2nd International Conference on Engineering Optimization
September 6 - 9, 2010, Lisbon, Portugal

Simultaneous Topology Optimization and Optimal Control for Vibration Suppression in Structural Design

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Abstract
This work presents a structural topology optimization methodology for a cantilever beam, which includes an optimal control design for reducing vibrations. The topology optimization in this paper uses homogenization design method, based on the concept of optimizing the material distribution, through a density distribution. A Continuum finite elements modeling is applied to simulate the dynamic characteristics of the structure. The modal basis is used to derive an optimal control. The cost functional is the strain energy of the structure and the control energy. The location for the actuator in the beam was chosen based in a known fact that the best place for one actuator is as close as possible to the fixed size of the structure, which bears the maximum stress induced by the first and most significant mode. Results of numerical simulations for a cantilever beam model are presented and discussed.
Keywords: Topologic optimization, optimal control, vibrations, dynamics.

1. Introduction

Structural topology optimization and structural vibration control have called attention both in theoretical research and practical applications in engineering. Structural vibration control is a particularly important consideration in the design of dynamic systems. The main idea of the structural optimization is to obtain an optimal material layout of a load-bearing structure. Usually, continuum topology optimization problems are formulated to minimize the structural material volume or to optimize the structural performance. A typical example is to raise the first fundamental frequency of a structure while obeying a volume constraint [5]. Meanwhile, structural dynamics control is used to minimize or suppress vibration effects.

There are always fundamental interest in designs with efficient structural control system from both structural and control engineers. However, these groups have been working independently. Traditionally, the structural designer develops his design based on strength and stiffness requirements, and the control designer creates the control algorithm to reduce the dynamic response of a structure [3]. In this work we are designing the structure and controls simultaneously, meaning that the cost function includes not only the strain energy, but also the control energy.

The reason why topology optimization is becoming a very important research field is the necessity of efficient methodologies to design structures, thus saving material and time. The main objective of the topology optimization problem is to find a material distribution that minimizes a given objective functional, subjected to a set of constraints, achieved by a consistent parameterization of the material properties in each part of the design domain. A natural question is whether there exists or not material in a given point, which leads to a discrete problem. It is well-known that this integer parameterization leads to numerical difficulties, associated with the integer problem convergence [2, 6, 7]. Minimizing the vibration effects of the dynamic response is an important goal for the structural vibration control, and the effectivity of the control depends on the weighting matrices.

The objective of this paper is to present a structural design methodology considering the control effects, the change of the topology by a control force action, and design modal control for suitable fixed actuator locations. The structural optimization design is completed through a density design method, while the control force is obtained by the optimal control design for transient response and performed in the modal space.

The efficient structural control design needs a careful selection of actuator positions [3]. However, in this work the actuators locations are chosen arbitrarily prior to the structural design. In fact, it is well known that a good location for an actuator in a cantilever structure is close the fixed size of the structure, since it acts upon the first and most significant mode. The lower fundamental modes are responsible for the most of the tip displacement of the beam; therefore, the first two eigenfunctions are computed and considered in this work.

The additional dynamics and control design were included in a topology optimization code [10]. Simulations were conducted to assess the effectiveness and control model efficiency.

2. Formulation of Structural Topology Optimization Considering Control Action

In this work the homogenization design method [8] is the tool for the topology optimization considering a control action. This method is based on the concept of optimizing the material distribution, through a density distribution. A finite element mesh is defined to perform the structural modal analysis [4]. As a simplification, we assume that the density is constant in each finite element. An optimality criteria (OC) is derived from the necessary minimization conditions, and it is solved to update the density distribution. A number of simplifications are introduced to the implementation, as a regular mesh.

