Introduction to Optimal Estimation

MICHAEL ATHANS

MIT & ISR/IST
Last Revised: September 28, 2001
Ref. KF #1
The Basic Estimation Questions

- Where am I?
- How much do I believe the “where am I” estimate
The Concept of Sensor Fusion

- Different sensor types yield different position estimates (and uncertainty volumes)
  - **SENSOR FUSION** combines the measurements from all the different sensors to yield
    - "better" updated position estimate
    - reduced volume of uncertainty about fused estimate
  - Sensor fusion is a centralized decision problem
Never Forget ...

- Obtaining only an estimate of a quantity is never enough
- Being “right” on the average is nice, but not enough
- We must also obtain a measure of the quality (or believability) of the estimate
- We need something like a standard-deviation or some bounding measure
- Example. An estimate of, say, 5.2 with ±23% uncertainty is “worse” than an estimate of, say, 5.12 with ±7% uncertainty
• We shall deal with the **dynamic evolution of uncertainty**, so we must combine
  • **stochastic processes** (time-varying random variables)
  • **linear and nonlinear dynamic systems**, whose state-evolution depends on stochastic processes
• We shall define the three common classes of estimation problems
  • filtering
  • prediction (or forecasting)
  • smoothing (or interpolation)
• We shall employ **“optimal”** methods for extracting information from uncertain measurements, and avoid ad-hoc processes
• Expert understanding of the filtering problem is essential for the solution of the prediction and/or the smoothing problem
Dynamic Systems and “Where am I?”

- Recall that the state variable description of dynamical systems is useful because
  - knowledge of the present state summarizes past behavior
  - knowing the present state and future inputs is sufficient to determine exactly all future states and output (sensor) variables
- The question “where am I?” in dynamic systems (plants) corresponds to the estimation of the entire state vector
  - positions, velocities, accelerations ... in mechanical systems
  - Pressures, temperatures ... in thermodynamic systems
  - inductor currents, capacitor voltages ... in linear electric circuits
- We shall deal with state estimation problems where the plant is subject to stochastic disturbances, and on the basis of noisy sensor measurements
The only real-time signals that are available are the control(s) and the noisy sensor measurement(s).

Cannot directly measure the actual state variables or plant disturbances or sensor noise.
Example: A Sailboat

STATE VARIABLES
• 3D positions and velocities
• Roll, yaw, and pitch angles
• Roll, yaw, and pitch rates

CONTROL VARIABLES
• Rudder angle
• Sail area and angle

PLANT DISTURBANCES
• Wind and wave forces and moments (including fluctuations from mean)

NOISY MEASUREMENTS
• Heading angle, yaw angle and rate(?), roll angle and rate(?), ...
The Physical System: Uncertainties

- Exogenous uncertainties
  - initial state is random vector
  - plant disturbance is vector-valued random process (or sequence)
  - sensor noise is vector-valued random process (or sequence)
- Mathematical models of plant dynamics and sensors are inaccurate
Open-Loop Prediction

• Make a mathematical model of the plant and sensors, driven by known control
• Use average values of initial state, plant disturbance and sensor noise
Estimator (Filter) Structure

- Use real-time information to provide input signals to the plant model, so as to improve state estimates.
Filtering, Prediction and Smoothing

**FILTERING PROBLEM**
- Given input time function $U[t_0, t]$ and measurement time function $Y[t_0, t]$, find "best" estimate of the state $x(t)$

**PREDICTION PROBLEM**
- Let $T$ be a prediction time.
- Given $U(t_0, t + T)$ and $Y(t_0, t)$
- Determine "best" estimate of (future) state $x(t + T)$

**SMOOTHING PROBLEM**
- Let $\tau$ be any time, $t_0 \leq \tau \leq t$
- Given $U(t_0, t)$ and $Y(t_0, t)$
- Determine "best" estimate of (past) state $x(\tau)$
The Nature of Mathematical Models

- Dynamic models of the physical plant, with finite number of state variables
- Static models of sensor measurements

CONTINUOUS- TIME MODELS: \( t_0 \leq t \)

- State dynamics described by ordinary vector differential equations
  \[
  \frac{d}{dt} x(t) = f(x(t), u(t), \xi(t), t)
  \]
  \[y(t) = g(x(t), u(t), \theta(t), t)\]

DISCRETE - TIME MODELS: \( t = 0, 1, 2, \ldots, \)

- State dynamics described by vector difference equations
  \[x(t + 1) = f(x(t), u(t), \xi(t), t)\]
  \[y(t) = g(x(t), u(t), \theta(t), t)\]
Linear Dynamical Systems

Linear Time-Varying (LTV) Systems

- Continuous-time
  \[
  \frac{d}{dt} x(t) = A(t)x(t) + B(t)u(t) + L(t)\xi(t) \\
  y(t) = C(t)x(t) + \theta(t)
  \]

- Discrete-time
  \[
  x(t+1) = A(t)x(t) + B(t)u(t) + L(t)\xi(t) \\
  y(t) = C(t)x(t) + \theta(t)
  \]

Linear Time-Invariant (LTI) Systems

- Continuous-time
  \[
  \frac{d}{dt} x(t) = Ax(t) + Bu(t) + L\xi(t) \\
  y(t) = Cx(t) + \theta(t)
  \]

- Discrete-time
  \[
  x(t+1) = Ax(t) + Bu(t) + L\xi(t) \\
  y(t) = Cx(t) + \theta(t)
  \]
Computational Considerations

- The filter must operate in real-time.
- We must solve in real-time the plant-model dynamical equations and sensor-model equations.
- The types of the implemented transformations (that map the residuals to plant model corrections) are dictated by real-time computer constraints:
  - optimal
  - suboptimal
In dynamic estimation the state-vector estimate evolves with time.
The covariance matrix of the state estimation vector can be used to quantify the volume of uncertainty about the state estimate.
Thus we must find the dynamic equations that govern the dynamic evolution of BOTH the state estimate and error-covariance matrix.
Radar (Ladar) Tracking

Estimate (3D) positions, velocities, and perhaps accelerations of moving aerospace target, based upon noisy measurements of range, azimuth and elevation

\[ r(t) = \text{range}, \quad \alpha(t) = \text{azimuth}, \quad \beta(t) = \text{elevation} \]
Passive Sonar Tracking

- Towed-array (an array of microphones) deployed behind our submarine can measure azimuth angle to enemy submarine
- No range measurement available, for stealth
Basic Oxygen Furnace

- Steel production
- Desired steel strength requires a certain % of carbon in iron
- Iron ore is mixed with carbon, calcium etc in pressurized vessel
- Superheated oxygen melts mixture and burns carbon
- Mass spectrometer measurements are used to estimate % carbon as a function of time
Combined Plant-Identification and State-Estimation

- Example of simultaneous system identification and state estimation
- True plant is one (or close to one) of N possible models
- A parallel bank of N filters is constructed, each corresponding to a specific model
- It is possible to evaluate the posterior probabilities of each model being the true plant
- Global state estimate is generated by probabilistic weighting the state estimates of each model
Concluding Remarks

- We must understand stochastic sequences and stochastic processes.
- We must study the response of dynamic systems to uncertain initial states, modeled as random vectors.
- We must study the response of dynamic systems to both deterministic time-functions and stochastic processes.
- We must specify precisely the optimization philosophy of generating “best” estimates.
- Real-time computational requirements dictate whether we use a truly optimal estimation algorithm or resort to a suboptimal one.
  - solving in real-time differential or difference equations is feasible.
  - solving in real-time partial (or integro-partial) differential equations is NOT feasible.
References


