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ON THE CHARACTERIZATION OF PARAMETRIC UNCERTAINTY ON FGM PLATES

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Summary: *Composite materials with their intrinsic tailor-made capabilities can be strong candidates to improve the mechanical performance of structures, either by partially or totally replacing other traditional materials. These easily tailored features can be thought not only in terms of the more often used fibre reinforced laminated composites but also in the context of particulate composites. In general, the mechanical performance of composite structures can be, intentionally or not, influenced through the manipulation of geometric properties, the selection of material's phases and its disposition in the composite, as well as, the spatial distribution of reinforcement agents, such as fibres or particles. The uncertainty associated to all these different aspects can be considered as the main source of variability to the mechanical behaviour of a given structure. It is therefore important to characterize the relations between the geometric and material parameters and a set of some relevant structural responses. The quantification of uncertainty is often related to the experimental behaviour of a given structure, although it can also be assessed within the design perspective, where it is useful to understand and identify the parameters with a greater influence on the uncertainty associated to the model simulations.*

In the present work, one considers functionally graded plates, where different material and geometric characteristics are assumed to be uncertain. The mechanical behaviour of such plates is modelled using Lagrange- and Kriging-based finite element models, developed according to the assumptions of the first order shear deformation theory. A set of numerical results is presented and discussed in order to identify the most significant modelling parameters for the description of the output variability, in this case the maximum deflection.

1. INTRODUCTION

The uncertainty associated to real physical quantities, that characterize a functionally graded material (FGM), can be, among others, a crucial issue on the assessment of an eventual failure in a FGM structure. This uncertainty can be thought at the microscale of the composite, for instance if one thinks about the real geometry of the inclusions or the adhesion conditions to the agglomerating phase, or at the macroscale if one considers the average values assigned to these characteristics/properties, where one assumes a variability based on the manufacturer's technical sheets. This latter case is the focus of the present work.

There is a significant number of published works carried out based on the assumption of deterministic geometric and material characteristics, that provide predictions on the expected behaviour of a given structure. This is also the case of structures made of FGMs.

The term FGM has only appeared by the mid 1980s [1] and characterize the continuous variation of the materials' properties in a 3D structure domain. These materials are known to provide superior thermal and mechanical performance, because of their properties variation characteristic [2, 3]. Many approaches on FGM structures design have been introduced. Meshless method and third-order shear deformation combined with different homogenization schemes, such as, the rule of mixtures or the Mori-Tanaka approach have been used in [4]. Lee et al. [5] used a higher-order shear deformation theory (HSDT) considering different nature of volume fraction distributions, as the ones based on exponential, power-law and sigmoid functions. Nguyen et al. [6] present a study about the requirement of a shear correction factor when the first-order shear deformation theory (FSDT) is used and how this factor can affect the model and its results. Those authors quantify the influence of a unique correction factor on the model outputs. Note that one of the major difficulties of modelling a composite structure is the accurate determination of the material properties [7]. According to [2], it is necessary to understand the effects of varying the relative proportion of the material phases involved in the FGM constitution, if one intends to obtain an optimized material, which may correspond to a set of specific operation needs. The comprehension of these effects enable a better prediction of the responses of a given structure when submitted to external loads. Thus, knowing that both extrinsic and intrinsic characteristics and factors can affect the mechanical behaviour of a structure, namely a FGM one, the main question is to quantify how much the uncertainty related to these parameters affects the outputs of a FGM model and which ones are more important relevant on the description of the variability on the results.

The present work is focused on a stochastic approach that assumes that one can describe the relation between the variability of inputs and outputs based on a sample of model responses obtained by finite elements analysis. This approach has the purpose of identify the most significant parameters in the description of the variability of the model results, by the use of a multi-variable regression model, validating all its assumptions. It is here performed a comparative analysis between the input-output correlation of a model based on Lagrange interpolation and a Kriging-based one, since different models were created and their results compared.

2. SHEAR DEFORMATION THEORY AND FINITE ELEMENT MODELS

In the present work, one uses the FSDT to model the behaviour of the plates that will be studied. This theory is adequate considering not only the geometric characteristics of the FGM plates, but also taking into account the reasonable computational cost associated to the significant number of finite element analysis that will be required. The displacement field is described as:

$$\begin{aligned} u(x,y,z) &= u^0(x,y) + z\theta_x^0(x,y) \\ v(x,y,z) &= v^0(x,y) + z\theta_y^0(x,y) \\ w(x,y) &= w^0(x,y) \end{aligned} \tag{1}$$

where u^0, v^0, w^0, θ_x^0 and θ_y^0 are the generalized displacements associated to the plane mid-surface [8, 9]. A shear correction factor of 5/6 is used.

