A GENERAL SOLUTION FOR ANTI-PLANE ELASTIC FIELD IN ANISOTROPIC MEDIA CONTAINING AN ESHELBY’S ELLIPTIC INHOMOGENEITY

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Summary: A general complex variable approach is presented for anti-plane problem for a system of one anisotropic elliptic inhomogeneity embedded in an infinite anisotropic medium (matrix). The system is subject to far-field shear stresses in the matrix at infinity and eigenstrains inside the inhomogeneity. The solution is derived based on conformal mapping for the complex variable and Laurent series expansion for the stress functions in which the coefficients in the series are determined by using continuous conditions at the interface. The shear stress distribution at the interface and displacements for the entire system are shown for the case of polynomial eigenstrains with the forms of uniform, linear and quadratic distributions, respectively. The translational deformation of the inhomogeneity and the effect of the orientation of principal axes of the anisotropic materials on the displacements of both the matrix and inhomogeneity are illustrated.

1 INTRODUCTION

Due to imperfect emanation from manufacturing processes and utilization, various defects and damages exist in composite materials and structures, even in the absence of any external loading. Microscopic defect such as particle, void or crack are referred as inhomogeneity or inclusion. Around the late 1950’ [1-3], Eshelby studied the problem of an inclusion embedded in an infinite isotropic elastic medium. Later, some related work was systematically investigated by many scholars [4-6]. For polynomial distributions of eigenstrains, Asaro and Barnett [7] showed that if an anisotropic ellipsoidal inclusion embedded in an infinite linear elastic medium undergoes eigenstrains, which are a polynomial of degree M in the spatial coordinates x, the final stress and strain state in the transformed inclusion is also a polynomial of degree M in x. Kinoshita [8] and Mura [9] analyzed elastic fields and

For anisotropic medium containing inhomogeneities, complex representation has been widely used in problems of 2-D plane elasticity. Using Stroh's formulation and Greens function integral, Bacon et al. [12] studied anisotropic medium containing defects, such as dislocations, inclusions and point defects. Ting [6], and Hwu & Yen [13] studied the problem of elliptic inclusion in anisotropic media. Ru [14] analyzed Eshelby's problem for an inclusion of arbitrary shape within an anisotropic plane or half-plane of the same elastic constant. Recently, Nie et al. [15-18] developed some analytic solutions of elastic fields for the plane and anti-plane problems of elliptical inhomogeneity with polynomial eigenstrains by means of the principle of minimum potential energy and complex variable method.

In this paper, based on complex representative theory of Lekhnitskii [19], a general method is presented for the anti-plane problem of anisotropic elliptic inhomogeneity in an infinite anisotropic medium under the action of far-field loading and polynomial eigenstrains. Based on continuous conditions of the stress and displacement at the interface, sets of algebraic governing equations are derived for the unknown coefficients in the series for complex stress functions. Specifically, all of the unknown coefficients can be analytically obtained by solving the equations for polynomial eigenstrains. The solution is verified with available results for the special case of eigenstrains. Numerical examples are given to illustrate the shear stress distribution at the interface and displacements for the entire elastic field, and some related results are discussed.

2 FUNDAMENTAL EQUATIONS

Neglecting body forces, the reduced equilibrium equation for an anti-plane problem is written as

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0,$$

(1)

where the stress components ($\tau_{xz}, \tau_{zy}$) are expressed in terms of a stress function $\phi(x,y)$ such that

$$\tau_{xz} = -\frac{\partial \phi}{\partial y}, \quad \tau_{zy} = \frac{\partial \phi}{\partial x}.$$

(2)

The constitutive relations in the anti-plane problem can be expressed as

$$\gamma_{xz} = \frac{\partial w}{\partial x} = \alpha_{44} \tau_{xz} + \alpha_{45} \tau_{zy}, \quad \gamma_{zy} = \frac{\partial w}{\partial y} = \alpha_{45} \tau_{xz} + \alpha_{55} \tau_{zy},$$

(3)

where $\alpha_{44}$, $\alpha_{45}$ and $\alpha_{55}$ are compliance elements, and $w = w(x,y)$ is displacement in the $z$-direction. Using Eq.(2) and eliminating $w(x,y)$ in Eq.(3), one gets

$$\alpha_{44} \frac{\partial^2 \phi}{\partial y^2} - 2 \alpha_{45} \frac{\partial^2 \phi}{\partial x \partial y} + \alpha_{55} \frac{\partial^2 \phi}{\partial x^2} = 0,$$

