absolute cost
- C_s : unit cost of reinforcing steel
- C_c : unit cost of concrete

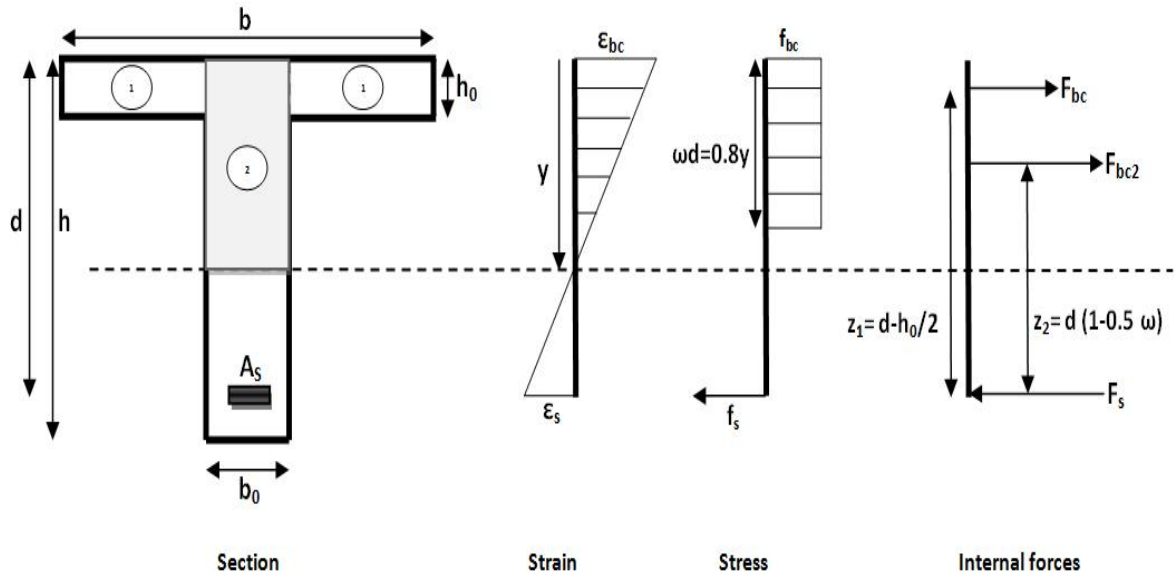


Figure 1: Typical T-Cross section, strain, stress and internal force distributions under ultimate design load.

Cost components such as concrete formwork and steel forming are herein implicitly included in the objective function as appropriate percentages of the unit costs of concrete and steel respectively.

It should be equally important to note that in a cost optimization problem, the optimal values of the design variables are affected by the relative cost values of the objective function only, but not by the absolute cost values. In other words, the absolute cost values affect the final value of the objective function but not the optimal values of the design variables. Thus, the objective function to be minimized can be written as follows:

$$C = b_0 d + (b-b_0)h_0 + (C_s / C_c) A_s \longrightarrow \text{Min}$$

The absolute cost C_0 can then be recovered from the optimized relative cost C by using the relation

$$C_0 = C L C_c.$$

The value of the cost ratio C_s / C_c varies from one country to another and may eventually depend from one region to another for certain countries.

In general, the behavior constraints are based on different design codes. For illustrative purposes, the design constraints are herein defined in accordance with the BAEL-99 design code specifications [4], a slightly different version of the current EC2 provisions. In addition, the problem, the case of compressive steel is not included in the present illustration and it is assumed that stress in concrete under ultimate design loads is uniformly distributed.