To enable the characterization of the mechanical behaviour of functionally graded plates, whose geometric and material parameters can be affected by uncertainty, two types of quadratic quadrilateral plate finite elements were implemented. These elements have both nine nodes, differing in the nature of the interpolation functions, which can be the often used Lagrange functions or Kriging-based functions [10]. These latter ones can be observed in Figure 1, associated to each node considering the usual numbering counter-clockwise numbering scheme.

To obtain the Kriging-based shape functions, it is assumed that for a generic function $q(x, y)$, say a degree of freedom, one can approximate it by a linear combination of interpolation functions $\phi(x, y)$ and the values assumed by each function in the nodal points are given by

$$\bar{q}(x,y) = \phi(x,y)q(x,y) \tag{2}$$

as also happens in other type of approximations. In this particular case, the derivation of these interpolation functions can be summarized as

$$\phi(x,y) = \boldsymbol{s}^T(x,y)\mathbf{A} + \boldsymbol{r}^T(x,y)\mathbf{B} \tag{3}$$

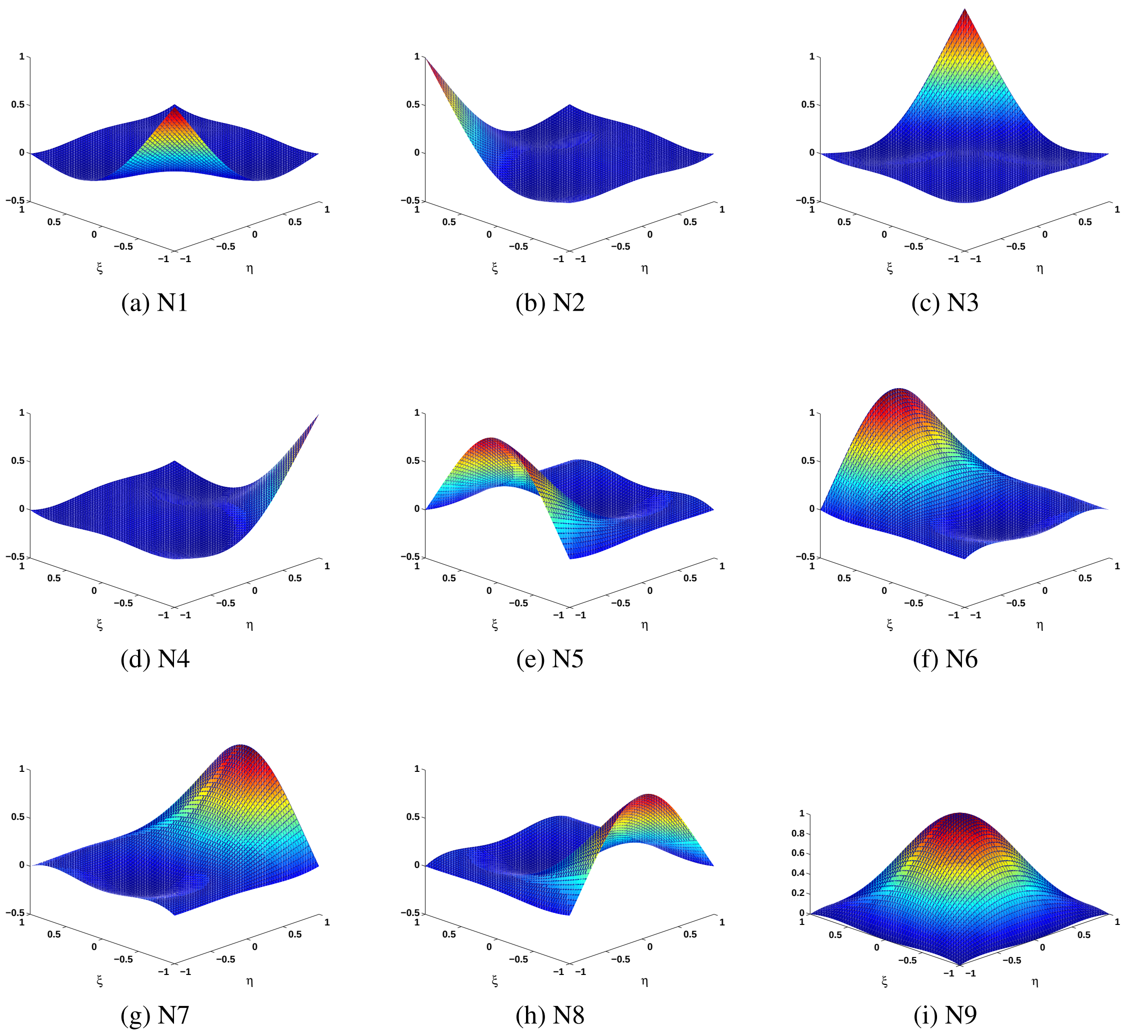
where matrices $\mathbf{A}$ and $\mathbf{B}$ are given by:

$$\mathbf{A} = \left(\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S}\right)^{-1}\mathbf{S}^T\mathbf{R}^{-1} \tag{4}$$

$$\mathbf{B} = \mathbf{R}^{-1}\left(\mathbf{I} - \mathbf{S}\mathbf{A}\right) \tag{5}$$

with $\boldsymbol{s}(x, y)$ being a vector that contains the different monomials that constitutes the polynomial basis used, $\boldsymbol{r}(x, y)$ is the Euclidean distances vector, and $\mathbf{R}$ and $\mathbf{S}$ are respectively a covariance matrix and a rectangular matrix where each monomial coefficient assumes a value associated to a point position.

The finite element analyses are carried out by using the usual equilibrium equations, widely disseminated in the literature [8, 9].



(a) N1 (b) N2 (c) N3

(d) N4 (e) N5 (f) N6

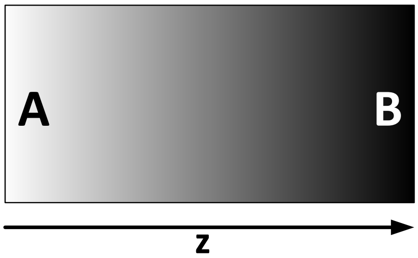
(g) N7 (h) N8 (i) N9

Figure 1: Kriging-based interpolating functions.

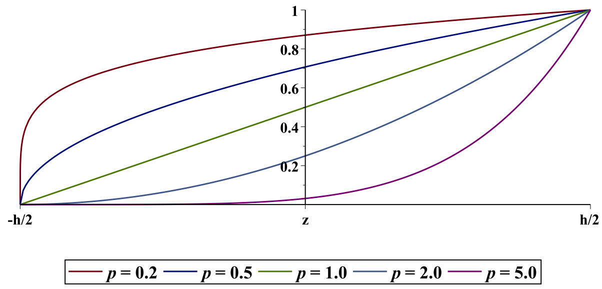
2.1 Functionally graded materials

FGMs are usually particulate composite materials which composition may vary in a 3D space according to a specified phase mixture distribution law, possessing a continuous profile.

Figure 2a illustrates a mixture distribution with a single variation in the z (thickness) direction for a dual-phase particulate composite, where in its surface on the left end side one has only phase A, whereas in the surface on the right end side there is only phase B. The profile distribution can be expressed in different ways, being however the more common, the one used



(a) Schematic representation.



(b) Volume fraction distribution profiles for different exponent values.

Figure 2: FGM mixture through the composite thickness.

in [11, 12, 13] known as the power exponent law,

$$V_r = \left(\frac{z}{h} + \frac{1}{2}\right)^p \tag{6}$$

where the volume fraction of the reinforcement particles is denoted by V_r, and h and p are, respectively, the thickness of the composite plate and the exponent that dictates a faster or slower incorporation of the reinforcement particles, nearer its outer surfaces. In Figure 2b one can observe this volume fraction evolution within the composite thickness for the cases, $p = 0.2$, $p = 0.5$, $p = 1.0$, $p = 2.0$ and $p = 5.0$, being possible to have a qualitative appreciation on the distribution of the elastic properties through thickness.

