Time optimal control of multibody systems

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In the field of race car engineering the performance of a mechanical system is typically measured by the duration of a driving maneuver, which is closely related to time optimal control. An approach often used for solving optimal control problems is based on an adjoint gradient computation of the cost function which has to be minimized. Using the gradient information, a (local) minimum can be found by applying appropriate optimization techniques, such as line search algorithms. This approach has the advantage that it is more robust than solving the underlying two point boundary value problem and, hence, applicable also to complex multibody systems described by differential algebraic equations. In this contribution the approach for fixed final time presented in [1] is extended by terms needed to solve problems with variable final time. To illustrate the problem description and the proposed solution strategy for this case, we consider the dynamical system of a simplified vehicle model having the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1}$$

where $\mathbf{x}(t)$ denotes the vector of state variables and $\mathbf{u}(t)$ the vector of control inputs. The extension to differential algebraic systems is straightforward and shown in [1].

The final time t_1 of a simulation run is reached, when the states satisfy a scalar equation of the form

$$\Phi(\mathbf{x}_1) = 0, \quad \text{where} \quad \mathbf{x}_1 = \mathbf{x}(t_1). \tag{2}$$

Recalling the racing car example, this condition may describe the crossing of the race track's finish line. Note, that this is only one form of a time optimal control problem. In other problems, the final state may be either fully prescribed (such as in robotics) or even completely free .

Now, the goal is to find control inputs $\mathbf{u}(t)$ restricted by $0 \le \mathbf{u}_i(t) \le \bar{\mathbf{u}}_i$ which minimize the cost function

$$J = \int_{t_0}^{t_1} (1 + p(\mathbf{x}))\, \mathrm{d}t$$

where $p(\mathbf{x})$ is a penalty function in order to introduce some state constraints. Without $p(\mathbf{x})$, the cost function is simply the length of the time interval $t_1 - t_0$.

From Pontryagin's minimum principle [2] follows, that – in the non-singular case – the optimal control for bounded inputs, which appear linear in the Hamiltonian, is a bang-bang control, where only the switching points are unknown. In the vehicle dynamics example, this may be the case for the accelerator and the brake input. However, also some control inputs can appear, which are not of the bang-bang type (like the steering in the vehicle example).

In order to apply a solution strategy for our time optimal control problem similar to the method proposed in [1], we try to compute the gradient of the cost function by first expanding it with Eq. (1). For arbitrary adjoint variables $\mathbf{y}(t)$, the cost function does not change if we augment it in the following way:

$$J = \int_{t_0}^{t_1} \Big(1 + p(\mathbf{x}) + \mathbf{y}^\mathsf{T} (\mathbf{f}(\mathbf{x},\mathbf{u}) - \dot{\mathbf{x}})\Big)\, \mathrm{d}t. \tag{3}$$

We now consider an infinitesimal variation of the control inputs $\delta\mathbf{u}(t)$ causing an infinitesimal change of the states $\delta\mathbf{x}(t)$ and of the final time δt_1. Hence, the variation of the cost function Eq. (3) reads

$$\delta J = \int_{t_0}^{t_1} \Big\{ p_\mathbf{x}^\mathsf{T} \delta\mathbf{x} + \mathbf{y}^\mathsf{T} (\mathbf{f}_\mathbf{x} \delta\mathbf{x} + \mathbf{f}_\mathbf{u} \delta\mathbf{u} - \delta\dot{\mathbf{x}}) \Big\}\, \mathrm{d}t + \{1 + p(\mathbf{x}(t_1))\}\, \delta t_1. \tag{4}$$

Since the final state must satisfy Eq. (2), δt_1 and $\delta \mathbf{x}(t_1)$ cannot be chosen independently. Up to first order, the variation of the final state at t_1 is given by

$$\delta \mathbf{x}_1 = \mathbf{x}(t_1+\delta t_1) + \delta \mathbf{x}(t_1+\delta t_1) - \mathbf{x}(t_1) \approx \mathbf{x}(t_1) + \dot{\mathbf{x}}(t_1)\delta t_1 + \delta \mathbf{x}(t_1) - \mathbf{x}(t_1) = \dot{\mathbf{x}}(t_1)\delta t_1 + \delta \mathbf{x}(t_1).$$