Thus, the formulation of the minimum cost design of reinforced concrete T-beams under ultimate loads can be mathematically stated, without loss of generality, as follows:

Find the decision variables b , b_0 , h , d , h_0 , A_s , and ω such that:

$$C = b_0 d + (b-b_0)h_0 + (C_s / C_c)A_s \longrightarrow \text{Min} \quad (1)$$

Subject to:

a) Behavior constraints:

$$M_u \leq f_{bc} (b-b_0)h_0 (d-0.5h_0) + f_{bc} b \cdot d^2 \cdot \omega (1 - 0.5\omega) \quad (2)$$

$$\omega = (f_s / f_{bc})(A_s / b_0 d) - (b-b_0) h_0 / b_0 d \quad (3)$$

$$p_{\min} \leq A_s / (b_0 h + (b-b_0) h_0) \leq p_{\max} \quad (4)$$

Conditions on strain compatibility in steel and concrete

$$f_c / \gamma_s E_s \leq 0.0035(0.8-\omega) / \omega \leq 10/1000 \quad (5)$$

$$0.259 \leq \omega / 0.8 \leq 0.668 \quad (\text{steel class HA FeE400}) \quad (6)$$

$$0.186 \leq \omega(1-0.5\omega) \leq 0.392 \quad (\text{steel class HA FeE400}) \quad (7)$$

b) Shear stress constraint:

$$(V_u / \bar{\tau}_u) \leq b_0 d \quad (8)$$

c) Design variables constraints including rules of current practice:

$$h \geq L / 16 \quad (9)$$

$$d / h = 0.90 \quad (10)$$

$$0.20 \leq b_0 / d \leq 0.40 \quad (11)$$

$$(b-b_0) / 2 \leq L / 10 \quad (12)$$

$$b/h_0 \leq 8 \quad (13)$$

$$h_0 \geq h_{\min} \quad (14)$$

Where:

C : relative cost value

M_u : ultimate bending moment capacity

V_u : ultimate shear capacity

ω : relative depth of compressive concrete zone

p : steel percentage

f_{bc} : allowable compressive stress of concrete

f_s : allowable compressive stress of steel

$\bar{\tau}_u$: allowable shear stress

$p_{\min}$: minimum steel percentage

$p_{\max}$: maximum steel percentage

γ_s : partial safety factor for steel

E_s : modulus of elasticity of reinforcing steel

The objective function equation (1) and the constraints equations, Eq.(2) through Eq.(14), together form a nonlinear optimization problem. The reasons for the non linearity of this optimization problem are essentially due to the expressions for the beam cross sectional area, the bending moment capacity and other constraints equations as well as the requirement to update iteratively the self weight of the element, both in the constraints functions and the objective function. Both the objective function and the constraint functions are non linear in terms of the decision variables.

The decision variables of the model are the geometrical dimensions of the T- cross section, ratio of the amount of steel A_s and ω , the relative depth of the compressive concrete zone (equal to the internal axial force in steel over the total internal compressive force in the concrete).

In order to solve this non linear optimization problem, the GRG (Generalized Reduced Gradient) method was used for the following reasons:

- i) the GRG method is widely recognized as an efficient method for solving a relatively wide class of non linear optimization problems.
- ii) the program can handle up to 200 constraints, which is suitable for reinforced concrete T-beam optimal design problems.
- iii) if necessary, the program itself can estimate the Hessian matrix by using finite differences techniques.
- iv) the Generalized Reduced Gradient (GRG) method used by the program allows for a systematic selection between the Quasi-Newton Method and the Conjugate Gradient (GG) method, depending on the available computer storage capacity.

Finally, it should also be noted that in the case of serious cracking, the optimization constraints should be reformulated in accordance with the serviceability design loads as prescribed by the design code criteria to be used.

3. Numerical results and discussion

In order to illustrate the applicability of the optimal design obtained by using the present formulation, the minimum cost design procedure is now applied to design a reinforced concrete beam with T-section for which a detailed design solution based on the BAEL design code has been reported in reference [4].

The study reinforced concrete T-beam with pinned supports, along with the overhead crane (shown in most critical position so as to produce maximum bending moment under live load) are presented in Figure 2 below.

