Higher – order Rodrigues dual vectors. Kinematic equations and tangent operator

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The special Euclidean displacement Lie group $\mathrm{S}\mathbb{E}_3$ is isomorfic with the Lie group of orthogonal dual tensors $\underline{S\mathbb{O}}_3$[4], [5]. Forwards, we will denote with $\underline{\mathbb{R}}$ the set of dual numbers, with $\underline{\boldsymbol{V}}_3$ the set of dual vectors, with $\mathbf{L}(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3)$ the set dual tensors. We will write a generic element from $\underline{S\mathbb{O}}_3 = \{\underline{\boldsymbol{R}} \in \mathbf{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3) | \underline{\boldsymbol{R}\boldsymbol{R}}^T = \underline{\boldsymbol{I}}, \det\underline{\boldsymbol{R}} = 1\}$ as:

$$\underline{\boldsymbol{R}} = (\mathbf{I} + \varepsilon\widetilde{\boldsymbol{\rho}})\boldsymbol{R}, \varepsilon^2 = 0, \varepsilon \neq 0 \tag{1}$$

where $\boldsymbol{\rho}$ is the translation vector of the rigid displacement ($\widetilde{\boldsymbol{\rho}}$ denotes the skew-symetric dual tensor associated to the dual vector $\boldsymbol{\rho}$) and $\boldsymbol{R} \in \mathrm{S}\mathbb{O}_3$ the rotation tensor. Also we will write:

$$\underline{\boldsymbol{R}}(\underline{\alpha}, \underline{\mathbf{u}}) = \underline{\boldsymbol{I}} + \sin\underline{\alpha}\underline{\widetilde{\mathbf{u}}} + (1 - \cos\underline{\alpha})\underline{\widetilde{\mathbf{u}}}^2 = \exp(\underline{\alpha}\underline{\widetilde{\mathbf{u}}}) \tag{2}$$

where the unit dual vector $\underline{\mathbf{u}}$ designate the screw-axis of the rigid displacement and $\underline{\alpha} = \alpha + \varepsilon d$ is the corresponding dual angle [1], [3], [6]. As it was proved in the previous works of the author [5], using the n-th order Cayley transform:

$$cay_n(\quad): \underline{V}_3 \to \underline{S\mathbb{O}}_3, cay_n(\underline{\mathbf{v}}) = (\underline{\boldsymbol{I}} + \underline{\widetilde{\mathbf{v}}})^n (\underline{\boldsymbol{I}} - \underline{\widetilde{\mathbf{v}}})^{-n}, \forall n \in \mathbb{N}^* \tag{3}$$

a minimal vectorial parameterization for $\mathrm{S}\mathbb{E}_3$ can be given. The inverse transformation of cay_n is a multifunction with n branches given by the below equation [5]:

$$cay_n{}^{-1}(\quad): \underline{S\mathbb{O}}_3 \to \underline{\boldsymbol{V}}_3, \; cay_n{}^{-1}[R(\underline{\alpha}, \underline{\mathbf{u}})] = \left(\tan\frac{\underline{\alpha} + 2k\pi}{2n}\right)\underline{\mathbf{u}}, k = \overline{0, n-1} \tag{4}$$

The vectors denoted with $\mathbf{v}_\mathrm{k} = \left(\tan\frac{\underline{\alpha}+2k\pi}{2n}\right)\underline{\mathbf{u}}$, with $k = \overline{0, n-1}$ represents a minimal parameterization of the rigid displacent and will be named **higher-order Rodrigues vectors**. If $\underline{\mathbf{v}}$ is one of the Rodrigues dual vectors, then the dual orthogonal tensors that corespons to the rigid displacement can be computed using the below equation [5].

$$\underline{\boldsymbol{R}} = \underline{\boldsymbol{I}} + \frac{2p_n(|\underline{\mathbf{v}}|)q_n(\underline{\mathbf{v}})}{\left(1+|\underline{\mathbf{v}}|^2\right)^n}\underline{\widetilde{\mathbf{v}}} + \frac{2q_n^2(|\underline{\mathbf{v}}|)}{\left(1+|\underline{\mathbf{v}}|^2\right)^n}\underline{\widetilde{\mathbf{v}}}^2 \tag{5}$$

where p_n and q_n are the polynomials given by the following equations:

$$p_n(X) = \sum_{k=0}^{[n/2]}(-1)^k \binom{2k}{n} X^{2k} \tag{6}$$

$$q_n(X) = \sum_{k=0}^{[(n-1)/2]}(-1)^k \binom{2k+1}{n} X^{2k} \tag{7}$$

In the case of a rigid body motion given by a curve $\underline{\mathbf{R}} = \underline{\mathbf{R}}(t) \in \mathrm{S}\mathbb{O}_3, \forall t \in \mathbb{R}$, which poses the problem of Rodrigues vector recovery, knowing the dual angular velocity vector (dual twist) in space $\underline{\boldsymbol{\omega}}$ or body frame $\underline{\boldsymbol{\omega}}^\mathrm{B}$, we have the follwing result:

Theorem: *There are* $\underline{\boldsymbol{S}}$ *and* $\underline{\boldsymbol{T}}$ *two dual non-singular tensors so as:*

$$\begin{cases} \dot{\underline{\mathbf{v}}} = \underline{\boldsymbol{S}}\underline{\boldsymbol{\omega}} \\ \dot{\underline{\mathbf{v}}} = \underline{\boldsymbol{S}}^{\mathrm{T}}\underline{\boldsymbol{\omega}}^{\mathrm{B}} \end{cases} \tag{8}$$

$$\begin{cases} \underline{\boldsymbol{\omega}} = \underline{\boldsymbol{T}}\dot{\underline{\mathbf{v}}} \\ \underline{\boldsymbol{\omega}}^{\mathrm{B}} = \underline{\boldsymbol{T}}^{\mathrm{T}}\dot{\underline{\mathbf{v}}} \end{cases}. \tag{9}$$

The following equations will give the tensors $\underline{\boldsymbol{S}}$ *and* $\underline{\boldsymbol{T}}$:

$$\begin{gathered} \underline{\boldsymbol{S}} = \frac{p_n(|\underline{\mathbf{v}}|)}{2q_n(\underline{\mathbf{v}})}\underline{\boldsymbol{I}} - \frac{1}{2}\underline{\tilde{\mathbf{v}}} + \frac{\left(1+|\underline{\mathbf{v}}|^2\right)q_n(|\underline{\mathbf{v}}|) - np_n(|\underline{\mathbf{v}}|)}{2n|\underline{\mathbf{v}}|^2 q_n(\underline{\mathbf{v}})}\underline{\mathbf{v}} \otimes \underline{\mathbf{v}} \\ \underline{\boldsymbol{T}} = \frac{2q_n(|\underline{\mathbf{v}}|)p_n(|\underline{\mathbf{v}}|)}{\left(1+|\underline{\mathbf{v}}|^2\right)^n}\underline{\boldsymbol{I}} + \frac{2q_n^2(|\underline{\mathbf{v}}|)}{\left(1+|\underline{\mathbf{v}}|^2\right)^n}\underline{\tilde{\mathbf{v}}} + 2\frac{n\left(1+|\underline{\mathbf{v}}|^2\right)^{n-1} - q_n(|\underline{\mathbf{v}}|)p_n(|\underline{\mathbf{v}}|)}{|\underline{\mathbf{v}}|^2\left(1+|\underline{\mathbf{v}}|^2\right)^n}\underline{\mathbf{v}} \otimes \underline{\mathbf{v}} \end{gathered} \tag{10}$$

In Eq. (10), we've noted the dual tensor product of two vectors with $\underline{\mathbf{v}} \otimes \underline{\mathbf{v}}$.
From **the Rodrigues dual vector parametrization** $\underline{\mathbf{v}} = \mathrm{tg}\frac{\underline{\alpha}}{2}\underline{\mathbf{u}}$, $n = 1$, we will have

$$\begin{gathered} \boldsymbol{S} = \frac{1}{2}\underline{\boldsymbol{I}} - \frac{1}{2}\underline{\tilde{\mathbf{v}}} + \frac{1}{2}\underline{\mathbf{v}} \otimes \underline{\mathbf{v}} \\ \boldsymbol{T} = \frac{2}{1+|\underline{\mathbf{v}}|^2}\left[\underline{\boldsymbol{I}} + \underline{\tilde{\mathbf{v}}}\right] \end{gathered} \tag{11}$$

From **the Wiener-Milenkovic dual vector parametrization** $\underline{\mathbf{v}} = \mathrm{tg}\frac{\underline{\alpha}}{4}\underline{\mathbf{u}}$ $\left(\underline{\mathbf{v}_{\boldsymbol{S}}} = -\mathrm{ctg}\frac{\underline{\alpha}}{4}\underline{\mathbf{u}}\right)$, $n = 2$, we will obtain:

$$\begin{gathered} \boldsymbol{S} = \frac{1-|\underline{\mathbf{v}}|^2}{2}\underline{\boldsymbol{I}} - \frac{1}{2}\underline{\tilde{\mathbf{v}}} + \frac{1}{2}\underline{\mathbf{v}} \otimes \underline{\mathbf{v}} \\ \boldsymbol{T} = \frac{4}{\left(1+|\underline{\mathbf{v}}|^2\right)^2}\left[\left(1 - |\underline{\mathbf{v}}|^2\right)\boldsymbol{I} + 2\underline{\tilde{\mathbf{v}}} + 2\underline{\mathbf{v}} \otimes \underline{\mathbf{v}}\right] \end{gathered} \tag{12}$$

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