Dynamic Stochastic Estimation, Prediction and Smoothing
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PhD. Programme

Implementation Methods

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Implementation Methods

Up to this point there has been presented the Kalman filters as they are supposed to behave.

Facts from computer implementation examples:

- When implemented in computers, the observed mean-square estimation errors are (much) larger than the values predicted by the covariance matrix (EVEN with simulated data);
- The variances of the filter estimation errors observed diverge from the theoretical values;
- Solutions obtained for the Riccati equation have negative variance!

Implementation Methods

A ***real*** problem:

The Kalman filter is defined in terms of the real number system, which has infinite precision...

and then it is implemented on digital computers with finite precision.

Moreover, the REAL arithmetic of computers is not the arithmetic of the real numbers. It is an arithmetic of floating-point number which are but a finite subset of the rational numbers.

Floats (4 bytes) can have a roundoff error in the order of 10^{-8}

Error Analysis of Numerical Methods

For a procedure to solve a given problem, some features should be taken in consideration:

- Numerical stability refers to robustness against roundoff errors;
- Precision is also influenced by the procedural details of the implementation method;
- Implementation methods cannot always be ordered (results can depend on the problem at hand);
- Ill conditioned problems, or problems where the output of a procedure (solution) is very sensitive to noise in the input data (problem).

Error Analysis of Numerical Methods

Example: The sensitivity of the solution to the linear problem

$$Ax = b$$

to uncertainties in A and b and roundoff errors is characterized by the condition number of A (for nonsingular A matrices)

$$cond(A) = \frac{\max_x \|Ax\| / \|x\|}{\min_x \|Ax\| / \|x\|}$$

It also equals the ratio of the largest and smallest singular value of A.

Rule of thumb for the maximum relative error $\delta = ||x-x||/||x||$:

$$\delta = c_A \, \varepsilon \, cond(A)$$

where ε is the unit roundoff error and c_A depends on the dimension of A.

A problem is ill conditioned if *cond(A)+1=cond(A)* evaluates to true.

Ill conditioned Kalman Filtering problems

For Kalman filtering problems the solution of the associated Riccati equation should equal the covariance matrix of actual estimation uncertainty.

Factors that contribute to ill conditioning:

- large uncertainties in the values of A, Q, C and R.
- large ranges on the actual values of the systems matrices due to poor choice of scaling or dimensional units;
- ill conditioning of the intermediate result $R^{*}=C\Sigma C^{T}+R$;
- large matrix dimensions;
- poor machine precision.

Effects of Roundoff errors on Kalman Filtering

Floating-point Roundoff Errors

$$\pm .d_1 d_2 \cdots d_p \times b^{\pm e}$$

b – base

d_i – digit $0<d_i<b$

p – precision

e – exponent

Intrinsic characteristics and errors associated to a representation:

- underflow limit $b^{e_{min}-1}$
- overflow limit $b^{e_{max}}$
- rounding (to the nearest floating point number)

$$\varepsilon = \frac{1}{2} b^{1-p}$$

- chopping (towards zero)

$$\varepsilon = b^{1-p}$$

Typical Floating-point Formats

Floating-point attributes	Machine format				
	ANSI/IEEE		CRAY-1	IBM370	
	Single	Double		Single	Double
b	*2*	*2*	*2*	*16*	*16*
p	*24*	*53*	*48*	*6*	*14*
e_{min}	*-126*	*-1022*	*-16384*	*-64*	*-64*
e_{max}	*127*	*1023*	*8191*	*63*	*63*
ε	$\approx 6\times 10^{-8}$	$\approx 10^{-16}$	$\approx 3\times 10^{-15}$	$\approx 5\times 10^{-7}$	$\approx 10^{-16}$

