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Sensitivity Analysis of Time-Delayed Systems

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ABSTRACT — *The regenerative effect in machining processes can be modeled by time-delayed differential equations. Stability analysis of these time-periodic systems can be done by Semi-Discretization resulting in an approximation of the monodromy matrix. Finally, the stability of the system can be determined if all its eigenvalues are located inside the unit circle in the complex plane. The caluclation of the stability lobe diagrams, where the stability of machining processes is depicted in dependency of operating conditions, is connected with high computational effort. Hence, we present an analytical approach for the calculation of the derivative of eigenvalues in machining systems.*

1 Introduction

Time-delayed systems are used for mathematical modeling of the surface regeneration of the workpiece which may lead to an instability called chatter [1]. For milling processes these systems are periodic and frequently in addition non-smooth. Both time-delay and periodicity of systems can cause dynamic instabilities [2]. Based on Floquet theory, the stability can be analyzed by an eigenvalue analysis of an approximated monodromy matrix by time discretization [3]. If all eigenvalues of the discrete system are located inside of the unit circle in the complex plane, the system is stable.

Stability lobe diagrams, representing the stability limit in dependency of machining parameters such as spindle speed and axial immersion and the specific cutting-force coefficient, respectively, are employed for the choice of appropriate operating points. For this purpose, the magnitude of the critical eigenvalue can be evaluated on a full grid for instance and, subsequently, the implicitly defined stability limit can be interpolated. However, this approach exhibits often not the desired efficiency.

Efficient calculation schemes for stability lobe diagrams in the literature are based on bisection [4] or curve tracking as marching squares [5], both acting on regular grids. Another class of methods represent continuation methods based on circular search [6, 7], for instance. If the time-delay system is smooth, software packages are available for continuation of time-delayed systems, see [8, 9]. These software packages include continuation methods using Newton's iteration.

The focus of our work is the application of stability analysis methods of time-delayed systems which are capable of non-smoothness as generated by milling for Newton methods in continuation approaches. Hence, we have to provide an analytical derivative of the (critical) eigenvalue of the time-discretized, time-delayed system. In literature there are a few investigations on this topic, thus, a semi-analytical approach has been presented [10], where the numerical differentiation is applied for the monodromy matrix, approximated by Time Finite Element Analysis. In [11], sensitivity analysis based on the so-called Numerical Integration Methods has been presented. Here, we investigate differentiation of Semi-Discretization and Spectral Elements [2, 12], which are frequently applied and well known methods.

2 Time-Delayed System

Milling processes are interesting time-delayed systems since there is the possibility of highly complex coupling of current and delayed states due to the regenerative effect. Hence, we focus on a milling process as an example model.

The linearized first-order dynamic system describing the perturbation of the nonlinear time-periodic DDE reads

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{x}(t-\tau) \quad . \tag{1}$$

The matrices are time-periodic, $\boldsymbol{A}(t) = \boldsymbol{A}(t+T)$ and $\boldsymbol{B}(t) = \boldsymbol{B}(t+T)$, with the periodicity correlating to the time delay $T = \tau = 60/(N\Omega)$ for symmetric tools. Spindle speed Ω is commonly given in $1/\text{min} = \text{rpm}$. Furthermore, the system matrices read

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & 1 \\ -\left(\omega_\text{n}^2 + HG(t)\right) & -2\zeta\omega_\text{n} \end{bmatrix} \quad , \qquad \boldsymbol{B}(t) = \begin{bmatrix} 0 & 0 \\ HG(t) & 0 \end{bmatrix} \quad . \tag{2}$$

The dynamic cutting-force coefficient of a straight fluted tool can be approximated according to [2] by

$$G(t) = \sum_{j=1}^{N} g_j(t) \sin^q \varphi_j(t) \left(\frac{K_\text{t}}{K_\text{r}} \cos \varphi_j(t) + \sin \varphi_j(t) \right) \quad . \tag{3}$$

Here, the natural angular frequency $\omega_\text{n} = 2\pi f_\text{n}$, the damping ratio ζ, the specific cutting-force coefficient H, the tangential and radial cutting-force coefficient $K_\text{t,r}$ and the cutting-force exponent q are taken into account. The angular position of the N teeth is denoted by $\varphi_j(t)$ and $g_j(t)$ screens cutting teeth.