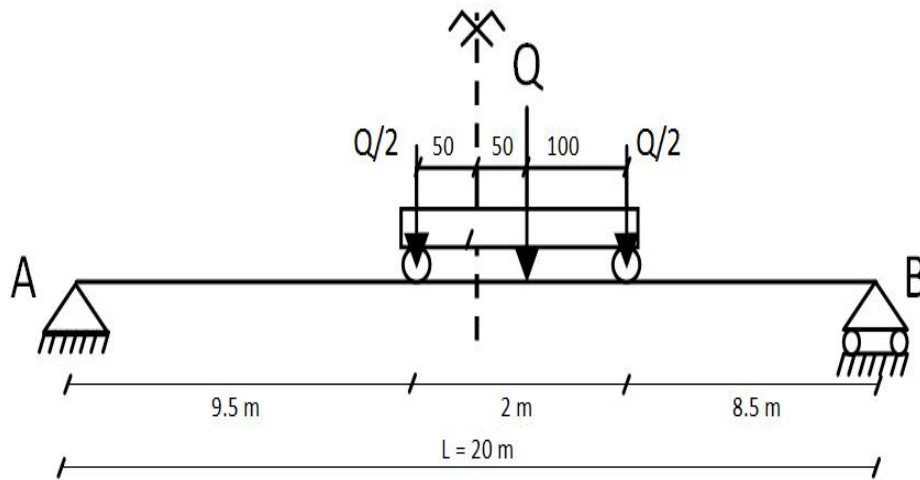


Figure 2: Critical position of overhead crane on study T-beam

The objectives of this application test are:

- i) to evaluate the performance of the minimum cost design model and the solution methodology in terms of convergence to an optimal solution.
- ii) to examine the characteristics of the solution in terms of the identification of binding and non-binding constraints.
- iii) to provide minimum cost design solutions of a T-beam under ultimate loads that can be used for comparison purposes in future investigations.

As previously mentioned, the design constraints are defined in accordance with the code design specifications of the French design code BAEL-99 (Béton Armé aux Etats Limites, version 99) which represents a slightly different version of the EC 2 provisions.

The corresponding pre-assigned parameters of the system are defined as follows:

- beam span : $L = 20\text{ m}$
- dead load : $G = 19.5\text{ KN/m}$ (including initial self weight)
- live load : $Q = 450\text{ KN}$

Combination rule for ultimate beam capacities in bending and shear respectively

$$M_u = 1.35 M_G + 1.5 M_Q$$

$$M_u = 4.991\text{ MNm}$$

$$V_u = 1.35 V_G + 1.5 V_Q$$

$$V_u = 1.39 \text{ MN}$$

Input data for concrete characteristics :

- compressive strength of concrete at 28 days : $f_{c28} = 20 \text{ MPa}$
- partial safety factor for concrete : $\gamma_b = 1.5$
- allowable compressive stress: $f_{bc} = (0.85 f_{c28}) / \gamma_b = 11.3 \text{ MPa}$
- allowable shear stress: $\bar{\tau}_u = \text{Min}\{(0.20 f_{c28}) / \gamma_b ; 5 \text{ MPa}\} = 2.60 \text{ MPa}$

Input data for steel characteristics :

- elastic limit : $f_e = 400 \text{ MPa}$
- partial safety factor for steel: $\gamma_s = 1.15$
- allowable tensile stress : $f_s = f_e / \gamma_s = 348 \text{ MPa}$
- Young's elastic modulus: $E_s = 2 \times 10^5 \text{ MPa}$
- minimum steel percentage: $p_{\min} = 0.1\%$
- maximum steel percentage: $p_{\max} = 4\%$

Input data for unit costs of construction materials:

The unit costs of steel and concrete are generally determined in accordance with builders price books or unit price data bases for construction materials e.g. references [6], [7] and [8]. In the present work, the unit cost ratio $C_s / C_c = 36$ is used.