Bounds on Roundoff errors

Given a variable represented as a floating-point

$$v + \delta v;$$

The magnitude of the roundof error will be bounded by

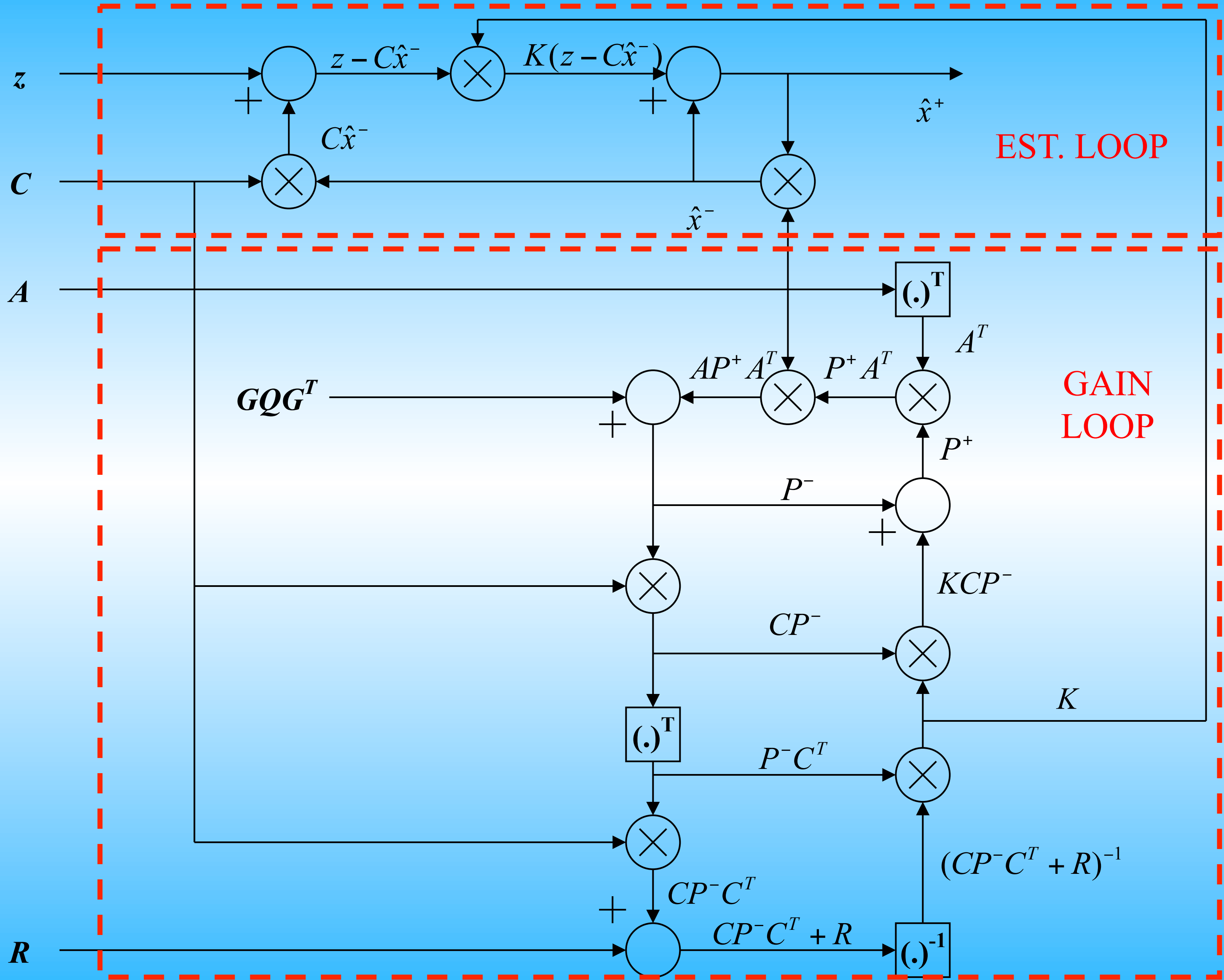
$$|\delta v| < \varepsilon |v|.$$

The bound also holds for a *m* x *n* matrix A if the Frobenius norm or the 2-norm are used.

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2} \qquad \|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

Roundoff Errors Propagation in Kalman Filters

Data flow:



Roundoff Errors Propagation in Kalman Filters

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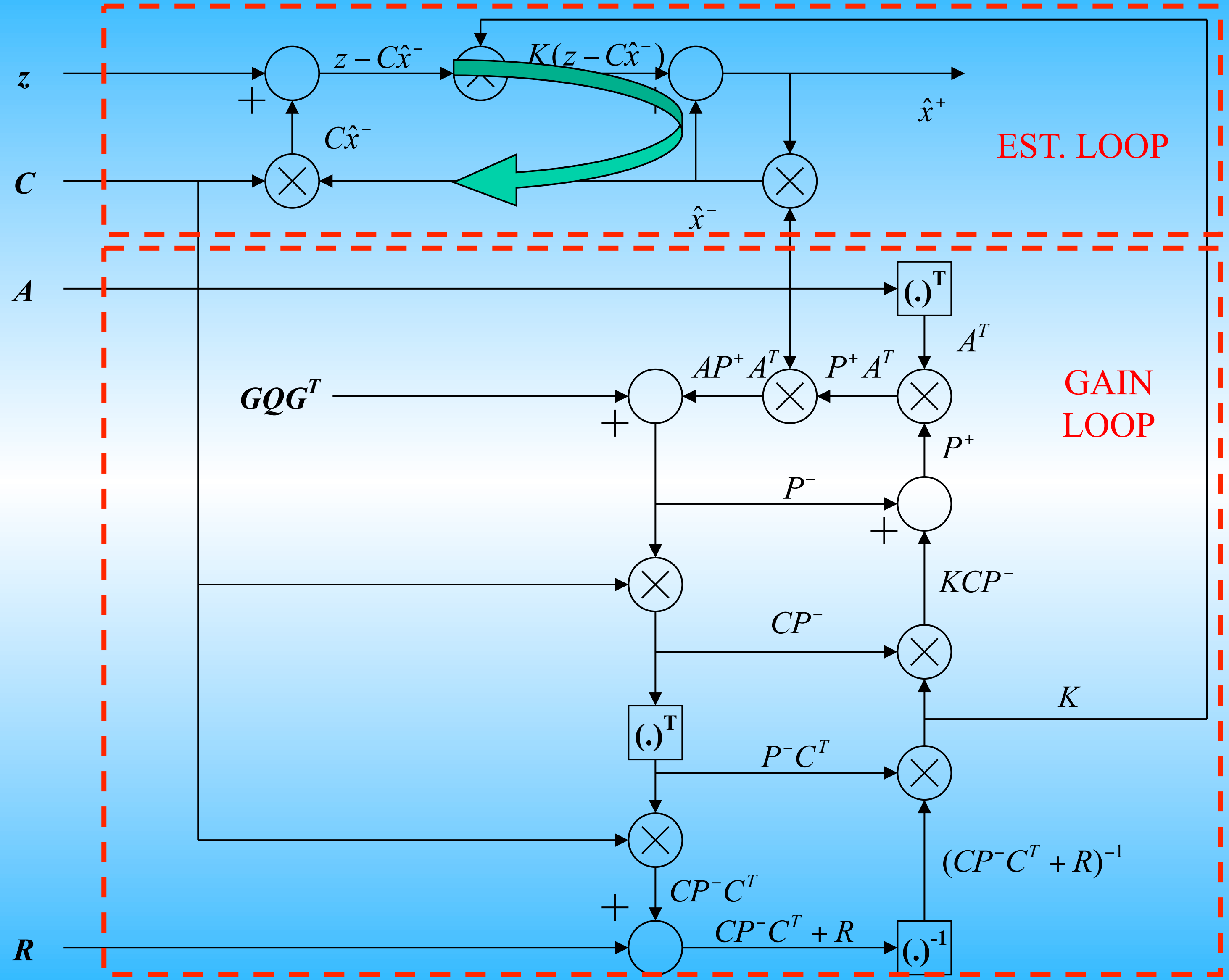