In milling processes, frequently the dynamic cutting-force coefficient is not smooth since the teeth are entering and exiting the workpiece. Especially for low radial immersion and small numbers of teeth, the period can be divided in two sections $T = T_\text{d} + T_\text{o}$, the first section $t \in [0, T_\text{d}]$ where at least one tooth is cutting, thus, $G(t) \neq 0$ and the second section of the period $t \in (T_\text{d}, T]$ where no tooth is cutting, hence, $G(t) = 0$ resulting in $\boldsymbol{A}_\text{c}$. When no tooth is cutting, the DDE becomes a time-invariant ODE.

For stability analysis in this paper, we chose a two-fluted tool, thus $N = 2$, the cutting-force ratio $K_\text{t}/K_\text{r} = 1/0.3$, the cutting-force exponent $q = 3/4$ and the damping ratio $\zeta = 0.02$. Different radial immersion a_e, commonly posed as ratio with respect to the diameter of the tool D, as well as the setup of up- or down-milling affect the screen function $g_j(t)$. We choose $a_\text{e}/D = 0.9$ and down-milling. Finally, stability lobe diagrams can be depicted in the plane of the dimensionless spindle speed $\tilde{\Omega} = N\Omega/(60 f_\text{n})$ and the dimensionless specific cutting-force coefficient $\tilde{H} = H/\omega_\text{n}^2$, for details of the model see [2].

3 Stability Analysis

Stability analysis of time-delayed systems is a popular field in research. There is a magnitude of different methods for stability analysis, [3]. Two important principles have to be mentioned, stability analysis in frequency domain and by time discretization. Using time discretization, a single period of the system is discretized for the approximation of the monodromy matrix. Based on Floquet theory, the stability of the system is given if all eigenvalues of the system are positioned inside the unit circle in the complex plane.

In this paper we focus on two methods based on time-discretization, Semi-Discretization [2] and Spectral Elements [12]. To emphasize our points we state a few assumptions for simplification of the stability analysis methods. Here, milling with constant spindle speed as well as a symmetrically straight-fluted tool is considered, hence, the time-delay is constant and equals the period of the system.

3.1 Semi-Discretization Method

There are different orders of approximation in the Semi-Discretization scheme, frequently improved zeroth-order and first-order Semi-Discretization are used. Due to clarity, here shortly the calculation scheme of improved zeroth-order Semi-Discretization is presented.

Considering $T=\tau=$ const. results in $p=r$ and with direct composition the monodromy matrix can be stated as

$$\underbrace{\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{P}_{p-1} & & & & & \\ & \boldsymbol{I} & -\boldsymbol{P}_{p-2} & & & & \\ & & \ddots & \ddots & & & \\ & & & \boldsymbol{I} & -\boldsymbol{P}_1 & & \\ & & & & \boldsymbol{I} & -\boldsymbol{P}_0 & \\ & & & & & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{\Psi}} \begin{bmatrix} \boldsymbol{y}_p \\ \boldsymbol{y}_{p-1} \\ \boldsymbol{y}_{p-2} \\ \vdots \\ \boldsymbol{y}_2 \\ \boldsymbol{y}_1 \\ \boldsymbol{y}_0 \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{R}_{p-1} & \boldsymbol{R}_{p-1} & & & & & \\ & \boldsymbol{R}_{p-2} & \boldsymbol{R}_{p-2} & & & & \\ & & \ddots & & & & \\ & & & \boldsymbol{R}_1 & \boldsymbol{R}_1 & & \\ & & & & \boldsymbol{R}_0 & \boldsymbol{R}_0 & \\ \boldsymbol{I} & & & & & & \end{bmatrix}}_{\boldsymbol{\Phi}} \begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{y}_{-1} \\ \boldsymbol{y}_{-2} \\ \vdots \\ \boldsymbol{y}_{-p+2} \\ \boldsymbol{y}_{-p+1} \\ \boldsymbol{y}_{-p} \end{bmatrix} \tag{4}$$

using the approximation of p time-steps as well as $\boldsymbol{y}_0=\boldsymbol{y}_0$. In the i-th time-step the approximation reads

$$\boldsymbol{y}_{i+1} = \boldsymbol{P}_i\boldsymbol{y}_i + \boldsymbol{R}_i\left(\boldsymbol{y}_{i-r+1}+\boldsymbol{y}_{i-r}\right) \tag{5a}$$