(4)

which has solution of the form

$$\phi(x,y) = F_1(x + \mu_1 y) + F_2(x + \mu_2 y),$$

(5)
where \( \mu_i, i = 1,2 \), are two roots of the resulting characteristic equation
\[
\alpha_{44} \mu^2 - 2\alpha_{45} \mu + \alpha_{55} = 0. \tag{6}
\]
For ideal elastic materials, the two roots consist of one pair of complex conjugates such that \( \mu_1 = \mu = \alpha + i \beta \), \( \beta > 0 \) and \( \mu_2 = \bar{\mu} \). Eq. (6) can thus be rewritten as
\[
\varphi = 2 \text{Re}[F(z_1)], \tag{7}
\]
where
\[
z_1 = x + \mu y. \tag{8}
\]

The components of stress and displacement can be expressed as
\[
\tau_{zx} = -2 \text{Re}[\mu F'(z_1)], \quad \tau_{zy} = 2 \text{Re}[F'(z_1)], \tag{9}
\]
and
\[
w = -2 \text{Re}[r F(z_1)], \tag{10}
\]
where
\[
r = \alpha_{44} \mu - \alpha_{45}. \tag{11}
\]

3 TRANSFORMATION AND CONTINUOUS CONDITION AT THE INTERFACE

Consider a system of infinite anisotropic media (matrix) containing an anisotropic elliptic inhomogeneity. As shown in Fig.1, semi-major and semi-minor axes of the ellipse are denoted by \( a \) and \( b \) in the \( x \) and \( y \) directions respectively. The interface between the inhomogeneity and matrix is denoted by \( \Gamma \). \( \gamma_{zx}^* \) and \( \gamma_{zy}^* \) are the prescribed components of the eigenstrain in the inhomogeneity, and \( \tau_{zx}^\infty \) and \( \tau_{zy}^\infty \) are the components of stress at infinity in the matrix.

![Figure 1: Elliptic inhomogeneity in an infinite anisotropic matrix](image)

3.1 Transformation from the \( z_1 \) -plane to \( \zeta \) -plane

Although affine transformation \( z_1 = x + \mu y \) from the physical \( z \)-plane for an anisotropic medium containing an elliptic inhomogeneity to the complex \( z_1 \)-plane is not conformal, the elliptic region on the physical plane remains elliptic on the complex \( z_1 \)-plane, as shown in
Fig. 2. The elliptic region on the $z_1$-plane can be obtained by

$$\frac{x_1^2}{a_1^2} + \frac{y_1^2}{b_1^2} = 1,$$

where

$$a_1^2 = \frac{2\beta^2 a^2 b^2}{(\beta^2 b^2 + a^2 + \alpha^2 b^2 - \Delta)}, \quad b_1^2 = \frac{2\beta^2 a^2 b^2}{(\beta^2 b^2 + a^2 + \alpha^2 b^2 + \Delta)},$$

$$\Delta = \sqrt{(a^2 - \beta^2 b^2 + \alpha^2 b^2) + (2\alpha \beta b^2)},$$

with inclined angles of

$$\tan(2\theta) = \frac{2\beta ab^2}{a^2 - \beta^2 b^2 + \alpha^2 b^2}.$$  \hspace{1cm} (13)

As shown in Fig. 3, the ellipse on the $z_1$-plane can be mapped into a unit circle on the $\zeta$-plane by conformal mapping such that

$$z = \omega(\zeta) = \left(\frac{a - i \mu b}{2}\right)\zeta + \left(\frac{a + i \mu b}{2}\right)\frac{1}{\zeta} = \frac{l}{2}(T\zeta + \frac{1}{T\zeta}),$$

where

$$l = \sqrt{a^2 - (i \mu b)^2}, \quad T = \frac{a - i \mu b}{a + i \mu b}.$$ \hspace{1cm} (15)

The line $-|l| \leq x_1 \leq |l|$ inside the ellipse on the $z_1$-plane is mapped to the inner circle of radius $|\frac{1}{T}|$ on the $\zeta$-plane. Therefore, the ellipse is actually mapped to an annulus with outer radius of 1 and inner radius of $|\frac{1}{T}|$. Along the outer boundary of this region, $\zeta = \sigma = e^{i\theta}$. 
3.2 Series expansions for the complex function

Denoting

\[ S^{c,0} = \frac{T^{c,0} - T^{c,0}}{2} = \frac{a-i\mu^{c,0}b}{2}, \quad \tau^{c,0} = \frac{1}{(T^{c,0})^2} = \frac{a+i\mu^{c,0}b}{a-i\mu^{c,0}b}, \]

Eq.(14) can be expressed as

\[ z'^{0,0}_{i} = o^{0,0}(\zeta^{c,0}) = S^{c,0}(\zeta^{c,0} + \tau^{c,0} \frac{1}{\zeta^{c,0}}), \tag{16} \]

where the superscripts \( c \) and 0 are used for the matrix and inhomogeneity respectively.