Roundoff Errors Propagation in Kalman Filters

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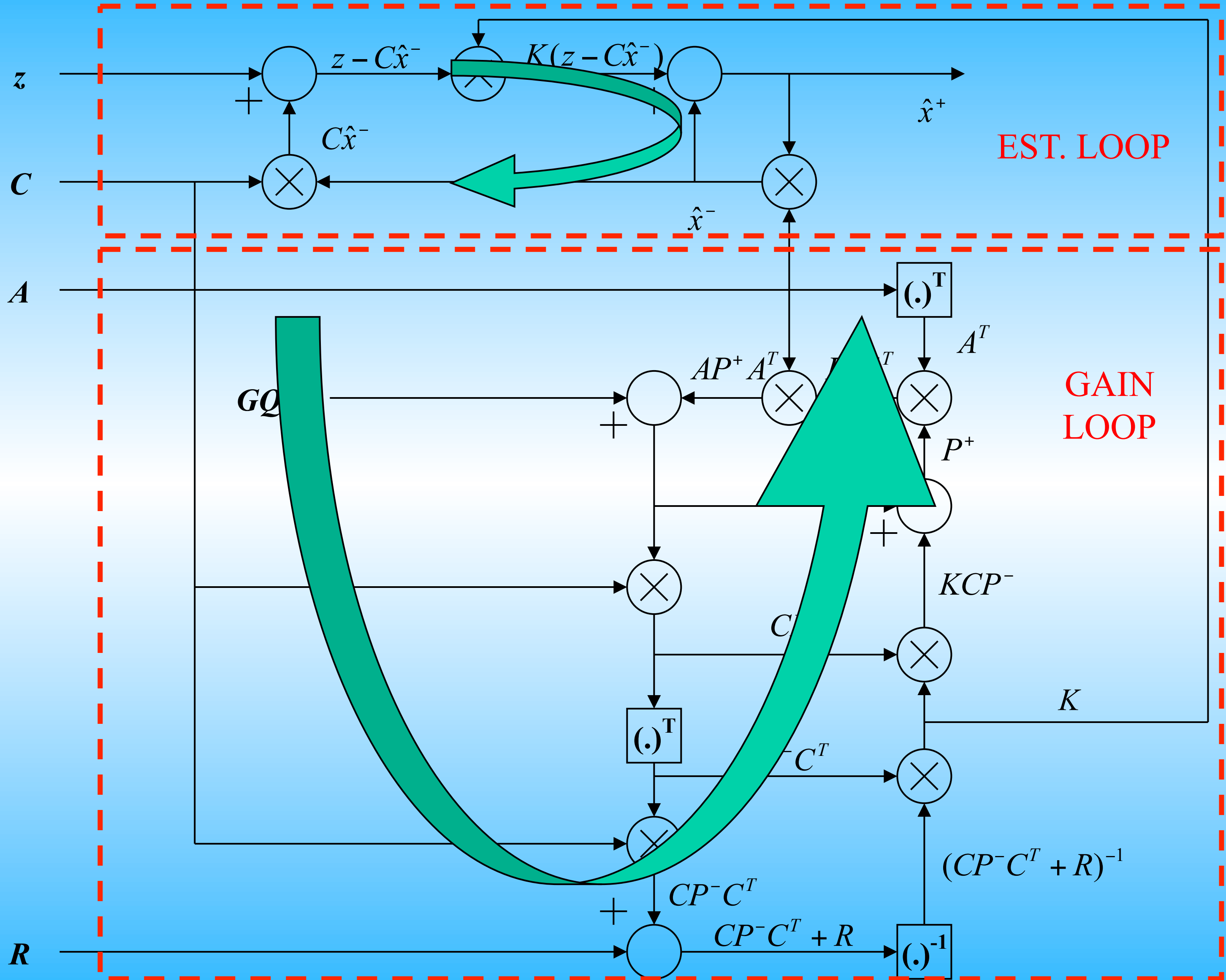

Numerical Analysis

The *a priori* value of the covariance P^- is the one used in the Kalman gain computation. Its first order propagation error is considered next:

$$\delta x_{k+1}^- = (A - KC)\left\lfloor \delta x_k^- + \delta P_k^- (CP^- C^T + R)^{-1} \left(z - Cx_k^-\right) \right\rfloor + \Delta x_{k+1} (A - KC)\delta P_k^-$$
$$\delta P_{k+1}^- = (A - KC)\delta P_k^- (A - KC)^T + \Delta P_{k+1} + A(\delta P_k^- - \delta P_k^{-T})A^T - A(\delta P_k^- - \delta P_k^{-T})(A - KC)^T$$

where Δ refers to the roundoff error added at each recursion step. The norm of added roundoff errors is given by:

$$\left|\Delta x_{k+1}^-\right| = \varepsilon_1 \left(\left|A - KC\right|\left|x_k^-\right| + \left|K\right|\left|z\right|\right) + \left|\Delta K\right|\left|z\right| + \left|\Delta K\right|\left(\left|C\right|\left|x_k^-\right| + \left|z\right|\right)$$
$$\delta P_{k+1}^- = \varepsilon_2 cond\left(CP_k^- C^T + R\right)^2 \left|P_{k+1}^-\right| ,$$

Where ε_1, ε_2 are constant multiples of ε (the unit roundoff error).

Example of filter divergence due to numerical errors

Given the scalar system
$$\begin{cases} x_{k+1} = x_k \\ z_k = x_k \end{cases},$$

where $Q=0$ and such that $P_o >> R$.