$$\boldsymbol{P}_i = \mathrm{e}^{\boldsymbol{A}_i h} \tag{5b}$$

$$\boldsymbol{R}_i = \frac{1}{2}\left(\boldsymbol{P}_i-\boldsymbol{I}\right)\boldsymbol{A}_i^{-1}\boldsymbol{B}_i \tag{5c}$$

with the matrices $\boldsymbol{A}_i=\boldsymbol{A}\left(t_i\right)$ and $\boldsymbol{B}_i=\boldsymbol{B}\left(t_i\right)$, and the time-step h. Note, direct composition of the monodromy matrix is not the original proceeding, however, for sensitivity analysis it exhibits benefits.

3.2 Spectral Element Method

Towards higher-order approximation, the well-conditioned Spectral Element Method, [12], has been developed for DDEs with Legendre polynomials and Legendre-Gauss-Lobatto (LGL) points, which are used for trial and test functions. Furthermore, the resulting weighted residual integral is solved by LGL quadrature which uses the interpolation points as well, thus only information at the LGL nodes is necessary.

According to that, the system discretized by a single element reads

$$\underbrace{\begin{bmatrix} \boldsymbol{N}_{1,n+1} & \cdots & \boldsymbol{N}_{1,i} & \cdots & \boldsymbol{N}_{1,1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{N}_{p,n+1} & \ldots & \boldsymbol{N}_{p,i} & \ldots & \boldsymbol{N}_{p,1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{N}_{n,n+1} & \ldots & \boldsymbol{N}_{n,i} & \ldots & \boldsymbol{N}_{n,1} \\ \boldsymbol{0} & \ldots & \boldsymbol{0} & \ldots & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{\Psi}} \begin{bmatrix} \boldsymbol{y}^{\mathrm{c}}_{n+1} \\ \vdots \\ \boldsymbol{y}^{\mathrm{c}}_{i} \\ \vdots \\ \boldsymbol{y}^{\mathrm{c}}_{1} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{P}_{1,n+1} & \cdots & \boldsymbol{P}_{1,i} & \cdots & \boldsymbol{P}_{1,1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{P}_{p,n+1} & \ldots & \boldsymbol{P}_{p,i} & \ldots & \boldsymbol{P}_{p,1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{P}_{n,n+1} & \ldots & \boldsymbol{P}_{n,i} & \ldots & \boldsymbol{P}_{n,1} \\ \boldsymbol{P}_{\mathrm{o}} & \ldots & \boldsymbol{0} & \ldots & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{\Phi}} \begin{bmatrix} \boldsymbol{y}^{\mathrm{d}}_{n+1} \\ \vdots \\ \boldsymbol{y}^{\mathrm{d}}_{i} \\ \vdots \\ \boldsymbol{y}^{\mathrm{d}}_{1} \end{bmatrix} \tag{6a}$$

with the matrices

$$\boldsymbol{N}_{p,i} = \boldsymbol{I}\frac{1}{T_{\mathrm{d}}}\sum_{k=1}^{n+1}\left(\Phi_i'(\eta_k)\Psi_p(\eta_k)\omega_k\right) - \boldsymbol{A}(\eta_i T_{\mathrm{d}})\frac{\Psi_p(\eta_i)\omega_i}{2} \tag{7a}$$

$$\boldsymbol{P}_{p,i} = \boldsymbol{B}(\eta_i T_{\mathrm{d}})\frac{\Psi_p(\eta_i)\omega_i}{2} \tag{7b}$$

$$\boldsymbol{P}_{\mathrm{o}} = \mathrm{e}^{\boldsymbol{A}_{\mathrm{c}} T_{\mathrm{o}}} \tag{7c}$$

and the relation $\boldsymbol{y}^{\mathrm{c}}_1 = \boldsymbol{P}_{\mathrm{o}}\boldsymbol{y}^{\mathrm{d}}_{n+1}$, for details see [12, 13, 14].

4 Sensitivity Analysis

In this section, we discuss an approach to compute derivatives of the (critical) eigenvalue of time-discretized DDEs. The approach is presented using two exemplary stability analysis methods in this field, Semi-Discretization and Spectral Elements.

Firstly we summarize the calculation of derivatives of eigenvalues, secondly the derivation of the discretized systems is presented and thirdly special terms are discussed.

