Time integration of multibody systems using nonlinear domain decomposition techniques with mixed interface conditions

Eva-Maria Dewes[1] and Daniel J. Rixen[1]

[1] *Chair of Applied Mechanics, Technical University of Munich, {eva-maria.dewes,rixen}@tum.de*

In engineering applications, one can often assume that the deformations within flexible bodies of a multibody system remain small, but the overall motion of the system, which is typically described by a co-rotated floating frame, is highly non-linear. The application of domain decomposition methods to this problem class seems natural, since each body can be considered as a subdomain. The challenge is then to solve the global non-linear system efficiently during time integration.

To decompose the global system, one has mainly three possible choices to impose the boundary conditions or interface transmission: via prescribing the boundary degrees of freedom (dof) leading to a primal formulation (primal Schur complement methods), or prescribing the interface forces, which leads to a dual formulation (dual Schur complement methods such as the Finite Element Tearing and Interconnecting (FETI) method[1]). A third possible formulation is achieved via a combination of the two former approaches, leading to mixed or so called Robin interface conditions and thus to mixed Schur complement methods, on which we want to concentrate in this contribution.

In the context of multibody dynamics, this means we need to formulate Robin interface condtions to ensure the nonlinear kinematic constraints of the mechanical system. Hence, we have to find a linear combination of compatibility conditions, in which an additional optimizable interface parameter appears which is of the magnitude of a stiffness matrix and holds information from the neighbouring subdomains. With this, we are able to introduce additional information about the overall system behaviour into the local subdomain problems. Especially when solving the subdomain problems in parallel with a nonlinear system solver (i.e. independant of each other) we aim to accelerate convergence significantly via additional information about the neighbouring subdomains.

In this contribution, we introduce a new variant of our locally nonlinear FETI integrator where we formulate the mechanical problem with Robin interface conditions resulting in a mixed dual Schur complement method for the solution of the Newton problems in each timestep. We use an academical test model to compare the new mixed formulation with the dual one and validate our results numerically.

We consider mechanical systems with equations of motion represented by a system of differential-algebraic equations (DAE) in index-3-formulation

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{f}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{B}(\mathbf{q})^T\boldsymbol{\lambda} = 0 \tag{1}$$

$$\mathbf{g}(\mathbf{q},t) = 0 \tag{2}$$

with dynamic equations (1) and kinematic constraints (2). The vector $\mathbf{q}$ collects the generalized coordinates of the system and the vector of Lagrange multipliers $\boldsymbol{\lambda}$ represents the constraint forces to enforce the nonlinear constraint equations (2). $\mathbf{B}(\mathbf{q}) := \frac{\partial \mathbf{g}(\mathbf{q})}{\partial \mathbf{q}}$ denotes the non-constant Jacobian matrix and $\mathbf{M}$ denotes the symmetric but not necessarily constant mass matrix.

Applying the generalized-α time integration scheme [2] leads to the following linear system, where iteration indices n and k correspond to the time and Newton increments, respectively

$$\begin{bmatrix} \tilde{\mathbf{A}}^k_{n+1} & \mathbf{B}^T(\mathbf{q}^k_{n+1}) \\ \mathbf{B}(\mathbf{q}^k_{n+1}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta q^k \\ \Delta \lambda^k \end{bmatrix} = \begin{bmatrix} -\mathbf{r}^k_{\mathrm{eq},n+1} \\ -\mathbf{r}^k_{\mathrm{cons},n+1} \end{bmatrix}, \quad \tilde{\mathbf{A}}^k_{n+1} := \tilde{\beta}\mathbf{M}^k_{n+1} + \tilde{\gamma}\mathbf{C}^k_{n+1} + \mathbf{K}^k_{n+1}. \tag{3}$$

with scaled time stepping matrix $\tilde{\mathbf{A}}^k_{n+1}$, tangent stiffness matrix $\mathbf{K}^k_{n+1}$ and tangent damping matrix $\mathbf{C}^k_{n+1}$. For the choice of parameters $\tilde{\beta}, \tilde{\gamma}$ see [3].