P_0 is so much greater than R that

$R < \varepsilon P_0$

Exp	Value	
	Exact	Rounded
$P_0 C^T$	P_0	P_0
$CP_0 C^T$	P_0	P_0
$CP_0 C^T + R$	$P_0 + R$	P_0
$K_1 = P_0 C^T (CP_0 C^T + R)^{-1}$	$\frac{P_0}{P_0 + R}$	1
$P_1 = P_0 - K_1 C P_0$	$\frac{P_0 R}{P_0 + R}$	0
$\vdots$	$\vdots$	$\vdots$
$K_k = P_{k-1} C^T (CP_{k-1} C^T + R)^{-1}$	$\frac{P_0}{kP_0 + R}$	0
$P_k = P_{k-1} - K_k C P_{k-1}$	$\frac{P_0 R}{kP_0 + R}$	0

- Calculated variance equal to zero!
- Actual variance $\frac{P_0 R}{P_0 + R} \approx R$
- Theoretical variance in the exact case

$$\frac{P_0 R}{kP_0 + R} \approx \frac{R}{k}$$

An Overview of Factorization Tricks

The more numerically stable implementations of the Kalman filter use one or more of the following techniques to solve the associated Riccati equation:

1. Factoring the covariance matrix of state estimation into Cholesky factors (triangular factors $CC^T=M$);
2. Modified Cholesky decomposition algorithms ($M=DD_U U^T$ or $M=LD_L L^T$);
3. Factoring the covariance matrices of the measurement and state noises, Q and R, respectively;
4. Symmetric matrix square roots of element matrices (in the form $I-\sigma v v^T$);
5. Triangularization (QR decomposition);
6. Gram-Schmidt ortonormalization (ortho times triang.)

...

Cholesky Decomposition Methods and Applications

Objective: To find C such that

$$CC^T = M$$

where C is called Cholesky factor (not unique). If suitable constraints are used a unique solution can be found.

Decomposition methods

Example:

$$\begin{bmatrix} p_{11} & p_{21} & p_{31} \\ p_{21} & p_{22} & p_{32} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T = \begin{bmatrix} c_{11}^2 & & \\ c_{11}c_{21} & c_{21}^2 + c_{22}^2 & \\ c_{11}c_{31} & c_{21}c_{31} + c_{22}c_{32} & c_{21}^2 + c_{22}^2 + c_{33}^2 \end{bmatrix}$$

Solution:

$$c_{11} = \sqrt{p_{11}}$$
$$c_{21} = p_{21} / c_{11}$$
$$c_{31} = \sqrt{p_{22} - c_{21}^2}$$

$$c_{32} = \left(p_{32} - c_{21}c_{31}\right) / c_{22}$$
$$c_{33} = \sqrt{p_{33} - c_{31}^2 - c_{32}^2}$$

Decorrelating the Components of Vector-Valued Measurements

Suppose that the observations are given by

$$z=Hx+v$$

where $E[vv^T]=R$ is not a diagonal matrix, but can be factored as

$$R = UDU^T,$$

Where D is diagonal and U is an unit upper triangular matrix.

Redefine

$$\begin{aligned} \bar{z} &= U^{-1}z \\ &= (U^{-1}H)x + (U^{-1}v) \\ &= \overline{H}\bar{x} + \bar{v}. \end{aligned}$$

Solve

$$\begin{aligned} U\bar{z} &= z \\ U\overline{H} &= H \end{aligned}$$

$$\begin{aligned} \overline{R} &= E\left[\bar{v}\bar{v}^T\right] \\ &= E\left[U^{-1}vv^T U^{T^{-1}}\right] \\ &= U^{-1}E\left[vv^T\right]U^{T^{-1}} \\ &= U^{-1}RU^{T^{-1}} \\ &= U^{-1}\left(UDU^T\right)U^{T^{-1}} \\ &= D \end{aligned}$$

An Overview of Factorization Tricks

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1. Factoring the covariance matrix of state estimation into Cholesky factors (triangular factors $CC^T=M$);
2. Modified Cholesky decomposition algorithms ($M=DD_UU^T$ or $M=LD_LL^T$);
3. Factoring the covariance matrices of the measurement and state noises, Q and R, respectively;
4. Symmetric matrix square roots of element matrices (in the form $I-\sigma vv^T$);
5. Triangularization (QR decomposition);
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...

Modified Cholesky (UD) Decomposition Algorithms

Objective: To find U (unit upper triangular matrix) and D (diagonal matrix) such that the mxm symmmetric matric M is factored as

$$M = UDU^T$$

With these constraints a unique solution can be found.

Solution:

```
for j=m,...,1
   for i=j,...1
      σ=M_ij
      for k=j+1,...,m
         σ=σ-U_ik D_kk U_jk
      if (i=j)
         D_jj=σ
         U_jj=1
      else
         U_ij=s/D_jj
```

Computational complexity

$$\frac{1}{6} m(m-1)(m+4) \quad \textit{flops}$$

No square roots!

Symmetric matrix square roots of elementary matrices

The square of a symmetric elementary matrix verifies

$$\begin{aligned}\left(I-\sigma vv^T\right)^2 &= \left(I-\sigma vv^T\right)\left(I-\sigma vv^T\right)\\ &= I-\left(2\sigma-\sigma^2|v|^2\right)vv^T\\ &= I-svv^T\end{aligned}$$

with $s = 2\sigma - \sigma^2 |v|^2.$

The symmetric square root of a symmetric elementary matrix

$$\left(I-svv^T\right)^{\frac{1}{2}} = \left(I-\sigma vv^T\right),$$

with

$$\sigma = \frac{1+\sqrt{1-s|v|^2}}{|v|^2}.$$

Symmetric Matrix Square Factorization*

Define the Cholesky factor of the covariance matrix *P:*

$$P^- = \Sigma^-\left(\Sigma^-\right)^T \quad \text{and} \quad P^+ = \Sigma^+\left(\Sigma^+\right)^T,$$

so that the observation update Riccati equation can be factored as

$$\Sigma^+\left(\Sigma^+\right)^T = \Sigma^-\left(\Sigma^-\right)^T - \Sigma^-\left(\Sigma^-\right)^T C^T\left(C\Sigma^-\left(\Sigma^-\right)^T C^T + R\right)^{-1} C\Sigma^-\left(\Sigma^-\right)^T$$

and define $V = \left(\Sigma^-\right)^T C^T$. The previous expression can be written as

$$\Sigma^+\left(\Sigma^+\right)^T = \Sigma^-\left(I_n - V\left(V^T V + R\right)^{-1} V^T\right)\left(\Sigma^-\right)^T.$$

For the case that the measurement is a scalar , Potter was able to write this expression as

$$I_n - v\left(v^T v + R\right)^{-1} v^T = WW^T,$$

resulting that

$$\Sigma^+\left(\Sigma^+\right)^T = \Sigma^- WW^T\left(\Sigma^-\right)^T \quad or \quad \Sigma^+ = \Sigma^- W.$$

a Cholesky factor !