The volume fraction distribution gives the phase's mixture composition at each point and it varies through thickness, so the corresponding average material properties (P_{ave}) will be also influenced by this variation. Although there are many other homogenization schemes to predict the average properties of a composite, the Voigt's rule of mixtures [14] is used in the present work. For a dual-phase composite, this rule is written as

$$P_{ave} = V_r P_r + (1 - V_r) P_m \tag{7}$$

where P_r and P_m are generic material properties of the reinforcement particle (r subscript) and of the matrix (m subscript), respectively.

In the following two sections, one presents a study on the variability of the FGM's properties, followed by the basics on multiple linear regression models.

3. FGM'S PROPERTIES VARIABILITY SIMULATION

By considering now a power-law exponent $p = 1$, one obtains a linear approximation of the volume fraction distribution, as presented in Figure 2b, for a plate with a thickness denoted by h. Instead of assuming that these parameters possess exact values, in the present work one starts by admitting that the volume fraction distribution is affected by uncertainty associated

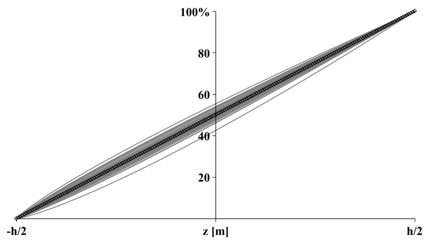
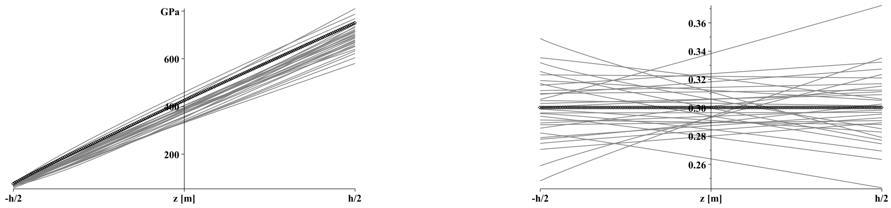


Figure 3: Volume fraction through thickness, considering variability.



(a) (b)

Figure 4: Elastic properties distributions through thickness, considering variability.

to its exponent p and also to the thickness h of the plate. For instance, one may consider that the exponent p has a mean value μ equal to 1, representing the linear volume fraction (Figure 3). It seems reasonable, from the engineering point of view, to simulate p according to a normal distribution with the mentioned mean value and a coefficient of variation of 7.5%. The simulated spread of p is shown in Figure 3. Note that if this spread is propagated to the average material properties through equation 7, it results on the spread shown in Figure 4.

This uncertainty applied to the exponent, which dictates the inclusion rate through thickness of the reinforcement particles (ceramic particles), and to the thickness, along with the uncertainly of the mechanical properties of each material reflects the global variability expected in a real condition. Each one of these variables will take a fundamental role on the characterization of the deflection variability. Thus, in order to simulate the variability on the FGM's properties, both geometric and mechanical ones, one has simulated them using a random multivariate normal distribution $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \Sigma)$, with the mean values given in Table 1 and a diagonal covariance matrix (ensuring the independence between modelling parameters) with standard deviations computed using a 7.5% coefficient of variation.

For sampling purposes, one used a Latin Hypercube Sampling (LHS), often considered to perform experiment simulations emulating physical systems. Note that the LHS used has the ability to ensure the independence between variables [15].

Table 1: Parameters used in the simulation (according to [6]).

Parameters	E_c	E_m	ν_c	ν_m	h	p
μ	696 GPa	70 GPa	0.3	0.3	0.05 m	1

4. MULTIPLE LINEAR REGRESSION

A simple linear regression model aims at build a probabilistic model that relates a dependent variable Y to a single predictor X. As the uncertainty on FGM plates may be due to several parameters, the use of a multiple linear regression model is more appropriated and it is given by

$$Y = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k + \varepsilon \tag{8}$$

where k is the number of independent variables used to explain the dependent variable Y, β_i are the regression coefficients and ε is the residual or error term. Note that in the case of multiple linear regression the predictor is a vector $\boldsymbol{X}$.

The coefficient β_0 is the intercept, which corresponds to the value predicted when the independent variables are zero, whereas β_i are the partial slopes, representing the influence of the variable X_i on the response Y. The term ε is assumed to have a normal distribution with zero mean and constant variance σ^2. Additionally, the independent variables used to predict Y should be uncorrelated. This means that, if the assumptions of the model are validated, a response value $\hat{y}$ can be estimated from the sampled values x_i with a random residual $\varepsilon \sim N(0, \sigma)$. So, the residuals $\varepsilon = y - \hat{y}$ can be used to estimate the regression coefficients and to validate the model assumptions. The regression coefficients are estimated using the least squares method [16].