4.1 Derivatives of Eigenvalues

The considered discretization schemes obtains an generalized eigenvalue problem with real, asymmetric $n \times n$-matrices. For the i-th eigenvalue λ_i, its associated right eigenvector $\boldsymbol{X}_i$ and left eigenvector $\boldsymbol{Y}_i$ can be stated

$$\lambda_i \boldsymbol{\Psi} \boldsymbol{X}_i = \boldsymbol{\Phi} \boldsymbol{X}_i \tag{8a}$$

$$\lambda_i \boldsymbol{\Psi}^\top \boldsymbol{Y}_i = \boldsymbol{\Phi}^\top \boldsymbol{Y}_i \quad . \tag{8b}$$

If n distinct eigenvalues are presupposed, then n independent right eigenvectors exist and are biorthogonal to n independent left eigenvectors, [15, 16]. Differentiation of the eigenvalue with respect to the parameter ρ yields

$$\frac{\partial \lambda_i}{\partial \rho} = \frac{\boldsymbol{Y}_i^\top \left(\frac{\partial \boldsymbol{\Phi}}{\partial \rho} - \lambda_i \frac{\partial \boldsymbol{\Psi}}{\partial \rho} \right) \boldsymbol{X}_i}{\boldsymbol{Y}_i^\top \boldsymbol{\Psi} \boldsymbol{X}_i} \tag{9}$$

where ρ is an arbitrary parameter which should be investigated. If multiple eigenvalues exist, enhanced strategies can be applied for the calculation of the derivative, see [17].

Since the focus of interest is here the change of the magnitude of the complex eigenvalue $|\lambda_i|$, we further consider the derivative of the magnitude of eigenvalues according to

$$\lambda_i' \equiv \frac{\partial |\lambda_i|}{\partial \rho} = \frac{\Re(\frac{\partial \lambda_i}{\partial \rho})\Re(\lambda_i) + \Im(\frac{\partial \lambda_i}{\partial \rho})\Im(\lambda_i)}{|\lambda_i|} = \frac{\Re(\frac{\partial \lambda_i}{\partial \rho}\bar{\lambda}_i)}{|\lambda_i|} = \frac{\lambda_i \frac{\partial \bar{\lambda}_i}{\partial \rho} + \bar{\lambda}_i \frac{\partial \lambda_i}{\partial \rho}}{2|\lambda_i|} \quad , \tag{10}$$

with the complex conjugated denoted by the bar and the real part $\Re$ as well as the imaginary part $\Im$.

4.2 Derivatives of Discretized Systems

The matrices of the time-discretized systems $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ have to be differentiated in the next step. The arbitrary parameter ρ is taken here as the dimensionless spindle speed $\tilde{\Omega} = N\Omega/(60 f_\mathrm{n})$ and the dimensionless specific cutting-force coefficient $\tilde{H} = H/\omega_\mathrm{n}^2$, since these parameters exhibit good exemplary character. While $\tilde{\Omega}$ affects the period of system, $\tilde{H}$ affects the time-varying regenerative coupling of the system.

4.2.1 Improved Zeroth-Order Semi-Discretization

Differentiation of the matrices obtained by Semi-Discretization (5) with respect to the dimensionless spindle speed $\tilde{\Omega}$ yields

$$\frac{\partial \boldsymbol{P}_i}{\partial \tilde{\Omega}} = \frac{\partial h}{\partial \tilde{\Omega}} \boldsymbol{P}_i \boldsymbol{A}_i \tag{11a}$$

$$\frac{\partial \boldsymbol{R}_i}{\partial \tilde{\Omega}} = \frac{1}{2} \frac{\partial h}{\partial \tilde{\Omega}} \boldsymbol{P}_i \boldsymbol{B}_i \tag{11b}$$

and with respect to dimensionless specific cutting-force coefficient $\tilde{H}$

$$\frac{\partial \boldsymbol{P}_i}{\partial \tilde{H}} = \frac{\partial}{\partial \tilde{H}} \mathrm{e}^{\boldsymbol{A}_i h} \tag{12a}$$