* Introduced by J. Potter in 1966.

Symmetric Matrix Square Factorization*

For the scalar case, an symmetric elementary matrix of the form

$$I_n - \frac{vv^T}{R + |v|^2},$$

where v is a column vector, is obtained.

Using the results for the symmetric elementary matrices

$$\sigma = \frac{1 + \sqrt{1 - s|v|^2}}{|v|^2}, \quad \text{and} \quad s = \frac{1}{R + |v|^2}.$$

The radicand verifies

$$1 - s|v|^2 = 1 - \frac{|v|^2}{R + |v|^2} = \frac{R}{R + |v|^2} \geq 0.$$

Bierman UD Factorization

Define the covariance matrix P using UD factors as:

$$P^- = U^- D^- \left(U^-\right)^T \quad \text{and} \quad P^+ = U^+ D^+ \left(U^+\right)^T,$$

so that the observation update equation can be factored as

$$U^+ D^+ \left(D^+\right)^T = U^- D^- \left(U^-\right)^T - \frac{U^- D^- \left(U^-\right)^T C^T C U^- D^- \left(U^-\right)^T}{\left(C U^- D^- \left(U^-\right)^T C^T + R\right)^{-1}}$$

and define $v = \left(U^-\right)^T C^T$ as an n-vector, where n is the state vector size.

The previous expression can then be written as

$$U^+ D^+ \left(D^+\right)^T = U^- \left(D^- - \frac{D^- v v^T D^-}{\left(v^T D^- v + R\right)^{-1}} \right) \left(U^-\right)^T.$$

Bierman UD Factorization

The following unfactored expression is present:

$$D^- - D^- v\left(v^T D^- v + R\right)^{-1} v^T D^-$$

If it is possible to write it in the form:

$$D^- - D^- v\left(v^T D^- v + R\right)^{-1} v^T D^- = B D^+ B$$

then D^+ is the *a posteriori* D factor of P, because the resulting equation

$$\begin{aligned} U^+ D^+ \left(U^+\right)^T &= U^- \left(B D^+ B\right)\left(U^-\right)^T \\ &= \left(U^- B\right) D^+ \left(U^- B\right)^T \end{aligned}$$

can be solved for the *a posteriori* U factor as

$$U^+ = U^- B.$$

Bierman UD Factorization

It suffices to find a numerically stable and efficient method for the UD factorization of a matrix of the form

$$D^- - D^- v\left(v^T D^- v + R\right)^{-1} v^T D^- .$$

Lemma – If $D^- - D^- v\left(v^T D^- v + R\right)^{-1} v^T D^- = BD^+ B$ then for $1 \leq j \leq n$ and for $1 \leq i \leq j < n$,

$$D_{jj}^+ = D_{jj}^- \left[\frac{R + \sum_{k=1}^{j-1} v_k^2 D_{kk}^-}{R + \sum_{k=1}^{j} v_k^2 D_{kk}^-} \right], \qquad B_{ij}^+ = -\frac{D_{jj}^- v_i v_j}{R + \sum_{k=1}^{j-1} v_k^2 D_{kk}^-}.$$

■

Numerical Analysis of Square Root Methods

The *a priori* value of the covariance P^- is the one used in the Kalman gain computation. Its first order propagation error using a square root covariance factorization method is:

$$\begin{aligned} \delta x_{k+1}^- &= (A-KC)\left[\delta x_k^- + \delta P_k^- (CP^-C^T+R)^{-1}\left(z - Cx_k^-\right)\right] + \Delta x_{k+1}(A-KC)\delta P_k^- \\ \delta P_{k+1}^- &= (A-KC)\delta P_k^-(A-KC)^T + \Delta P_{k+1} \end{aligned},$$

where Δ refers to the roundoff error added at each recursion step. The norm of added roundoff errors is given by:

$$\left|\Delta x_{k+1}^-\right| = \varepsilon_1\left(\left|A-KC\right|\left|x_k^-\right| + \left|K\right|\left|z\right|\right) + \left|\Delta K\right|\left|z\right| + \left|\Delta K\right|\left(\left|C\right|\left|x_k^-\right| + \left|z\right|\right)$$

$$\delta P_{k+1}^- = \frac{\varepsilon_3\left(1 + cond\left(CP_k^-C^T + R\right)\left|P_{k+1}^-\right|\right)}{\left|\Sigma_{Cholesky}\right|},$$

Where ε_1, ε_3 are constant multiples of ε (the unit roundoff error).