Following the problem formulation developed in the preceding section, the minimum cost design can be rewritten as follows:

Find the decision variables b, b_0, h, d, h_0, A_s and ω such that:

$$C = b_0 d + (b - b_0)h_0 + 36A_s \longrightarrow \text{Min} \quad (1')$$

Subject to :

a) Behavior constraints :

$$0.442 \leq (b - b_0)h_0 (d - 0.50h_0) + b_0 d^2 \omega (1 - 0.5\omega) \quad (2')$$

$$\omega = 30.715 A_s / b_0 d - (b - b_0) h_0 / b_0 d \quad (3')$$

$$0.001 \leq A_s / (b_0 h + (b - b_0)h_0) \leq 0.04 \quad (4')$$

Conditions on strain compatibility in steel and concrete

$$1.739/1000 \leq 0.0035(0.8 - \omega) / \omega \leq 10/1000 \quad (5')$$

$$0.59 \leq \omega / 0.8 \leq 0.668 \text{ (steel class HA FeE400)} \quad (6')$$

$$0.186 \leq \omega(1 - 0.5\omega) \leq 0.392 \text{ (steel class HA FeE400)} \quad (7')$$

b) Shear stress constraint:

$$0.40 \leq b_0 d \quad (8')$$

c) Design variables constraints including rules of current practice

$$h \geq 1.25 \quad (9')$$

$$d/h = 0.90 \quad (10')$$

$$0.20 \leq b_0 / d \leq 0.40 \quad (11')$$

$$b - b_0 \leq 4 \quad (12')$$

$$b/h_0 \leq 8 \quad (13')$$

$$h_0 \geq 0.15 \quad (14')$$

It should be noted that the optimal solution vector of the above nonlinear mathematical programming problem cannot be considered as the final solution of the minimum cost design problem. As a matter of fact, because of the requirement to update the geometric dimensions of the section with the new self weight of the optimized beam, the degree of nonlinearity of the resulting optimization problem enhances further. The final optimal solution is thus obtained in two phases.

Phase 1 is concerned with the determination of the optimal solution using the initial loading parameters (i.e. with initial self weight corresponding in the starting solution)

Phase 2 is concerned with the requirement to update the self weight of the beam (both in the constraints functions and the objective function) with the geometric dimensions of the optimized section obtained in phase 1. The modified forces due to the new self weight are computed, the new dimensions of the beam are optimized and the process continued until convergence is achieved. In the present example, the optimal solution vector is reached after 5 cycles of iteration.

The vector of decision variables including the geometric dimensions of the cross section as obtained from the optimal design solution using the proposed approach, and the classical design solution are shown in Table 1.

Table 1: Final optimal solution including self weight effects

Classical solution		Practical optimal solution	
b(m)	1.20	b (m)	1.20
b _o (m)	0.40	b _o (m)	0.30
h(m)	1.60	d _{opt} (m)	1.67
d(m)	1.46	h _{opt} (m)	1.50
h _o (m)	0.14	h _{o opt} (m)	0.15
A _s (m ²)	114.61x 10 ⁻⁴	A _{s opt} (m ²)	108.21 10 ⁻⁴
ω	0.308	ω _{opt}	0.441
C _{class}	1.108596	C _{opt}	0.97454

From the above results, it is clearly seen that the relative depth of the compressive associated with the optimal solution zone is 43% larger than that given by classical solution, thus leading to a much better use of the concrete. It is also seen that the ratio C_{class}/C_{opt} of relative costs (C_{class} as obtained using BAEI classical design method to the optimal relative cost V) is equal to 1.14, i.e. a significant cost saving of the order of 14% as compared to the classical design solution.

The practical geometric dimensions (in cm) and the diameter of the reinforcing steel (in mm) of the optimized T-section are represented in Figure 3.