$$\frac{\partial \boldsymbol{R}_i}{\partial \tilde{H}} = \frac{1}{2} \left(\frac{\partial \boldsymbol{P}_i}{\partial \tilde{H}} \boldsymbol{A}_i^{-1} \boldsymbol{B}_i + (\boldsymbol{P}_i - \boldsymbol{I}) \boldsymbol{A}_i^{-1} \left(\frac{\partial \boldsymbol{B}_i}{\partial \tilde{H}} - \frac{\partial \boldsymbol{A}_i}{\partial \tilde{H}} \boldsymbol{A}_i^{-1} \boldsymbol{B}_i \right) \right) \tag{12b}$$

where the derivation of the matrix exponential $\partial \boldsymbol{P}_i / \partial \tilde{H}$ is discussed later. The derivation of first-order Semi-Discretization can be done in the same manner, however, the terms get slightly more complex. Despite the matrix exponential, differentiation of Semi-Discretization does not result in complex equations and can be implemented in efficient algorithms.

4.2.2 Spectral Element Method

In case of the Spectral Element Method, differentiation of the obtained matrices (7) with respect to dimensionless spindle speed $\tilde{\Omega}$ reads

$$\frac{\partial \boldsymbol{N}_{p,i}}{\partial \tilde{\Omega}} = \frac{\partial \frac{1}{T_\mathrm{d}}}{\partial \tilde{\Omega}} \boldsymbol{I} \sum_{k=1}^{n+1} \left(\Phi_i'(\eta_k) \Psi_p(\eta_k) \omega_k \right) \tag{13a}$$

$$\frac{\partial \boldsymbol{P}_{p,i}}{\partial \tilde{\Omega}} = \mathbf{0} \tag{13b}$$

$$\frac{\partial \boldsymbol{P}_\mathrm{o}}{\partial \tilde{\Omega}} = \frac{\partial T_\mathrm{o}}{\partial \tilde{\Omega}} \mathrm{e}^{\boldsymbol{A}_\mathrm{c} T_\mathrm{o}} \boldsymbol{A}_\mathrm{c} \tag{13c}$$

and with respect to dimensionless specific cutting-force coefficient $\tilde{H}$

$$\frac{\partial \boldsymbol{N}_{p,i}}{\partial \tilde{H}} = -\frac{\partial \boldsymbol{A}\left(\eta_i T_\mathrm{d}\right)}{\partial \tilde{H}} \frac{\Psi_p(\eta_i)\omega_i}{2} \tag{14a}$$

$$\frac{\partial \boldsymbol{P}_{p,i}}{\partial \tilde{H}} = \frac{\partial \boldsymbol{B}\left(\eta_i T_\mathrm{d}\right)}{\partial \tilde{H}} \frac{\Psi_p(\eta_i)\omega_i}{2} \tag{14b}$$

$$\frac{\partial \boldsymbol{P}_\mathrm{o}}{\partial \tilde{H}} = \frac{\partial}{\partial \tilde{H}} \mathrm{e}^{\boldsymbol{A}_\mathrm{c} T_\mathrm{o}} \tag{14c}$$

where the period is split up in two sections, thus, derivation of the matrix exponential $\partial \boldsymbol{P}_\mathrm{o} / \partial \tilde{H}$ is required as well. Note, while there is no cutting tooth, the derivative of the matrix exponential is independent of parameters of the chipping dynamics, furthermore, exclusively the derivation of the step sizes $\partial(1/T_\mathrm{d})/\partial \tilde{\Omega}$ and $\partial T_\mathrm{o} / \partial \tilde{\Omega}$ are required and the derivation of the matrices $\partial \boldsymbol{A}_i / \partial \tilde{H}$ and $\partial \boldsymbol{B}_i / \partial \tilde{H}$ at the nodes i.

4.3 Derivatives of the Matrix Exponential

Differentiation of the matrix exponential with respect to $\tilde{H}$ i.e. $\partial \mathrm{e}^{\boldsymbol{A}t} / \partial \tilde{H}$, $\boldsymbol{A} = \boldsymbol{A}\left(\tilde{H}\right)$, is more challenging and got high attention in research.

