Measuring the Error in Mixed Linear Complementarity Problem Formulations for Multibody Systems with Contact

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Contact in multibody systems can be modelled through unilateral constraints. This often results in a dynamics formulation that can mathematically be stated as a *linear complementarity problem* (LCP) or *mixed linear complementarity problem* (MLCP). There is a wide range of algorithms in the literature to solve LCPs and MLCPs, which can be classified into two types: direct and iterative methods [1]. Iterative solvers are computationally efficient but not very accurate, whereas direct solvers give accurate solutions at a relatively high computational cost.

Choosing the solver that suits the problem at hand best is not an easy task, especially when there is a time constraint and a solution with the desired tolerance cannot be found. In this case, the simulation accuracy can be increased significantly if we measure the error of all computed solutions at algorithm runtime and choose the solution with the least error. In numerical optimization, the *natural residual* [2] is defined as a measure of closeness of the computed result to being a solution of the complementarity problem. Moreover, the Fischer-Burmeister merit function can be used to capture the error in dynamics simulations [3]. However, all these functions do not take into account the different physical nature of impulses and velocities and thus can suffer from unit inconsistency. In this paper, we present a unit-consistent energy-like error metric for MLCP solvers that does not require any reference solution and can be computed online in order to improve simulation accuracy.

Let us consider a multibody system with generalized velocities $\mathbf{v} \in \mathbb{R}^m$ and transformation $\mathbf{J}\mathbf{v} = \mathbf{w}$ that defines the constraint subspace where $\mathbf{w} \in \mathbb{R}^n$ represents the velocities in that subspace and $\mathbf{J} \in \mathbb{R}^{n\times m}$ is the constraint Jacobian. The dynamic equations using a finite difference approximation for the accelerations can be written as

$$\begin{bmatrix} \mathbf{M} & -\mathbf{J}^{\mathrm{T}} \\ \mathbf{J} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{v}^+ \\ h\boldsymbol{\lambda}^+ \end{bmatrix} + \begin{bmatrix} \mathbf{p} \\ \frac{1}{h}\boldsymbol{\Phi} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{w} \end{bmatrix} \tag{1}$$

where h is the time step size, $\mathbf{v}^+$ are the unknown velocities at the end of the step, $\mathbf{v}$ are the known velocities at the beginning of the step, $\mathbf{M}$ is the mass matrix, and $\mathbf{p} = \mathbf{M}\mathbf{v} + h\mathbf{f}$ depends on the generalized applied forces $\mathbf{f}$. The values of the generalized constraint forces $\boldsymbol{\lambda}^+$ are specified by the nature of the constraint (e.g., $\lambda_{\mathrm{n}}^+ \geq 0$ for a normal contact force, and $\lambda_{\mathrm{t}}^+ \in [-\mu\lambda_{\mathrm{n}}, +\mu\lambda_{\mathrm{n}}]$ for a friction force component if the box friction approximation is used). Moreover, the constraints can be regularized, which introduces the compliance matrix $\mathbf{C}$ and the constraint violations in $\boldsymbol{\Phi}$. The general form of the MLCP that needs to be solved at each step is

$$\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{w} \tag{2}$$

$$\left.\begin{aligned} \mathbf{0} \leq \mathbf{u} - \mathbf{x} \quad &\perp \quad \mathbf{w}_{\mathrm{u}} \geq \mathbf{0} \\ \mathbf{0} \leq \mathbf{x} - \boldsymbol{l} \quad &\perp \quad \mathbf{w}_{\mathrm{l}} \geq \mathbf{0} \end{aligned}\right\} \tag{3}$$

where $\mathbf{A} = \mathbf{J}\mathbf{M}^{-1}\mathbf{J}^{\mathrm{T}} + \mathbf{C}$ is the lead matrix, $\mathbf{b} = \mathbf{J}\mathbf{M}^{-1}\mathbf{p} + \frac{1}{h}\boldsymbol{\Phi}$, and the variables $\mathbf{x} = h\boldsymbol{\lambda}^+ \in [\boldsymbol{l}, \mathbf{u}]$ are the generalized constraint force impulses subject to lower and upper bounds $\boldsymbol{l}$ and $\mathbf{u}$. The non-negative components of the constraint-space velocity $\mathbf{w} = \mathbf{w}_{\mathrm{l}} - \mathbf{w}_{\mathrm{u}}$ are complementary to the lower and upper bounds, denoted by operator $\perp$.

In order to solve the MLCP, the algorithm performs a series of iterations and computes some intermediate solutions, which satisfy Eq. (2) so that error occurs only in the conditions in Eq. (3). To compute the error, we approximate the lead matrix $\mathbf{A}$ by its diagonal matrix $\tilde{\mathbf{A}} = \mathrm{diag}\{a_{11}, \ldots, a_{nn}\}$ to break down Eq. (2) into scalar equations $a_{ii}x_i + b_i = w_i$ with two unknowns each, x_i and w_i. This is equivalent to not considering any coupling effects between constraints in order to analyze each constraint i separately and compute the error δe_i associated with it. Figure 1 illustrates the isolines of the error in the plane $x - w$.