We now consider that the objective function is the sum of the strain energy and the control energy. Then, the topologic optimization problem in steady state has the form

$$\min_{\mathbf{x}} J,\ J(\mathbf{x}) = \mathbf{f}^{\mathrm{T}}\mathbf{R}\mathbf{f} + \mathbf{U}^{\mathrm{T}}\mathbf{Q}\mathbf{U}$$

$$st \begin{cases} \dfrac{V(\mathbf{x})}{V_0} = V_{min} \\ \mathbf{KU} = \mathbf{Hf} + \mathbf{F} \\ \mathbf{0} < \mathbf{x}_{min} \le \mathbf{x} \le \mathbf{1} \end{cases}, \qquad (1)$$

where **U** is an $nx1$ displacement vector, **H** is an nxm location matrix for the control force, m is the number of action control forces and **F** is an $nx1$ applied external force vector, **f** is an $mx1$ control force vector. The magnitudes of the matrices **Q** and **R** are assigned according to the relative importance of the state variables and the control force in the minimization procedure. The matrix **Q** can be adjusted by **Q**=**K**, where **K** is the finite element global stiffness matrix. **x** is the vector of design variables, $\mathbf{x}_{min}$ is a vector of minimum relative densities. $V(\mathbf{x})$ and V_0 is the material volume and design domain volume, respectively and V_{min} is the prescribed volume function. Considering the discretization,

$$\mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U} = \sum_{e=1}^{N} (x_e)^p\, \mathbf{u}_e^T \mathbf{k}_e \mathbf{u}_e\ , \qquad (2)$$

where N is the number of elements, p is the penalization exponent, $\mathbf{u}_e$ and $\mathbf{k}_e$ are the element displacement vector and stiffness matrix, respectively.

The optimization problem is solved using the Optimality criterion (OC), and this criterion is derived from the Karush-Kuhn-Tucker conditions [6]. The Lagrangian function of the minimization problem is

$$L(\mathbf{x}) = J(\mathbf{x}) + \lambda_0 (V(\mathbf{x}) - V_{min} V_0) + \boldsymbol{\lambda}_1^T (\mathbf{KU} - (\mathbf{Hf} + \mathbf{F})) + \sum_{e=1}^{N} \lambda_{2e} (x_{\min} - x_e) + \sum_{e=1}^{N} \lambda_{3e} (x_e - x_{\max})\,. \qquad (3)$$

To locate a stationary point, it is necessary that $\partial L / \partial x_e = 0$, then

$$\frac{\partial L}{\partial x_e} = \frac{\partial J}{\partial x_e} + \lambda_0 \frac{\partial V}{\partial x_e} + \boldsymbol{\lambda}_1^T \frac{\partial(\mathbf{KU} - (\mathbf{Hf} + \mathbf{F}))}{\partial x_e} - \lambda_{2e} + \lambda_{3e} = 0\,, \qquad (4)$$

where $\lambda_0, \lambda_1, \lambda_{2e}$ and λ_{3e} are Lagrangian multipliers.

Assume that constrains of the design variables are not active, $\lambda_{2e} = \lambda_{3e} = 0$, and that the load **F** is design independent. With some expanding of the terms, also simplification of the equations and heuristics scheme for the design variables [10] we can obtain the new x_e for each iteration. More details about the expanding of the terms can be seen in the follow subsection, Sensitivity Analysis. The mesh-independent filter is the same as in Sigmund (2001) [10].

The feedback requires a full knowledge of states. By using the displacement closed-loop feedback control we can assume

$$\mathbf{f} = -\mathbf{R}^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{U}\ , \qquad (5)$$

then the equilibrium constraint from Eq. (1) becomes

$$\mathbf{K}_c \mathbf{U} = \mathbf{F}\,, \tag{6}$$

where

$$\mathbf{K}_c = \mathbf{K} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}^T\,. \tag{7}$$

We can note that this $\mathbf{K_c}$ is the modified matrix under control effect and the modification appears where the force control is applied, which affects also the eigenvalues and displacement of the stucture. The problem can be solved as the conventional static finite element method in standard form $\mathbf{K}_c\mathbf{U} = \mathbf{F}$.

The influence of the weighting matrix **R** is an important aspect to consider. To have significant effect on the topology of the structure, the matrix $\mathbf{R}^{-1}$ need an equivalent magnitude compatible with the stiffness matrix. Since the stiffness is modified on each iteration, then **R** is chosen as $\mathbf{R} = diag(\mathbf{w}/\boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ are the eigenvalues (the smallest to the largest) and **w** are weighting constants with the same order of magnitude of the stiffness terms.