Based on a specific sample, one can find estimates for each β_i as well as for the coefficient of multiple determination, R^2, which gives the proportion of variability of the response that is explained by the model (usually, this is an output of the linear model, as well as the R^2-adjusted). Using inferential statistics, the sample results can be generalize for the population. The ANOVA (analysis of variance) gives the significance of the model, based on the p-value of the F-test. If the model is significant, it means that at least one of the slopes is not zero, meaning that those predictors are useful. In that situation, the t-test gives the significance of each individual independent variable. Moreover, it is possible to construct confidence intervals for the slopes. Once a model is chosen, one must validate the Gauss-Markov assumptions made for the residuals [16].

5. RESULTS

Based on the methodology presented in section 3, the FGM's properties were simulated using a sampled of $n = 30$, which is a a sufficiently large sample size to support the significance of the results, keeping the problem at a reasonable size to deal with experimental tests.

Figure 5 is the matrixplot of the sampled FGM properties. As expected, the individual histograms show a Gaussian behaviour according to the values presented in Table 1. As it can be observed, the variables (material and geometric properties) are uncorrelated, as shown by the scatterplots and corresponding correlation coefficients in the same Figure.

With the sampled modelling parameters, one has proceeded to the finite element analysis aiming at characterize the maximum transverse displacement of the FGM plate. These analyses were carried out using both the Lagrange- and Kriging-based plate finite elements. The frequency histograms corresponding to each one of these models are presented in Figure 6, in

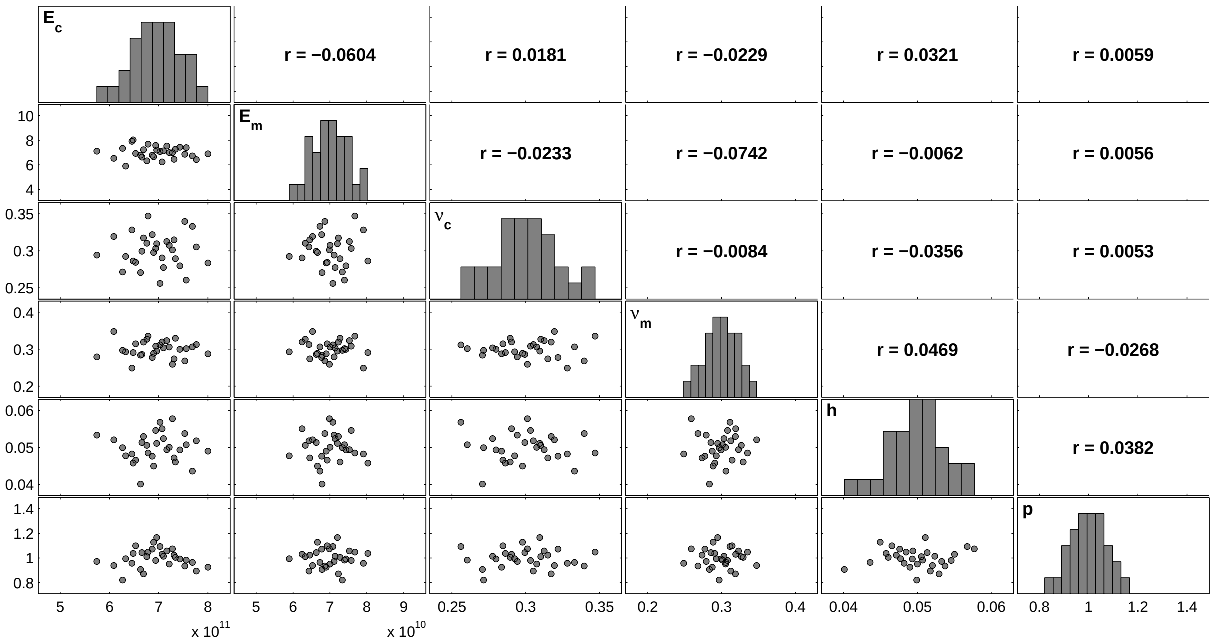
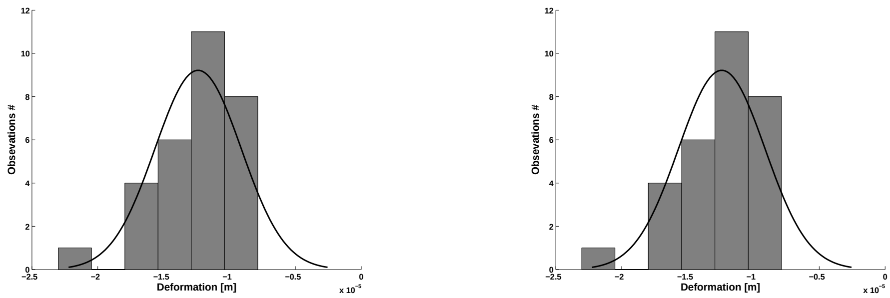