The two complex functions for the inhomogeneity and matrix can be expressed by Laurent series expansions as

\[ F^0(z'_i) = \sum_{m=0}^{\infty} A_m (\zeta^{0,m} + i\mu^{0,m} \frac{1}{\zeta^{0,m}}), \tag{17} \]

and

\[ F^c(z'_i) = \sum_{m=0}^{\infty} B_m \frac{1}{\zeta^{c,m}} + b_i^\infty S^c \zeta^c, \tag{18} \]

respectively. In the above equation, \( b^\infty_i = \frac{\tau^z_{x^i} + \tau^z_{y^i}}{\mu - \mu^c} \), and \( \tau^z_{x^i} \) and \( \tau^z_{y^i} \) are the far-field shear stresses.

3.3 Continuous condition for the stresses and displacement at the interface

For the perfect interface \( \Gamma \), stresses and displacements are continuous. The continuous condition for stress can be expressed by an equation for the resultant traction such that

\[ 2 \text{Re}[F^0(z'_i)]_| - 2 \text{Re}[F^c(z'_i)]_| = 0. \tag{19} \]

In the inhomogeneity, \( w^0 \) is the displacement induced by tractions at interface \( \Gamma \), \( w^* \) is the displacement induced by the eigenstrain \( (\gamma^s_{xx}, \gamma^s_{yy}) \) for stress-free state. The total displacements \( w' \) is then written as \( w' = w^0 + w^* \). It can be seen that rigid-body translation can be determined by the constraint of displacement although it has no effect on the stress distribution. Therefore, the translation at the centre of the inhomogenity is determined by \( w_0 = w'|_{z=0} \), where \( w^* \) is obtained by integrating the eigenstrains satisfying the compatibility equation without translation. The continuous condition for the displacement at interface \( \Gamma \),

\[ w'|_r = (w^0 + w^*)_r, \]

\[ -2 \text{Re}[r^c F^c(z'_i)]_r + 2 \text{Re}[r^0 F^0(z'_i)]_r = w^*_r. \tag{20} \]

4 DETERMINATION OF COEFFICIENTS IN THE SERIES

Substituting \( x = \frac{a}{2}(\sigma + \frac{1}{\sigma}) \) and \( y = -\frac{bi}{2}(\sigma - \frac{1}{\sigma}) \) into the resulting displacement \( w^* (x, y) \) from the eigenstrains \( (\gamma^s_{xx}, \gamma^s_{yy}) \), and using expansion of the Fourier series, we have
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\[ w^*(x, y)\big|_{\Gamma} = \sum_{m=1}^{\infty} d'_m \sigma^m + d'_0 + \sum_{m=1}^{\infty} d'_m \frac{1}{\sigma^m}, \]  

(21)

where

\[ d'_0 = \frac{1}{2\pi} \int_{0}^{2\pi} w^*(x, y)\big|_{\Gamma} d\theta \quad d'_m = \frac{1}{2\pi} \int_{0}^{2\pi} w^*(x, y)\big|_{\Gamma} e^{-im\theta} d\theta. \]

Substituting Eqs.(17) and (18) into Eqs.(19) and (20) yields

\[ 2\text{Re}\left[ \sum_{m=0}^{\infty} A_m (\sigma^m + t^0m \frac{1}{\sigma^m}) \right] - 2\text{Re}\left( \sum_{m=0}^{\infty} B_m \frac{1}{\sigma^m} \right) = 2\text{Re}(B^c S^c \sigma), \]

(22)

\[ 2\text{Re}\left[ r^0 \sum_{m=0}^{\infty} A_m (\sigma^m + t^0m \frac{1}{\sigma^m}) \right] - 2\text{Re}(r^c \sum_{m=0}^{\infty} B_m \frac{1}{\sigma^m}) = w^*\big|_{\Gamma} + 2\text{Re}(r^c B_1 S^c \sigma). \]

(23)

At the interface \( \Gamma \), \( \zeta^c = \zeta^0 = \sigma = e^{i\sigma} \).