2.1. Sensitivity Analysis

Sensitivities are defined as the derivatives of the objective function and the constraints with respect to the design variables, and is often the major computational cost of the optimization. In this work the objective function sensitivity requires differentiating displacements (which implies stiffness differentiation) and eigenvalues. The objective function can be simplified using Eq. (2) into Eq. (1), then $J = \mathbf{f}^T\mathbf{R}\mathbf{f} + \mathbf{U}^T(\mathbf{H}\mathbf{f} + \mathbf{F})$.Using Eq. (5), this yields $J = \mathbf{F}^T\mathbf{U}$. Taking the derivative of the objective function, on each element, one can obtain

$$\frac{\partial J}{\partial \mathbf{x}_e} = \mathbf{F}^T \frac{\partial \mathbf{U}}{\partial \mathbf{x}_e}\,, \tag{8}$$

and substituting

$$\frac{\partial \mathbf{U}}{\partial \mathbf{x}_e} = -\mathbf{K}_c^{-1}\frac{\partial \mathbf{K}_c}{\partial \mathbf{x}_e}\mathbf{U} \tag{9}$$

and

$$\frac{\partial \mathbf{K}_c}{\partial \mathbf{x}_e} = \frac{\partial \mathbf{K}}{\partial \mathbf{x}_e} + \frac{\mathbf{1}}{\mathbf{w}}\mathbf{H}\frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{x}_e}\mathbf{H}^T \tag{10}$$

into (8), we have

$$\frac{\partial J}{\partial \mathbf{x}_e} = -\left(\mathbf{U}^T\frac{\partial \mathbf{K}}{\partial \mathbf{x}_e}\mathbf{U} + \frac{\mathbf{1}}{\mathbf{w}}\mathbf{U}^T\mathbf{H}\frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{x}_e}\mathbf{H}^T\mathbf{U}\right). \tag{11}$$

The sensitivity of the each eigenvalue can be seeing in Haftka *et al.* (1990) [9], and is computed by

$$\frac{\partial \lambda}{\partial \mathbf{x}_e} = \phi^T\left(\frac{\partial \mathbf{K}}{\partial \mathbf{x}_e} - \lambda\frac{\partial \mathbf{M}}{\partial \mathbf{x}_e}\right)\phi = \phi_e^T\left(\frac{\partial \mathbf{k}_e}{\partial \mathbf{x}_e} - \lambda\frac{\partial \mathbf{m}_e}{\partial \mathbf{x}_e}\right)\phi_e\,, \tag{12}$$

where ϕ is the mass-normalized eigenvector and **M** is the mass matrix, on each element $\phi_\mathbf{e}$ and $\mathbf{m_e}$.

3. Control Effects on Strucural Topology

It is clear that control forces acting in different locations on the structure should influence the optimized design. To exemplify this fact, we assume a design domain as a cantilever beam shows in Fig. 1.

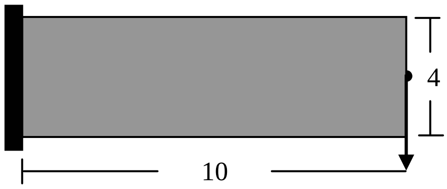
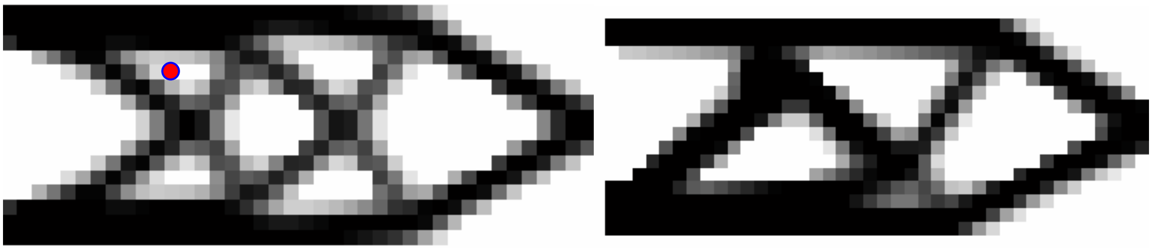


Figure 1. Design domain.