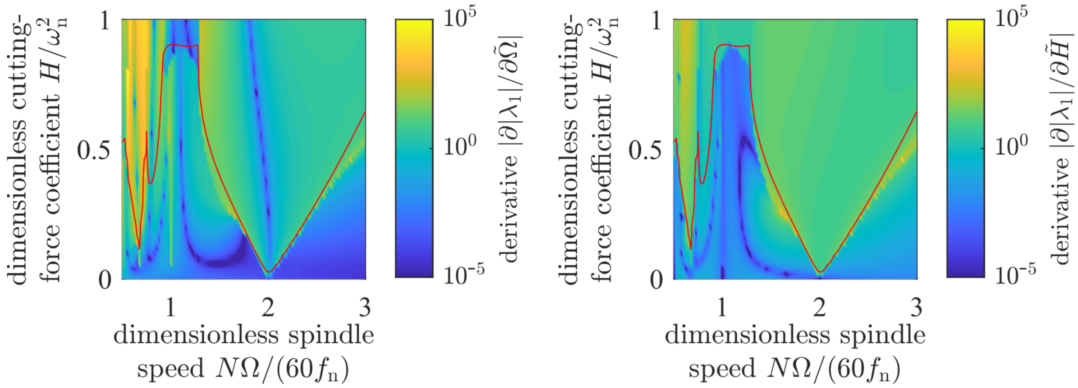
Diagonalization of $\boldsymbol{A}$ [18] is difficult for repeated eigenvalues, however, approaches based on the Wilcox's equation

$$\frac{\partial}{\partial p} \mathrm{e}^{\boldsymbol{A}t} = \int_0^t \mathrm{e}^{\boldsymbol{A}(t-u)} \frac{\partial \boldsymbol{A}}{\partial \rho} \mathrm{e}^{\boldsymbol{A}u} \mathrm{d}u \tag{15}$$

are used in our approach, [19], since robust implementations are available. The integral can be solved by quadrature or Padé approximation, for details see [15, 16].

5 Error Analysis

The gradient of the magnitude of the critical eigenvalue as well as the stability limit are shown in Fig. 1. Two characteristics have to be mentioned, the gradient is continuous and smooth in wide areas, however, where the critical eigenvalue is switched, e.g. in the vicinity of the first flip lobe at the right hand side, the gradient of the critical eigenvalue is discontinuous. Stability analysis as well as sensitivity analysis is performed by Spectral Elements of order $n = 25$ in this section.



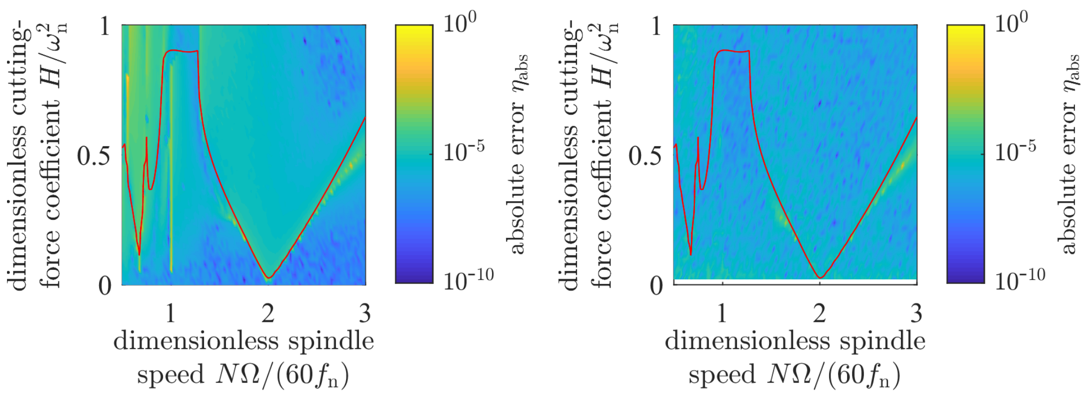
(a) derivative $\frac{\partial|\lambda_1|}{\partial\tilde{\Omega}}$ (b) derivative $\frac{\partial|\lambda_1|}{\partial\tilde{H}}$

Fig. 1: Gradient of the magnitude of the critical eigenvalue

In order to get an idea whether the obtained formulas of the derivatives of the eigenvalues of time-discretized delay-differential equations can be correct, the forward difference quotient

$$\frac{\partial\lambda(\rho)}{\partial\rho} \approx \frac{\lambda(\rho+\varepsilon\rho)-\lambda(\rho)}{\varepsilon\rho} \tag{16}$$

is employed with the relatively defined disturbance ε. The choice of the disturbance is a critical point in numerical differentiation, although there is no guarantee that the approximation is reasonable. Numerical differentiation with $\varepsilon = 10^{-6}$ results in an absolute error $\eta_\mathrm{abs} = |\lambda'_\mathrm{num} - \lambda'_\mathrm{ana}|$ which is show in Fig. 2.