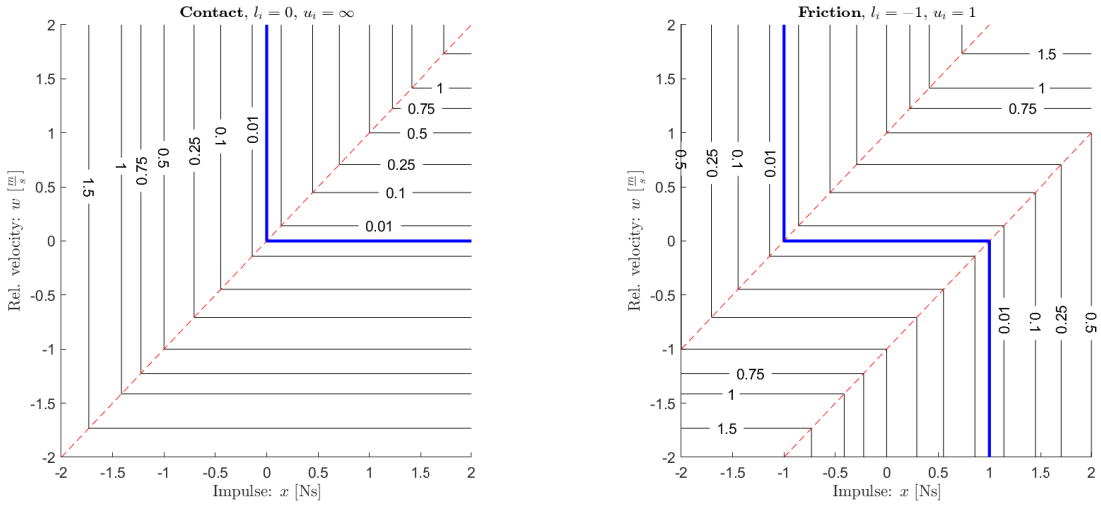

Fig. 1: LCP error isolines for normal (left) and friction (right) force constraints for unit effective mass and compliance $a_{ii} = 1$. Zero error for the feasible set of solutions (solid blue line) and quadratically increasing error along the dashed lines.

We decompose the computed constraint reaction impulse x_i and constraint-space velocity w_i per constraint into

$$x_i = x_{0,i} + \delta x_{u,i} - \delta x_{l,i} \quad \text{and} \quad w_i = w_{l,i} - w_{u,i}, \tag{4}$$

where $x_{0,i} = \max\big(\min\left(x_i, u_i\right), l_i\big)$, i.e. clamped to the upper and lower bound u_i and l_i. One of the impulse errors $\delta x_{u,i} \geq 0$ or $\delta x_{l,i} \geq 0$ is non-zero if x_i exceeds u_i or l_i, respectively. The constraint-space velocity is split into non-negative components $w_{l,i} \geq 0$, and $w_{u,i} \geq 0$, one of which must always be zero. Given $\sigma_{l,i} = (x_{0,i} + \delta x_{u,i}) - l_i$ and $\sigma_{u,i} = u_i - (x_{0,i} + \delta x_{l,i})$, we can define the error per constraint as

$$\delta e_i = \max\left(\tfrac{1}{2} a_{ii} \left(\delta x_{u,i}^2 + \delta x_{l,i}^2\right) \;,\; \min\left(\tfrac{1}{2a_{ii}} w_{l,i}^2, \tfrac{1}{2} a_{ii} \sigma_{l,i}^2\right) \;,\; \min\left(\tfrac{1}{2a_{ii}} w_{u,i}^2, \tfrac{1}{2} a_{ii} \sigma_{u,i}^2\right) \right). \tag{5}$$

This definition of the error takes the inverse effective mass and compliance a_{ii} associated with each constraint to compute the energy error. This allows us to combine the individual constraint error to establish a single energy-like error for the whole system at time t_k by taking the ℓ_1-norm of the component vector

$$\delta e_k = \|\delta \mathbf{e}_k\|_1 = \left\| \left[\delta e_1 \;\ldots\; \delta e_n\right]_k^{\mathrm{T}} \right\|_1 = \left(\sum_{i=1}^{n} |\delta e_i| \right)_k . \tag{6}$$

It can be easily seen that the unit-consistent MLCP metric can reach zero error if and only if a valid solution to the MLCP is found. This unit-consistent metric does not require any reference solution, which can be very expensive to compute for large-scale systems. Simulations were performed to analyze the error of direct and iterative solvers. The results show the expected different behaviours of each solver. Moreover, the proposed unit-consistent error metric agrees qualitatively with existing error functions which mix impulses and velocities [2, 3]. The error metric is inexpensive to compute. This can be done at runtime without compromising the performance noticeably. If the solver does not terminate due to an iteration limit, it is possible to determine the solution with the least error which improves the numerical stability and reduces the risk of simulation instabilities or "blow-ups".

References

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