A decomposition of the global system (1),(2) into N_s substructures is achieved by partitioning the submatrices of system(3) as follows

$$\tilde{\mathbf{A}} := \mathrm{diag}(\mathbf{A}^{(1)},\dots,\mathbf{A}^{(N_s)}), \quad \mathbf{B} := (\mathbf{B}^{(1)}|\dots|\mathbf{B}^{(N_s)}), \quad \Delta\boldsymbol{q} := (\mathbf{q}^{(1)},\dots,\mathbf{q}^{(N_s)})^T, \quad \mathbf{r}_{\mathrm{eq}} := (\mathbf{r}_{\mathrm{eq}}^{(1)},\dots,\mathbf{r}_{\mathrm{eq}}^{(N_s)})^T. \tag{4}$$

Using a dual formualtion, here the FETI method, leads to the following condensed interface problem on the Newton updates of the reaction forces $\Delta\boldsymbol{\lambda}$, which is solved iteratively via a preconditioned Conjugate Gradient method

$$\mathbf{F}_I\Delta\boldsymbol{\lambda} = \mathbf{d} \qquad \text{with} \quad \mathbf{F}_I := \mathbf{B}\tilde{\mathbf{A}}^{-1}\mathbf{B}^T, \quad \mathbf{d} := -\mathbf{B}\tilde{\mathbf{A}}^{-1}\mathbf{r}_{\mathrm{eq}} + \mathbf{r}_{\mathrm{cons}}. \tag{5}$$

Once a new $\Delta\boldsymbol{\lambda}$ is found, the Newton corrections $\Delta\boldsymbol{q}$ are computed locally i.e. independently on subdomain level.

To formulate the problem with Robin interface conditions, define new interface variables $\boldsymbol{\mu}_\Gamma^{(s)}$ and formulate the local equations for each subdomain seperately

$$\begin{bmatrix} \mathbf{A}_{ii}^{(s)} & \mathbf{A}_{i\Gamma}^{(s)} \\ \mathbf{A}_{\Gamma i}^{(s)} & \mathbf{A}_{\Gamma\Gamma}^{(s)} + \mathbf{Q}_{\Gamma\Gamma}^{(s)} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{q}_i^{(s)} \\ \Delta\boldsymbol{q}_\Gamma^{(s)} \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_{\mathrm{eq},i}^{(s)} \\ -\mathbf{r}_{\mathrm{eq},\Gamma}^{(s)} + \boldsymbol{\mu}_\Gamma^{(s)} \end{bmatrix}, \qquad \begin{matrix} \boldsymbol{\mu}_\Gamma^{(s)} & := -\mathbf{b}^{(s)^T}\Delta\boldsymbol{\lambda} + \mathbf{Q}_{\Gamma\Gamma}^{(s)}\Delta\boldsymbol{q}_\Gamma^{(s)} \\ \mathbf{b}^{(s)} & := \mathbf{B}^{(s)}\big|_\Gamma \end{matrix} \tag{6}$$

Condensation on the interface gives an explicit equation for the local boundary dof $\Delta\boldsymbol{q}_\Gamma^{(s)}$

$$\left(\mathbf{S}_{\Gamma\Gamma}^{(s)} + \mathbf{Q}_{\Gamma\Gamma}^{(s)}\right)\Delta\boldsymbol{q}_\Gamma^{(s)} = -\tilde{\mathbf{r}}_\Gamma^{(s)} + \boldsymbol{\mu}_\Gamma^{(s)}, \quad \tilde{\mathbf{r}}_\Gamma^{(s)} := \mathbf{r}_{\mathrm{eq},\Gamma}^{(s)} + \mathbf{A}_{\Gamma i}^{(s)}\mathbf{A}_{ii}^{(s)^{-1}}\mathbf{r}_{\mathrm{eq},i}^{(s)}, \quad \mathbf{S}_{\Gamma\Gamma}^{(s)} \text{ the Schur complements} \tag{7}$$