Numerical Analysis

The *a priori* value of the covariance P^- is the one used in the Kalman gain computation. Its first order propagation error is considered next:

$$\delta x_{k+1}^- = (A-KC)\left[\delta x_k^- + \delta P_k^-(CP^-C^T+R)^{-1}\left(z - Cx_k^-\right)\right] + \Delta x_{k+1}(A-KC)\delta P_k^-$$

$$\delta P_{k+1}^- = (A-KC)\delta P_k^-(A-KC)^T + \Delta P_{k+1} + A(\delta P_k^- - \delta P_k^{-T})A^T - A(\delta P_k^- - \delta P_k^{-T})(A-KC)^T$$

where Δ refers to the roundoff error added at each recursion step. The norm of added roundoff errors is given by:

$$\begin{aligned}\left|\Delta x_{k+1}^-\right| &= \varepsilon_1\left(\left|A-KC\right|\left|x_k^-\right| + \left|K\right|\left|z\right|\right) + \left|\Delta K\right|\left|z\right| + \left|\Delta K\right|\left(\left|C\right|\left|x_k^-\right| + \left|z\right|\right) \\ \delta P_{k+1}^- &= \varepsilon_2 cond^2\left(CP_k^-C^T+R\right)\left|P_{k+1}^-\right|\end{aligned},$$

Where ε_1, ε_2 are constant multiples of ε (the unit roundoff error).

Bibliography

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- *Applied Optimal Estimation*, A. Gelb, MIT Press, 1974;
- *Optimal Filtering*, B. Anderson and J. Moore, Prentice Hall, 1979.
- *Factorization Methods for Discrete Sequential Estimation*, G. J. Bierman, New York: Academic Press, 1977.

H_∞ Filtering and Smoothing

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Theme

- KF only corresponds to the optimal filtering strategy under restrictive assumptions, and for some objectives (functionals)
- The requirement on the knowledge of the power spectral density of the disturbances is too restrictive. The WNG assumption too.
- Unknown multimodal and/or skewed pdfs are common

However

• Optimality and stability still of great importance, in the presence of uncertainty (robustness)

• Other functionals / objectives can be used to formulate estimation problems. Minimization must be feasible

Norms of Signals

L_1 [0,T] norm

$$\|u\|_1 = \int_0^T |u(t)|dt < \infty$$

L_2 [0,T] norm

(energy)

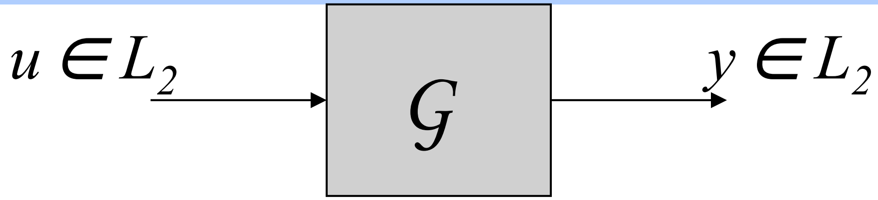
$$\|u\|_2 = \left(\int_0^T u(t)^T u(t) dt\right)^{\frac{1}{2}} < \infty \qquad (1)$$

L_∞ norm

(least upper bound)

$$\|u\|_\infty = \sup_t (|u(t)|) < \infty$$

Motivation for H_∞ Filtering



For finite energy signals in the input of system $\mathcal{G}$, how much is the minimum energy on the output?

Possible interpretation as a Min-max Nash game in estimation:

Maximum energy in the error is minimized.

For bounded systems, the H_∞ norm is defined as

$$\|\mathcal{G}\|_\infty = \sup_{u \in L_2, \|u\|_2 \neq 0} \frac{\|\mathcal{G}u\|_2}{\|u\|_2}$$

Denominated as the L_2 induced norm.

For LTI systems corresponds to the peak in the Bode diagram.

Norms of Systems

LTI Continuous - time model

$$\Sigma_{\mathcal{G}} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

Transfer function

$$G(s) = C(sI - A)^{-1} B$$

H_2 norm

$$\|\mathcal{G}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} tr\left(G(j\omega)G^*(j\omega)\right) d\omega \right)^{1/2}$$

$$= \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_i \sigma_i^2 \left(G(j\omega)\right) d\omega \right)^{1/2}$$

H_∞ norm

$$\|\mathcal{G}\|_\infty = \sup_\omega \sigma_{\max}\left[G(j\omega)\right]$$

*Input-output Relations**

	Stochastic	$\|y\|_2$	$\|y\|_\infty$
Stochastic	$\|\mathcal{G}\|_2$	∞	?
$u(t) = \delta(t)$	?	$\|\mathcal{G}\|_2$	$\|\mathcal{G}\|_\infty$
$\|u\|_2$	?	$\|\mathcal{G}\|_\infty$	$\|\mathcal{G}\|_2$
$\|u\|_\infty$	∞	∞	$\|\mathcal{G}\|_1$

*See [1] for details

Plant and Sensor Modeling

$$\Sigma_{\mathcal{G}} : \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)w(t) \\ y(t) = C(t)x(t) + D(t)w(t) \end{cases} \quad t \in [0, T] \qquad \begin{matrix} x(t) \in R^n \\ y(t) \in R^p \\ w(t) \in R^m \end{matrix}$$

$A(t), B(t), C(t), D(t)$ piecewise continuous bounded functions.

$z(t) = L(t)x(t)$ quantity of interest to be estimated.

(A, B) is stabilizable

(A, C) is detectable

$$D(t) \begin{bmatrix} B(t)^T \\ D(t)^T \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

Independent plant/measurement noises

Normalized measurement noise

H_∞ Filtering

Problem statement : For system $\mathcal{G}$, with known (unknown) initial conditions* and using the measurements $y(t)$, obtain an estimate $\hat{z}(t)$ of $z(t)$ that minimizes the (worst case) indeces

$$J_1 = \sup_{0 \neq w \in L_2} \frac{\|z - \hat{z}\|_2^2}{\|w\|_2^2} \quad or \quad J_2 = \sup_{0 \neq w \in L_2} \frac{\|z - \hat{z}\|_2^2}{\|w\|_2^2 + x_0' R x_0}, \qquad R^{**} > 0.$$

Important questions:

- Given $\gamma>0$, does there exist a filter with finite J_1 (or J_2)?
- Under the assumptions , does it verifies $J_1<\gamma^2$ (or $J_2<\gamma^2$)?
- How to find a realization for such filter?

* Considered 0 without loss of generality

** R^{-1} is a covariance matrix

H_∞ Filtering
Finite Horizon, Known Initial Conditions

Theorem[2]: Let the initial conditions be known and $T < \infty$.