Considering that there are no translation for infinite matrix, i.e., \( B_0 = 0 \), substituting Eq.(21) into Eqs.(22) and (23), and equating the coefficients of \( \sigma^m \) (\( m = 0, 1, 2, ... \)), yields

\[ 2A_0 + 2A_0 = 0 \]

\[ 2r^0 A_0 + 2r^0 A_{0} = d'_0, \]

(24)

for \( m = 0 \), and

\[ A_1 t^0 + A_1 - B_1 = \overline{b_1 S} \]

\[ r^0 A_1 t^0 + r^0 A_1 - r^c B_1 = \overline{d_1'} + r^c \overline{b_1'} S, \]

(25)

for \( m = 1 \) and

\[ A_m t^0m + A_m - B_m = 0 \]

\[ r^0 A_m t^0m + r^0 A_m - r^c B_m = d'_m, \]

(26)

for \( m \geq 2 \).

The unknown coefficients, \( A_m \) and \( B_m \), \( m = 0, 1, 2, ... \), in the resulting sets of algebraic equations Eqs.(24)-(26) can be obtained analytically by solving the equations for each \( m \). Based on the resulting expressions for \( F^0 (x_0') \) and \( F^c (x_0') \), the shear stresses and the displacements in the inhomogeneity and matrix can be derived by using Eqs.(9) and (10). In particular, tractions at the interface are obtained by

\[ \tau_{nz}^c,0 = \tau_{xz}^c \cos \psi + \tau_{yz}^c \sin \psi \big|_{\Gamma}, \]

where \( \psi \) is the angle between the outward normal and the x-axis, and

\[ \cos \psi = \frac{R \cos \theta}{\sqrt{\sin^2 \theta + R^2 \cos^2 \theta}}, \quad \sin \psi = \frac{\sin \theta}{\sqrt{\sin^2 \theta + R^2 \cos^2 \theta}}. \]

For the special case of identical isotropic materials for both the inhomogeneity and matrix, we have \( \alpha_{44}^{0} = \alpha_{55}^{0} = \alpha_{44} \) and \( \alpha_{45}^{0} = 0 \). If the eigenstrains are assumed to have the form of linear distribution

\[ \gamma_{x}^* = a_{00}^* + \frac{a_{10}^*}{a} x + \frac{a_{01}^*}{a} y, \quad \gamma_{y}^* = b_{00}^* + \frac{b_{10}^*}{a} x + \frac{b_{01}^*}{a} y, \]

the stresses have the reduced results

\[ \tau_{xz}^0 = -\frac{1}{2\alpha_{44}} a_{00} (1+h) + \tau_{xy}^0 = \frac{1}{4\alpha_{44} a} \left\{ \left( -a_{10}^* (1+h)^2 + b_{10}^* (1-h)^2 \right) x - 2(1-h^2) a_{00}^* y \right\}, \]
\[
\tau_{xy}^0 = -\frac{1}{2\alpha_{44}}(1-h)\gamma_{(0)0}^* + \tau_{xy}^\infty + \frac{1}{4\alpha_{44}}a \{ [a_{(1)0}^* (1+h)^2 - h_{(0)0}^* (1-h)^2] y - 2a_{(0)1}^* (1-h^2) x \},
\]

where \( h = \frac{a-b}{a+b} \). The above solution are similar to the results of Chen [11], and identical to the reduced results of Nie et al. [16] for \( \tau_{xy}^\infty = \tau_{xy}^0 = 0 \).

5 NUMERICAL EXAMPLES

In this paper, matrix (Glass/Epoxy) and inhomogeneity (Graphite/Epoxy), forming a symmetrical system, are chosen as anisotropic materials with material constants \( c_{44}^c = 4.58 \text{GPa} \), \( c_{45}^c = 3.40 \text{GPa} \), \( c_{44}^0 = 2.30 \text{GPa} \), \( c_{45}^0 = 0 \) and \( c_{55}^0 = 1.65 \text{GPa} \), where superscripts \( c \) and \( 0 \) are again used for the matrix and inhomogeneity respectively. The elliptic aspect ratio is taken as \( R = \frac{b}{a} = 0.5 \). Three cases of polynomial eigenstrains with degree 0, 1 and 2 are considered. The corresponding uniform, linear and quadratic eigenstrains are prescribed as

\[
\gamma_{zx}^* = a_{00}^* = 0.1, \quad \gamma_{zx}^* = \frac{a_{10}^*}{a} x = \frac{0.1}{a} x \quad \text{and} \quad \gamma_{zx}^* = \frac{a_{20}^*}{a^2} x^2 = \frac{0.1}{a^2} x^2 \quad \text{together with} \quad \gamma_{zy}^* = 0,
\]

respectively.