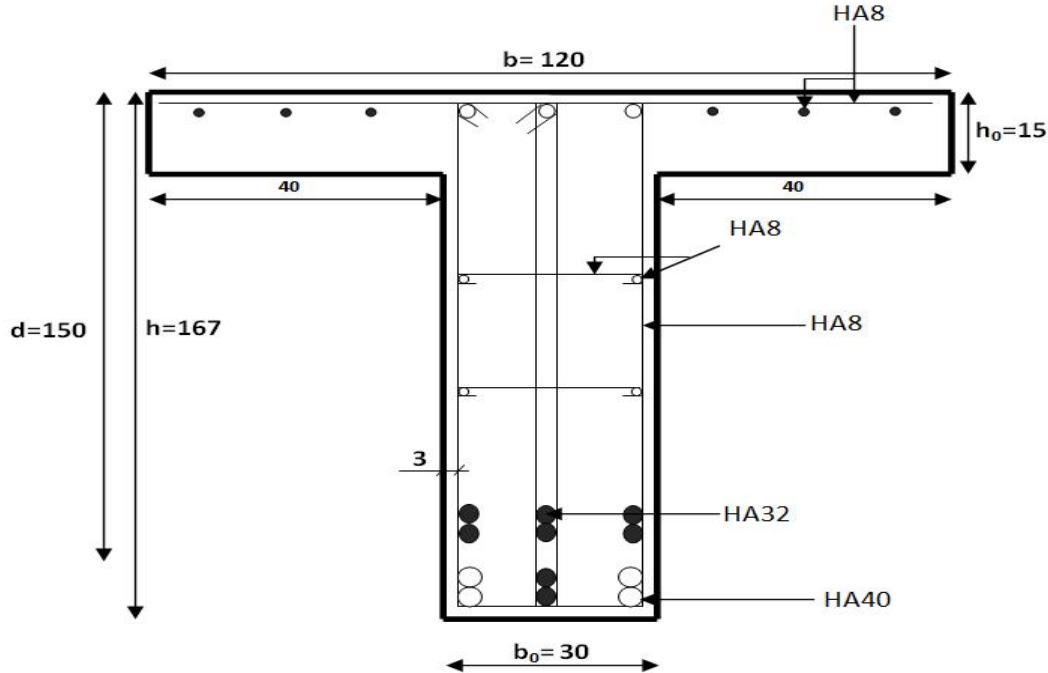


Figure 3: Final optimal solution for unit cost ratio $C_s/C_c=36$

A study of the constraints indicated that the design constraints of the beam are all non binding except for the behavior constraints associated with ultimate bending moment capacity (2)' and the geometrical design constraints ((10)'; (13)'; and (14)'). In addition, the results obtained by using various examples have shown that the optimal solutions are insensitive to changes in the shear stress constraints which can thus be excluded from

the problem formulation. All thickness values b_o and h_o are on the specified lower limit values. No effort was made to explore the reason for such a behavior of the solution results.

In order to further illustrate the variability of the optimal solution with the unit cost ratio C_s/C_c , optimal solutions have also been computed for various ratios $C_s/C_c = 20; 36; 50; 100; 150$. The corresponding optimal design solution vectors are shown in Table 2.

Table 2: Optimal solutions versus unit cost ratio C_s/C_c

C_s/C_c		20	36	50	100	150
Optimal solution	b_{opt} (m)	1.26	1.18	1.20	1.20	1.20
	$b_{o opt}$ (m)	0.28	0.29	0.30	0.33	0.35
	d_{opt} (m)	1.40	1.44	1.48	1.63	1.75
	h_{opt} (m)	1.55	1.60	1.65	1.81	1.94
	$h_{o opt}$ (m)	0.16	0.15	0.15	0.15	0.15
	A_{sopt} (m ²)	117.78×10^{-4}	115.69×10^{-4}	110.26×10^{-4}	97.97×10^{-4}	91.72×10^{-4}
	ω_{opt}	0.534	0.534	0.465	0.321	0.254
	C_{opt} (m ²)	0.78072	0.96703	1.12546	1.64172	2.11444
	C_{class} (m ²)	0.92522	1.10860	1.26905	1.84210	2.41515
	Gain in percent (%)	19	14	13	12	14

In this Table the gain in percent is defined as: $\text{Gain in percent (\%)} = ((C_{class} - C_{opt})/C_{class})100\%$
For the sake of clarity, the variation of the gain with respect to the unit cost ratio C_s/C_c is plotted in Figure 4.