To achieve global consistency, the two constraint equations of equilibrium of constraint forces and interface compatibility have to be fulfilled, for simplicity $N_s = 2$

$$1. \quad \mathbf{b}^{(1)}\Delta\boldsymbol{q}_\Gamma^{(1)} + \mathbf{b}^{(2)}\Delta\boldsymbol{q}_\Gamma^{(2)} = -\mathbf{r}_{\mathrm{cons}} \qquad\qquad 2. \quad \mathbf{b}^{(1)^T}\Delta\boldsymbol{\lambda} + \mathbf{b}^{(2)^T}\Delta\boldsymbol{\lambda} = \mathbf{0} \tag{8}$$

$$\Leftrightarrow \mathbf{Q}_{\Gamma\Gamma}^{(1)}\Delta\boldsymbol{q}_\Gamma^{(1)} - \mathbf{Q}_{\Gamma\Gamma}^{(2)}\Delta\boldsymbol{q}_\Gamma^{(2)} - \boldsymbol{\mu}_\Gamma^{(1)} + \boldsymbol{\mu}_\Gamma^{(2)} = \mathbf{0} \tag{9}$$

Reformulating equations (8),(9) gives together with equation (7) for $\Delta\boldsymbol{q}_\Gamma^{(s)}$ a new interface problem in mixed form, which only depends on the new interface variables $\boldsymbol{\mu}_\Gamma^{(s)}$

$$\begin{bmatrix} \mathbf{I} & -\mathbf{I} + \left(\mathbf{Q}_{\Gamma\Gamma}^{(1)}\mathbf{b}^{(1)^{-1}}\mathbf{b}^{(2)} + \mathbf{Q}_{\Gamma\Gamma}^{(2)}\right)\left(\mathbf{S}_{\Gamma\Gamma}^{(2)} + \mathbf{Q}_{\Gamma\Gamma}^{(2)}\right)^{-1} \\ \mathbf{I} - \left(\mathbf{Q}_{\Gamma\Gamma}^{(1)} + \mathbf{Q}_{\Gamma\Gamma}^{(2)}\mathbf{b}^{(2)^{-1}}\mathbf{b}^{(1)}\right)\left(\mathbf{S}_{\Gamma\Gamma}^{(1)} + \mathbf{Q}_{\Gamma\Gamma}^{(1)}\right)^{-1} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_\Gamma^{(1)} \\ \boldsymbol{\mu}_\Gamma^{(2)} \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} \left(\mathbf{Q}_{\Gamma\Gamma}^{(1)}\mathbf{b}^{(1)^{-1}}\mathbf{b}^{(2)} + \mathbf{Q}_{\Gamma\Gamma}^{(2)}\right)\left(\mathbf{S}_{\Gamma\Gamma}^{(2)} + \mathbf{Q}_{\Gamma\Gamma}^{(2)}\right)^{-1}\tilde{\mathbf{r}}_\Gamma^{(2)} - \mathbf{Q}_{\Gamma\Gamma}^{(1)}\mathbf{b}^{(1)^{-1}}\mathbf{r}_{\mathrm{cons}} \\ -\left(\mathbf{Q}_{\Gamma\Gamma}^{(1)} + \mathbf{Q}_{\Gamma\Gamma}^{(2)}\mathbf{b}^{(2)^{-1}}\mathbf{b}^{(1)}\right)\left(\mathbf{S}_{\Gamma\Gamma}^{(1)} + \mathbf{Q}_{\Gamma\Gamma}^{(1)}\right)^{-1}\tilde{\mathbf{r}}_\Gamma^{(1)} + \mathbf{Q}_{\Gamma\Gamma}^{(2)}\mathbf{b}^{(2)^{-1}}\mathbf{r}_{\mathrm{cons}} \end{bmatrix}. \tag{11}$$

The challenge is now to choose the interface impedance $\mathbf{Q}_{\Gamma\Gamma}$ appropriately in order to optimize and accelerate convergence of the local iterations in each subdomain. This is not a straightforward task, since it should approximate the Schur complements of the neighbouring subdomains good enough without beeing too costly to compute (the Schur complement of the remainder of the whole system would give the solution immediately). A simple approximation is achieved via a lumped approach, that is taking only into account the parts of the stepping matrices which correspond to the boundary dof of the neighbouring subdomains.

References

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