For a structural only design of this domain, we use the compliance as the objective function, and obtain the topology shown in Fig. 2a. Then we try to introduce a control force on this design layout. It is possible that on the desired location for the actuator there is no material. If the optimization is performed without considering the control force, then we need either to change the actuator location or to redesign the structure. In Fig. 2a we indicated with a point (small circle) the actuator location and designed the structure again, this time considering the control force. The new topology for this problem is shown in Fig. 2b. The mesh domain in the simulation is *16*x*40* finite elements.



(a) (b)

Figure 2. a-Topology optimization without control force action. b-Topology optimization with control force action.

In this simulation it can be noted that the structure design change completely with the control action effects. Additionally some attention for the actuator location is required to assure the controllability of the system.

4. Optimal Control Design in Modal Space

After computing the optimal structure we search for the vibration suppression for a transient response of the system. It is possible to design the control for the displacement of a particular point of the structure, but in this work we derive the control in independent modal space.

The formulation of independent modal space control, derived by the classical optimal theory [1], associated with the distributed-parameter system can be written briefly as follows. The modal formulation for the system is

$$\ddot{\boldsymbol{\eta}} + \boldsymbol{\omega}^2 \boldsymbol{\eta} = \phi^{\mathbf{T}} \mathbf{Hf} \ , \tag{13}$$

where ω are the angular frequencies.

The dynamic system defined by Eq. (13) can be parameterized in first order equations and written in the state-coefficient form

$$\dot{\mathbf{y}} = \mathbf{Ay} + \mathbf{Bf} \ , \tag{14}$$

where **y** is a state, time dependent variable, $\dot{\mathbf{y}} \in \Re^{2n}$ is the vector of the first order time derivates of the states in modal space, $\mathbf{f} \in S \in \Re^m$ is the control vector, *S* is the control constraint set. This system represents the constrains from the nonlinear regulator problem, together with $\mathbf{y}(\mathbf{t_0}) = \mathbf{y_0}$, $\mathbf{y}(\infty) = \mathbf{0}$, respectively the initial and final conditions.

The coefficient matrices, in modal space, without considering damping, are given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\boldsymbol{\omega}^2 & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \phi^{\mathrm{T}} \mathbf{H} \end{bmatrix}, \tag{15}$$

where $\mathbf{A} \in \Re^{2nx2n}$ and $\mathbf{B} \in \Re^{2nxm}$. It is assumed that **f(0)=0**, which imply that the origin is an equilibrium point.

A state feedback rather than output feedback is adopted to enhance the control performance. The quadratic cost function for the regulator problem is given by

$$J_c = \frac{1}{2} \int_{t_0}^{\infty} \left[\mathbf{y}^{\mathrm{T}} \mathbf{Q}_c \mathbf{y} + \mathbf{f}^{\mathrm{T}} \mathbf{R}_c \mathbf{f} \right] dt \ , \tag{16}$$

where $\mathbf{Q_c} \in \Re^{2nx2n}$ is semi-positive-definite matrix and $\mathbf{R_c} \in \Re^{mxm}$ positive definite. There are weighting matrices on the state and control inputs, respectively.

Assuming full state feedback, the control law is given by

$$\mathbf{f} = -\mathbf{R}_{\mathbf{c}}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\mathbf{y}\ . \tag{17}$$