Total displacement distributions along the \( x \)-axis for the three cases are shown in Fig.4. Results indicate that the displacement at the centre of the inhomogeneity is zero for uniform and quadratic eigenstrains. However, total displacement at the centre of the inhomogeneity is not zero for linear eigenstrains with the translation \( w_0' = -0.0166a \) along the \( z \)-axis.

![Figure 4: Total displacement (\( w^c \) and \( w' \)) along the \( x \)-axis](image.png)
Distributions of the shear stresses $\tau_{nz}^{0,c}$ along the interface of uniform and linear eigenstrains are shown in Figs.5 and 6. For uniform eigenstrains, results indicate that stresses along the interface and displacements for the entire field, as shown in Fig.5 and Fig.4, are anti-symmetrical about the semi-minor axis of the ellipse. It is due to the anti-symmetrical eigenstrains in the symmetrical system of the matrix/inhomogeneity. However, for linear eigenstrains, as shown in Fig.6 and Fig.4, both stresses and displacements are symmetrical about the semi-minor axis of the ellipse, due to symmetrical eigenstrains.

For orthotropic material, orientation of principal axes of the material is defined by an angle $\vartheta$. The angle $\vartheta$ in the $Oxy$ plane is measured from the positive principal axis to the positive $x$-axis. The compliance elements in the coordinate system can be expressed as

$$
\alpha_{44} = \alpha_{44} \cos^2 \vartheta + \alpha_{55} \sin^2 \vartheta,
$$

$$
\alpha_{55} = \alpha_{44} \sin^2 \vartheta + \alpha_{55} \cos^2 \vartheta,
$$

$$
\alpha_{45} = (\alpha_{44} - \alpha_{55}) \sin \vartheta \cos \vartheta,
$$

(27)
where $\alpha_{44}$, $\alpha_{45}$ and $\alpha_{55}$ are compliance elements in the coordinate system, while $\alpha_{44}$ and $\alpha_{55}$ are compliance elements in the principal axes system for the material.

In the Oxy coordinate system, the corresponding roots of Eq.(6) can be written as

$$\mu = \alpha_{45} \pm i \sqrt{\alpha_{44} \alpha_{55} - \alpha_{45}^2},$$

(28)

Hence, the corresponding parameter in Eq.(11) is expressed by

$$r = \alpha_{44} \mu - \alpha_{45}.$$  

(29)

Substituting Eqs.(27) and (28) into (29) results in

$$r = \pm i \sqrt{\alpha_{55} \alpha_{44}},$$

(30)

which indicates that the angle $\theta$ has no effect on the parameter $r$. It can thus be seen that angle $\theta$ of the matrix has no effect on the elastic fields in the inhomogeneity.

For the case of linear eigenstrains, total displacement distributions for both $\theta^c = 0^\circ$, $45^\circ$, $90^\circ$ and $\theta^0 = 0^\circ$, $45^\circ$, $90^\circ$, where superscripts $c$ and $0$ denote matrix and inhomogeneity respectively, are shown in Figs.7 and 8, respectively. It can be seen from Fig.7 that angle $\theta^c$ has no effect on displacement in the inhomogeneity along the $x$-axis. However, as shown in Fig.8, the angle $\theta^0$ has a large effect where displacements at the interface and the centre of the inhomogeneity decrease with an increase of $\theta^0$.
6 CONCLUSIONS

In this paper a general complex variable approach is given for an anti-plane problem of elliptic inhomogeneity-matrix system under the combined actions of polynomial eigenstrains inside the inhomogeneity and far-field shear stresses in the matrix at infinity. The solution is derived based on conformal mapping for the complex variables and Laurent series expansion for the stress functions. Coefficients in the series are determined by solving the resulting sets of algebraic equations with continuous conditions for stresses and displacement at the interface. The method and resulting solution can be used effectively to determine stress and displacement fields for the system, and to evaluate the translational deformation of the inhomogeneity and effects of the geometric and material parameters on the resulting fields.

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