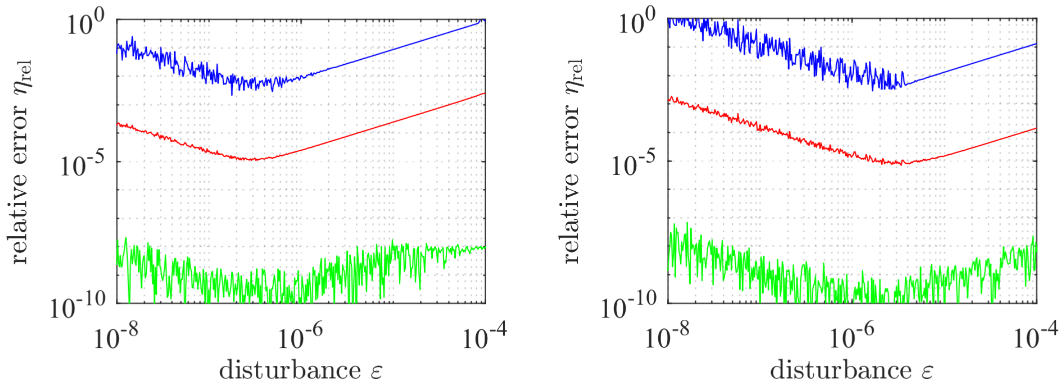
(a) derivative $\frac{\partial|\lambda_1|}{\partial\tilde{\Omega}}$ (b) derivative $\frac{\partial|\lambda_1|}{\partial\tilde{H}}$

Fig. 2: Absolute error of numerical differentiation with the disturbance $\varepsilon = 10^{-6}$

The absolute error η_abs is evaluated at 101×51 nodes of the stability lobe diagram for the dimensionless spindle speed $\tilde{\Omega}$ and at 101×50 nodes for the dimensionless specific cutting-force coefficient $\tilde{H}$, except of $\tilde{H}$=0. Both can be identified, regions where the rounding error is dominant and regions where the formula error is dominant. Furthermore, differentiation with respect to the dimensionless specific cutting-force coefficient $\tilde{H}$ is more affected by the rounding error than differentiation with respect to the dimensionless spindle speed $\tilde{\Omega}$ for the disturbance $\varepsilon = 10^{-6}$. The red line illustrates the stability limit.

The arithmetic mean, greatest and least relative error

$$\eta_\mathrm{rel} = \left|\frac{\lambda'_\mathrm{num} - \lambda'_\mathrm{ana}}{\lambda'_\mathrm{ana}}\right| \tag{17}$$



(a) derivative $\frac{\partial|\lambda_1|}{\partial\tilde{\Omega}}$ (b) derivative $\frac{\partial|\lambda_1|}{\partial\tilde{H}}$

Fig. 3: Arithmetic mean, greatest and least relative error of numerical differentiation in dependency of the disturbance ε

of the samples in Fig. 2 are shown in Fig. 3 for different disturbances ε.

The minimum of the arithmetic means can be identified at $\varepsilon_{\tilde{\Omega}} \approx 3 \cdot 10^{-7}$ and $\varepsilon_{\tilde{H}} \approx 3 \cdot 10^{-6}$ with $\eta_{\mathrm{rel}} \approx 10^{-5}$.

6 Conclusions

In this paper we presented an analytical approach for differentiation of eigenvalues in machining simulations, where time-delay differential equations arise. The (critical) eigenvalues can be approximated by time discretization. Two popular methods, Semi-Discretization and Spectral Elements, are considered, where Semi-Discretization is a highly robust method which can be applied comparatively easy. However, if the period of the system can be divided into sections where the time-varying characteristics are smooth, thus, discontinuities are at the boundary, the excellent convergence of the Spectral Elements gives a competitive edge.

Investigations of finite differences approximation confirm both the accuracy of the analytical results and the difficulty of effectively and efficiently using numerical differentiation. Considering Spectral Elements, differentiation is highly efficient, whereas the calculation of the matrix exponential as well as the calculation of its derivative is comparatively computationally expensive.

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