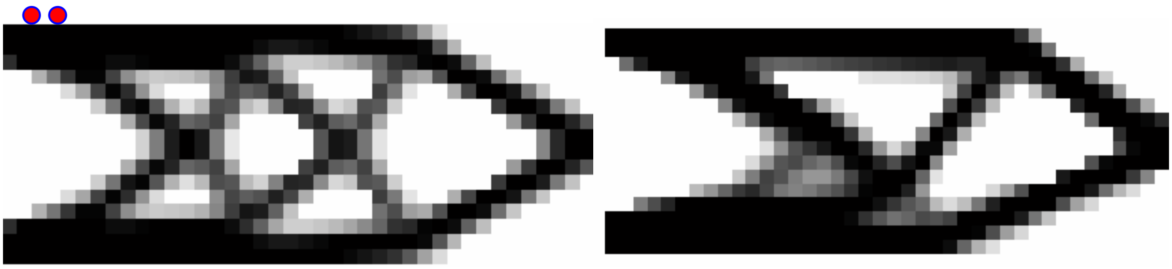
The state-dependent Riccati equation to obtain **P**, is given by

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}_{\mathrm{c}}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{Q}_{\mathrm{c}} = \mathbf{0}\,. \tag{18}$$

The computational cost is high if all modes are considered. But it can be dramatically reduced if only a few modes are dominant and their control is sufficient for the whole structure.

5. Results

The physical system considered in this work is composed by cantilevered steel beam shown in Fig. 1. The resulting topology for this problem is show in Fig. 3, where the locations of the horizontal control forces are indicate by points (small circles). This location for the actuator was chosen because it is a known fact that the best place for one actuator is as close as possible to the fixed size of the structure, which bears the maximum stress induced by the first and most significant mode.

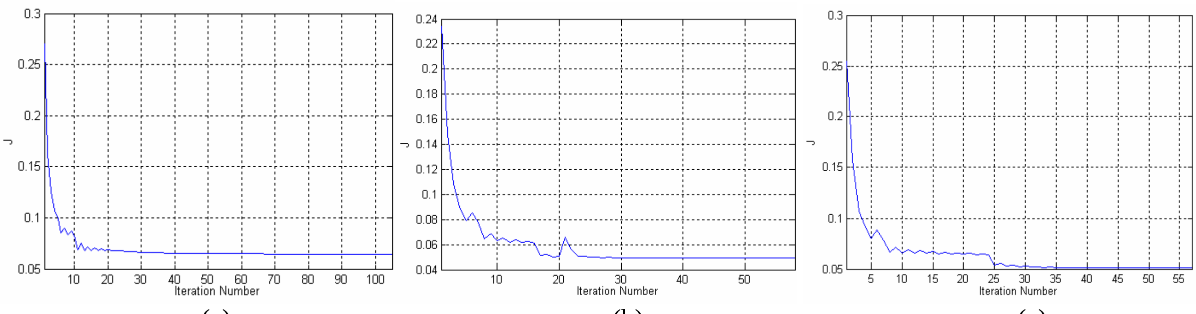


(c) (d)

Figure 3. c-Topology optimization without control force action. d-Topology optimization with control force action.

Some simplifications are introduced to the problem and its response analysis. We assume that the two control points can have different forces. This fact means that there are two external actuators. Embedded actuators (piezoelectric materials or hydraulic mechanisms), would generate equal magnitude opposing forces and need to be explicitly included in the model.

The convergence of the objective function is plotted in Fig.4.

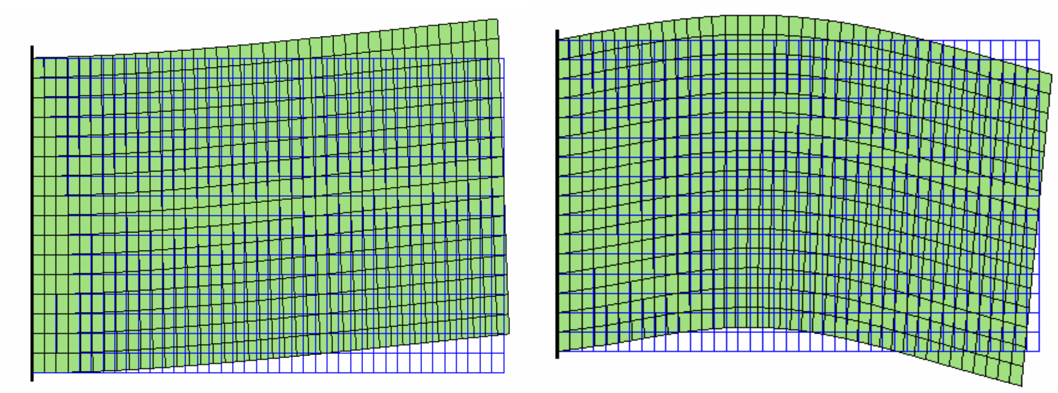


(a) (b) (c)