1) There exist a filter such that $J_1 < \gamma^2$ if and only if there exists a symmetric matrix $P(t)$ for $t \in [0,T]$ that satisfies

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C^T(t)C(t)P(t) + \frac{1}{\gamma^2}P(t)L^T(t)L(t)P(t) + B(t)B^T(t) \quad \text{with } P(0) = 0. \tag{2}$$

2) Moreover, if it exists, one filter for $J_1 < \gamma^2$ is given by

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + P(t)C^T(t)\left[y(t) - C(t)\hat{x}(t)\right] \quad \text{with } \hat{x}(0) = 0. \tag{3}$$

Null initial conditions considered without loss of generality.

Elements of Proof (I)

The value of the functional J_1 can be written as

$$\frac{\|z-\hat{z}\|_2^2}{\|w\|_2^2}=\gamma^2.$$

From $\Sigma_{\mathcal{G}}$, introducing $\hat{z}(t)=L(t)\hat{x}(t)$ and $\tilde{x}(t)=x(t)-\hat{x}(t)$

$$\frac{1}{\gamma^2}\|L(t)\tilde{x}(t)\|_2^2-\|w\|_2^2=0.$$

Using the $L_2[0,T]$ norm definition (1) we can write

$$\int_0^T\left\{\begin{bmatrix}\tilde{x}(t)^T & w(t)^T\end{bmatrix}\begin{bmatrix}\frac{1}{\gamma^2}L(t)^T L(t) & 0\\ 0 & -I\end{bmatrix}\begin{bmatrix}\tilde{x}(t)\\ w(t)\end{bmatrix}\right\}=0 \qquad (4)$$

Systems' Theory Digression

Dissipativity [6] - The system $G: w \rightarrow z$ with supply rate $s(t)$ is strictly dissipative if there exists a non-negative function $V: x \rightarrow R$ such that

$$V(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))dt > V(x(t_1))$$

for all $t_0 < t_1$ and for all trajectories of the system.

A system is dissipative if can not provide to the environment the same energy that was suplied by the exterior – energy losses.

Examples: electrical circuits, mechanical systems, thermodynamics...

Moreover, if $V(t)$ is differentiable, $\dot{V}(t) < s(t)$ holds.

From $\int_{t_0}^{t_1} -s(u(t), y(t)) + \dot{V}(x(t))dt < 0$, for any $t \in [t_0, t_1]$.

Lyapunov Stability – Second Method

Lyapunov Stability - An equilibrium point $x = 0$ is stable if

$$\forall \varepsilon, t > 0, \exists_{\delta(\varepsilon)>0} : \quad \|\mathrm{x}(0)\| < \delta \Rightarrow \|\mathrm{x}(\mathrm{t})\| < \varepsilon.$$

Lyapunov theorem (second method) - The equilibrium point $x = 0$ is stable if there exists a Lyapunov function that verifies

$$i) \quad V(0) = 0$$

$$ii) \; V(x) > \alpha\|x\|_2$$

$$iii) \; \dot{V}(x(t)) \leq 0, \text{ along all solutions of } S.$$

- Note that V(t)→ ∞ as $\|x\|_2$ → ∞.

- Stability of dynamic systems can be studied, whitout solving the differential equations. Sufficient conditions.

- No systematic method to find a Lyapunov function exists.

Elements of Proof (II)

Re - interpreting (4) and resorting to dissipativity concepts

the Lyapunov candidate function $V(x) = \tilde{x}^T(t)P^{-1}(t)\tilde{x}(t)$ is used

$$\int_0^T \left\{ \begin{bmatrix} \tilde{x}(t)^T & w(t)^T \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma^2} L(t)^T L(t) & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ w(t) \end{bmatrix} + \frac{d}{dt} V(t) dt \right\} = 0$$

From the definition of $\mathcal{G}$ and (3), the error dynamics is

$$\dot{\tilde{x}} = \left(A - PC^T C\right)\tilde{x} + \left(B - PC^T D\right)w$$

Therefore

$$\begin{aligned} \dot{V}(t) = & \ \dot{\tilde{x}}^T(t)P^{-1}(t)\tilde{x}(t) + \tilde{x}^T(t)\dot{P}^{-1}(t)\tilde{x}(t) + \tilde{x}^T(t)P^{-1}(t)\dot{\tilde{x}}(t) \\ = & \ \dot{\tilde{x}}^T(t)P^{-1}(t)\tilde{x}(t) - \tilde{x}^T(t)P^{-1}(t)\dot{P}^{-1}(t)P^{-1}(t)\tilde{x}(t) + \tilde{x}^T(t)P^{-1}(t)\dot{\tilde{x}}(t) \end{aligned}$$

Elements of Proof (III)

Re-arranging the terms results

$$\int_0^T \left\{ \begin{bmatrix} \tilde{x}(t)^T & w(t) \end{bmatrix}^T \bar{P} \begin{bmatrix} \tilde{x}(t) \\ w(t) \end{bmatrix} \right\} dt = 0,$$

where*

$$\bar{P} = \begin{bmatrix} \frac{1}{\gamma^2} L^T L + A^T P^{-1} - 2C^T C - P^{-1}\dot{P}P^{-1} + P^{-1}A & P^{-1}B - C^T D \\ BP - D^T C & -I \end{bmatrix}.$$

Using (2) and cancelling terms

$$\int_0^T \left\{ \begin{bmatrix} \tilde{x}(t)^T & w(t) \end{bmatrix}^T \begin{bmatrix} -C^T C - PBB^T P & P^{-1}B - C^T D \\ BP - D^T C & -I \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ w(t) \end{bmatrix} \right\} dt = 0$$

*Time-dependence omitted for simplicity.