Using this relation, the variation of Eq. (2) yields

$$\Phi_x^\mathsf{T}(\mathbf{x}_1)\delta \mathbf{x}_1 = \Phi_\mathbf{x}^\mathsf{T}(\mathbf{x}_1)\left(\dot{\mathbf{x}}_1\delta t_1 + \delta \mathbf{x}(t_1)\right) = 0 \quad \Rightarrow \quad \delta t_1 = -\frac{\Phi_\mathbf{x}^\mathsf{T}(\mathbf{x}_1)}{\Phi_\mathbf{x}^\mathsf{T}(\mathbf{x}_1)\dot{\mathbf{x}}_1}\delta \mathbf{x}(t_1) \tag{5}$$

using the abbreviations $\mathbf{x}_1$ and $\dot{\mathbf{x}}_1$ instead of $\mathbf{x}(t_1)$ and $\dot{\mathbf{x}}(t_1)$. After integration by parts of the term $\mathbf{y}^\mathsf{T}\delta\dot{\mathbf{x}}$ and inserting Eq. (5), Eq. (4) becomes

$$\delta J = \int_{t_0}^{t_1} \left\{ \mathbf{y}^\mathsf{T}\mathbf{f}_\mathbf{u}\delta\mathbf{u} + \left(p_\mathbf{x}^\mathsf{T} + \mathbf{y}^\mathsf{T}\mathbf{f}_\mathbf{x} + \dot{\mathbf{y}}^\mathsf{T}\right)\delta\mathbf{x} \right\} \mathrm{d}t - \left\{ \mathbf{y}_1^\mathsf{T} + \Phi_\mathbf{x}^\mathsf{T}(\mathbf{x}_1)\frac{1+p(\mathbf{x}_1)}{\Phi_\mathbf{x}^\mathsf{T}(\mathbf{x}_1)\dot{\mathbf{x}}_1} \right\} \delta\mathbf{x}(t_1).$$

If we now choose $\mathbf{y}(t)$ such that

$$\dot{\mathbf{y}} = -\mathbf{f}_\mathbf{x}^\mathsf{T}\mathbf{y} - p_\mathbf{x}, \quad \mathbf{y}(t_1) = -\Phi_\mathbf{x}(\mathbf{x}_1)\frac{1+p(\mathbf{x}_1)}{\Phi_\mathbf{x}^\mathsf{T}(\mathbf{x}_1)\dot{\mathbf{x}}_1} \tag{6}$$

the first variation of the cost function reduces to

$$\delta J = \int_{t_0}^{t_1} \mathbf{y}^\mathsf{T}\mathbf{f}_\mathbf{u}\delta\mathbf{u}\,\mathrm{d}t = \sum_{i=1}^{m}\int_{t_0}^{t_1} g_i(t)\delta u_i(t)\,\mathrm{d}t, \tag{7}$$

where $g_i(t) := \mathbf{y}^\mathsf{T}\mathbf{f}_{u_i}$ is used with u_i being the i-th component of $\mathbf{u}$. After solving the adjoint differential equation Eq. (6) backwards in time, an update of $\mathbf{u}$ may be computed.
If $u_i(t)$ is a continuous control input, the largest (local) decrease of J is obtained by setting $\delta u_i(t) = -\kappa g_i(t)$, if κ is a sufficiently small and positive number.
In case of $u_i(t)$ being a bang-bang control input, $\delta u_i(t)$ results from a variation of the switching times and the variation yielding the steepest descent of J can also be derived from Eq. (7). First, we introduce the switching points $\tau_1 \dots \tau_N$ to be identified. Moreover, we assume that either $u_i = 0$ or $u_i = \bar{u}_i > 0$. Shifting the switching points about $\delta\tau_k$ results in a variation $\delta u_i(t)$ which is different from zero only in the interval $[\tau_k, \tau_k + \delta\tau_k]$ and given by

$$\delta u_i(t) = \begin{cases} \pm\bar{u}_i & \text{if} \quad t \in [\tau_k, \tau_k+\delta\tau_k] \\ 0 & \text{else} \end{cases}$$

The negative sign has to be taken if the control switches from zero to $\bar{u}_i$ at τ_k and the positive sign for a switch from $\bar{u}_i$ to zero. For infinitesimal small shifts $\delta t_1 \dots \delta t_N$ we obtain from Eq. (7)

$$\delta J = \sum_{i=1}^{m}\int_{t_0}^{t_1} g_i\delta u_i\,\mathrm{d}t \approx \sum_{i=1}^{m}\sum_{k=1}^{N}\left\{\pm g_i(t_k)\bar{u}_i\right\}\delta t_k = \sum_{k=1}^{N}\left\{\sum_{i=1}^{m}\left(\pm g_i(t_k)\bar{u}_i\right)\right\}\delta t_k.$$

References

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[2] D. E. Kirk, *Optimal Control Theory: An Introduction*. Dover Books on Electrical Engineering Series, Dover Publications, 2004.