Figure 4. Objective function convergence a- without control; b- with one control force; c- with two control forces

We can observe in Fig. 4 that the convergence is faster in the initial 30 iterations, after there is a smaller change of the objective function value at each iteration.

The structure shown in Fig. 1 subject to transient forces produces initial deformation and so active the natural vibrations. The two free vibration modes of the model in Fig. 1, which finite element discretization are shown in the figures 5a and 5b,



(e) (f)

Figure 5. Deflections of first and second modes of the beam.

The results of the optimal control simulation in Matlab are shown in Fig. 6. The weighting matrices are $\mathbf{Q_c} = diag\{1\}$, $\mathbf{R_c} = diag\{0.5\}$. Here are considered the two first modes of the optimized structure. The position 1 is on the left point (small circle) and position 2 on the right, shown in Fig. 3. The fourth-order Runge-Kutta method was used to integrate the equations for a twenty seconds simulation.

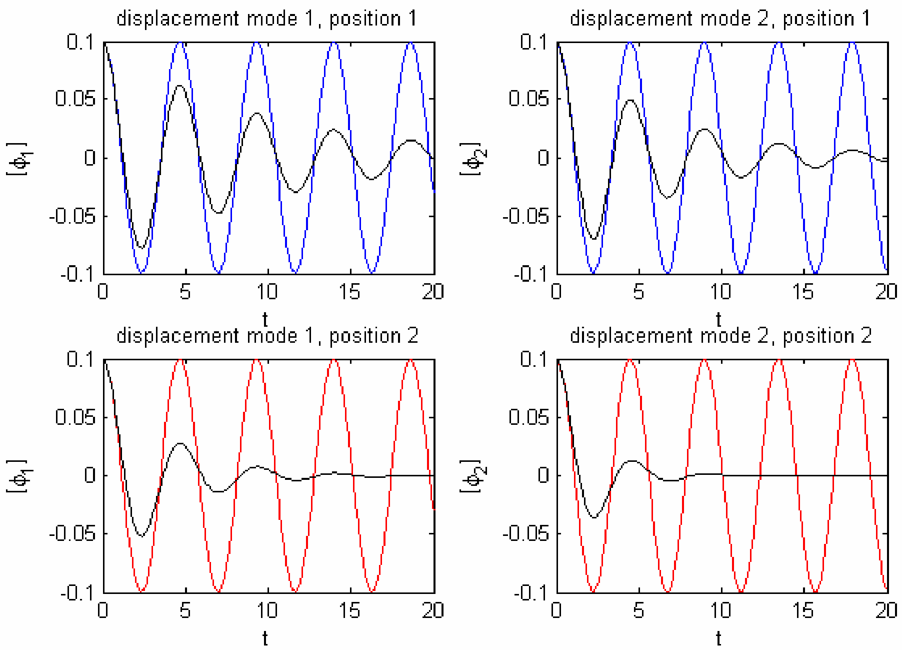


Figure 6. Deflections of first and second modes without independent modal control (blue and red) and with independent modal control (black).

It is possible observe that the modal displacement go quickly to zero, even without natural damping.

6. Conclusions and Considerations

In this work we introduced an integrated design procedure for a topology optimization and structural control system. This technique uses optimal design of a controlled structure and steady state control forces was achieved through the homogenization method and displacement feedback law. Optimal controls were applied to reduce the structural vibration within a reasonable few cycles. Active control can remove the vibration suppression from the structure effectively if it is carried out appropriately.

The simulations for the control system confirmed the effectiveness of this control technique. The numerical results indicate that combined structural topology design and optimal control can become an efficient methodology.

The present methodologies can be easily extended to other applications.

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