Figure 5: Characterization of the sampled modelling parameters.

both cases the shape of the histogram resembles a normal distribution. Although the p-values of the goodness-of-fit tests are not very high, this distribution was not rejected. Another important result is that there is no evidence to reject the hypothesis that both methods yield the same output (Figure 6).



(a) Lagrange (b) Kriging

Figure 6: Frequency histograms of the deformation obtained for both methods.

In order to identify the most significant modelling parameters for the description of the output variability, in this case the maximum deflection, a linear regression model was built using all the inputs. However, the built model is considered to be significant, the assumptions on its residuals are violated, being therefore the model invalid.

Thus, analysing the contribution of each variable, the maximum deflection variability is mostly explained by the thickness h. In fact, with just two input parameters, h and p, one is able to construct a valid model with a high value of explanation. As there is an interaction between these two variables, an extra input was added to cope with this effect. So, the proposed model is given by

$$Y = \beta_0 + \beta_1 h + \beta_2 p + \gamma_{12}\,(h\,p) + \varepsilon \tag{9}$$

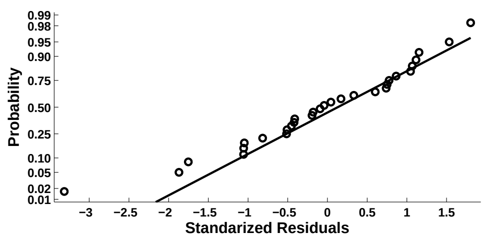
where the coefficient γ_{12} is related to the interaction effect.

As it can be observed by the model outputs of Table 2, the considered parameters are all significant, as well as the model of equation 9 is, with an explanation of 90% of the maximum deflection variability. The significance of the model is confirmed by a p-value < 0.0001 in the F-test and its fitting as an adjusted $R^2 \approx 90\%$. Regarding each parameter, individually, their significances are given in Table 2. As expected, regarding the result with all the modelling parameters, h is the most significant parameter in the description of the variability of the maximum deflection. To validate this model, the assumptions on its residuals are verified, as shown in Figure 7 and Table 4.

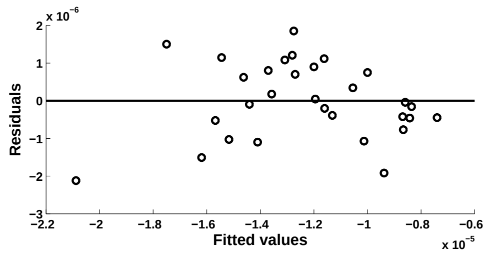
Table 2: Linear model summary - Lagrange.

Coefficients	**Estimate**	**Std. Error**	**t value**	**p-value**
β_0	-1.150e-04	3.360e-05	-3.424	0.00206 **
β_1	2.176e-03	6.713e-04	3.241	0.00326 **
β_2	6.291e-05	3.340e-05	1.884	0.07083 .
γ_{12}	-1.379e-03	6.671e-04	-2.068	0.04873 *

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1



(a) Probability plot.



(b) Residuals distribution.