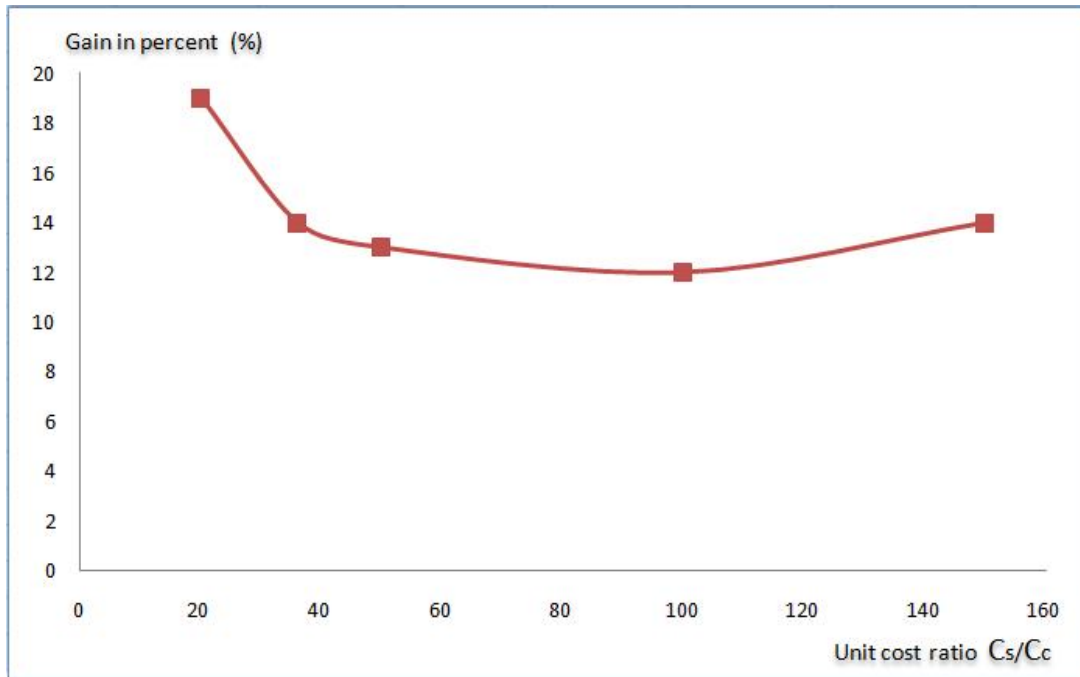


Figure 4: Variation of gain percentage versus unit cost ratio C_s/C_c

It can be observed that the relative gain decreases for increasing values of the cost ratio up to the value $C_s/C_c = 36$, and then seems to stabilize around an average value approximately equal to 13%.

4. Conclusions

This study deals with the minimum cost design of reinforced concrete T-beams at ultimate design loads. An analytical approach of the problem based on a minimum cost design criterion and a set of constraints including the nonlinear behaviour of concrete and steel is formulated. The set of constraints includes:

- i) restrictions on the structural behaviour in terms of ultimate bending moment capacity and percentage of reinforcing steel ;
- ii) various compatibility conditions in terms of strains in concrete and steel, depth of neutral axis and their limits
- iii) restrictions on ultimate shear stress ; and
- iv) limits of search for the decision variables

From the results of the present study, it is possible to draw the following major conclusions:

- 1 The problem formulation of the optimal cost design of reinforced concrete T-beams can be cast into a non linear programming problem, the numerical solution of which is efficiently determined using a direct search or the GRG (Generalized Reduced Gradient) method in a space of only a few variables representing the concrete cross section dimensions.
- 2 The space of feasible design solutions and the optimal solutions can be obtained using a reduced number of independent design variables.
- 3 Results obtained using various examples show that the optimal solutions are insensitive to changes in the shear stress constraints. Shear stress constraints are usually not critical in the optimal design of reinforced concrete T –beams and can thus be excluded from the problem formulation.
- 4 Optimal values of the design variables are affected by the relative cost values of the objective function only, but not by the absolute cost values.
- 5 The observations of the optimal solutions results reveal that the use of the optimization based on the minimum cost design concept may lead to substantial savings in the amount of the construction materials to be used in comparison to classical design solutions of reinforced concrete T-beams.
- 6 The objective and the constraints considered in the present paper are illustrative in nature. The present approach based on a nonlinear mathematical programming can be easily extended to other sections commonly used in structural design. More sophisticated objectives and considerations can be readily accommodated by suitable modifications of the present optimal cost design model.

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