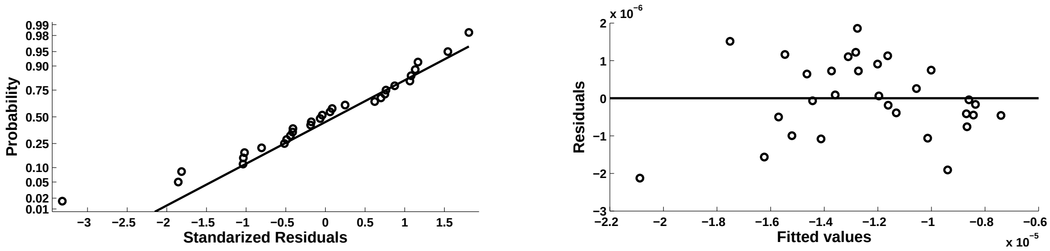
Figure 7: Diagnostic plots - Lagrange.

The results for the Kriging-based finite elements are very similar (see Table 3 and Figure 8), leading to an analysis analogous to the one presented for the Lagrange interpolation. This observation supports the already mentioned resemblance between the responses obtained by both methods. Moreover, there is no evidence to reject the hypothesis that both methods yield the same output.

Table 3: Linear model summary - Kriging-based.

Coefficients	**Estimate**	**Std. Error**	t **value**	p**-value**
β_0	-1.139e-04	3.363e-05	-3.387	0.00226 **
β_1	2.155e-03	6.721e-04	3.206	0.00355 **
β_2	6.174e-05	3.344e-05	1.847	0.07623 .
γ_{12}	-1.358e-03	6.678e-04	-2.033	0.05237 .

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1



(a) Probability plot. (b) Residuals distribution.

Figure 8: Diagnostic plots - Kriging-based.

As expected, due to resemblance between the outputs obtained by the two methods (Figure 6), the multiple regression model is the same for both cases and it is given by eq. 9, being the estimated coefficients quite similar, as shown in Tables 2 and 3. The normality test results for the residuals are given in Table 4 and the 95% confidence intervals are presented in Table 5.

Table 4: Normality test results.

Normality test	p**-value**	
	Lagrange	**Kriging**
Anderson-Darling	0.7761	0.7217
Cramer-von Mises	0.7614	0.6985
Lilliefors (Kolmogorov-Smirnov)	0.6471	0.5459
Pearson chi-square	0.3920	0.5304
Shapiro-Francia	0.8385	0.8127

Table 5: Confidence intervals on the regression coefficients.

Coefficients	**Lagrange**		**Kriging**	
	2.5%	**97.5%**	**2.5%**	**97.5%**
β_0	-1.8409e-04	-4.5969e-05	-1.8305e-04	-4.4775e-05
β_1	7.9573e-04	3.5556e-03	7.7326e-04	3.5362e-03
β_2	-5.7374e-06	1.3157e-04	-6.9873e-06	1.3047e-04
γ_{12}	-2.7506e-03	-8.2901e-06	-2.7304e-03	1.4992e-05

6. CONCLUSIONS

The uncertainty associated to geometric and material properties of a FGM structure can be responsible for the spread of its mechanical responses. These uncertainties can be considered at different levels, namely from the composite micro to macro scale. With the present work, focused at the macroscopic average properties/characteristics, it is intended to understand which properties are the ones with a predominant influence on the response variability. In this work, the analysis was limited to the maximum transverse deflection of a FGM plate. The evaluation of this physical quantity is carried out through finite element analyses, which consider quadratic quadrilateral plate elements based on Lagrange and Kriging interpolation functions.

To enable the identification of the most significant parameters on the description at a great extent of the static response of the FGM plate, one uses a multiple linear regression model. This approach yields statistical evidence that it could not be rejected the hypothesis that the formulation of the finite elements based on both Lagrange and Kriging interpolation produce quite similar results. It has demonstrated that the variability of the maximum transverse deflection can be described by only two parameters, if their interaction is considered, with an explanation around 90%, without violating any model assumption. These parameters are the thickness and the exponent of the rule of mixtures that describe the incorporation of the reinforcement particles in the FGM structure. Indeed, given these results, one may conclude that the maximum transverse deflection of a FGM plate is more sensitive to changes on geometric properties than on mechanical properties, regarding the considered variables. In this sense, preliminary studies show that, if the geometric properties are not randomised, the variability on the predicted deflection is due to the variability on the mechanical properties of the involved materials.

The presented approach can be applied to any structure, allowing to identify the contribution of each modelling parameters to the variability of the predicted responses. This study can be extended by including the analysis of the predicted natural frequencies. Note that one possible application of this approach is related to the parameter selection, regarding stochastic finite element analysis.

ACKNOWLEDGEMENTS

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