Elements of Proof (IV)

Schur Complements - Given matrices $U \in R^{nxn}$, $V \in R^{nxm}$, $W \in R^{mxn}$, and $Z \in R^{mxm}$, where $Z > 0$ the Schur complements of matrix $\begin{bmatrix} U & V \\ W & Z \end{bmatrix}$ is $U - VZ^{-1}W$.

Can be seen as a generalization to the matrix inversion lemma.

Using Schur complements

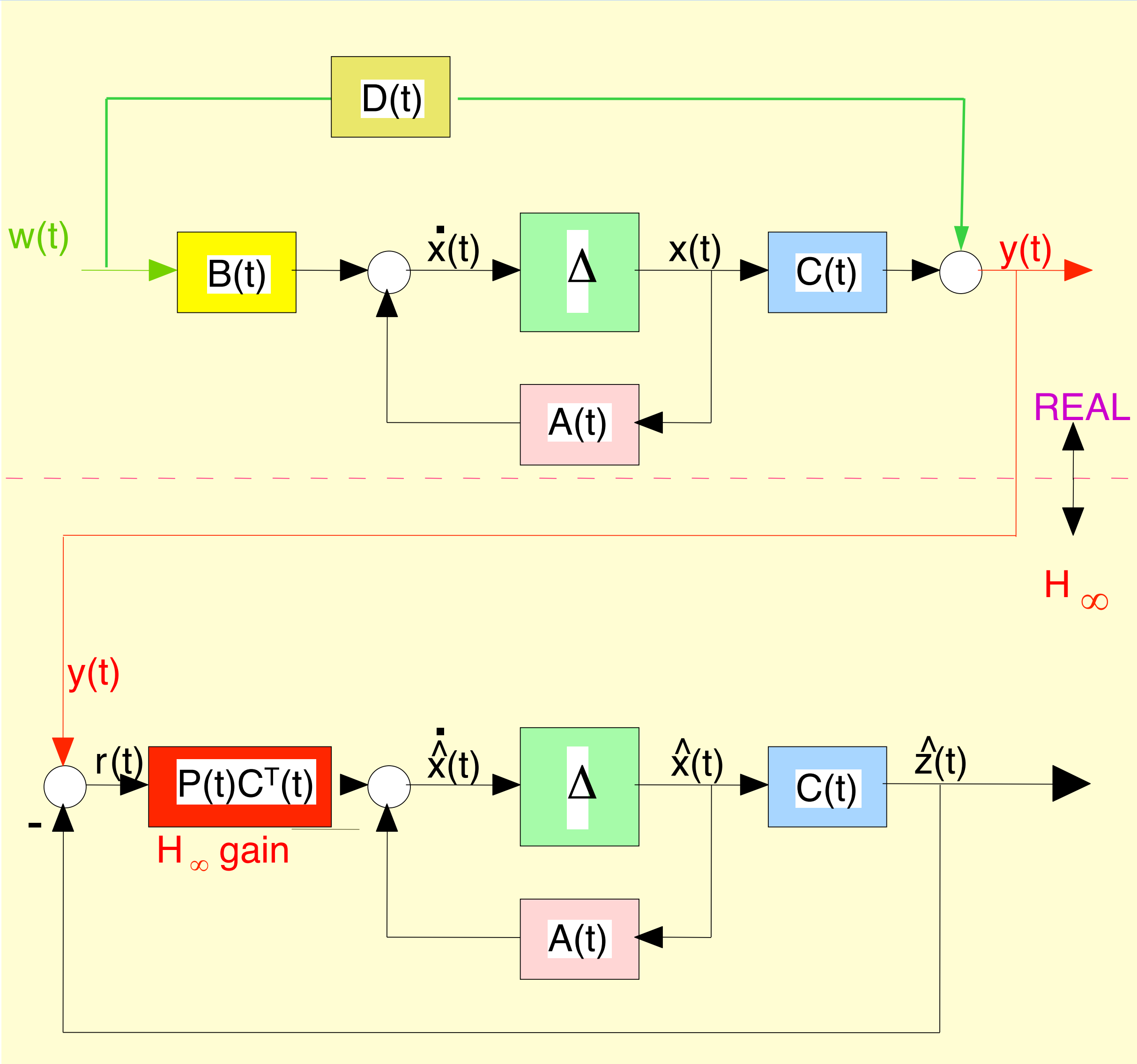
$$-C^T C - PBB^T P^{-1} + \left(B - C^T D\right)\left(BP - D^T C\right) = 0$$

$$-C^T C - PBB^T P^{-1} + PBB^T P^{-1} - P^{-1} BD^T C - C^T DB^T P^{-1} + C^T C = 0$$

using the noises independence and normalization assumptions

$$0 = 0 \quad . \qquad\qquad q.e.d.$$

Visualization of the H_∞ Filter



Discussion

- Optimal structure obtained, similar to LTV Kalman filter

- Unbiased estimator obtained (otherwise $J_1 \rightarrow \infty$, $J_2 \rightarrow \infty$)

- Complete proof is out of scope, but can be obtained

i. using systems' theory [2, 5];
ii. using estimation tools in Krein spaces [3];

- Stationary solutions can also be obtained (finite or infinite horizon cases)

- Modified Riccati equation that

  For $\gamma \Diamond \infty$ degenerates on the Riccati equation in KF
  Provides more robust solutions, for smaller γ
  Unfeasible for $\gamma < \gamma_{min}$!!

H_∞ Smoothing
Finite Horizon, Known Initial Conditions

Theorem[2]: Let the initial conditions be known and $T < \infty$.

1)There exist a smoother such that $J_1 < \gamma^2$ if and only if there exists a symmetric matrix $X(t)$ for $t \in [0, T]$ that satisfies

$$-\dot{X}(t) = A^T(t)X(t) + X(t)A(t) - X(t)B(t)B^T(t)X(t) - \frac{1}{\gamma^2}L^T(t)L(t) + C^T(t)C(t)$$

with $X(T) = 0$.

2)One smoother that minimizes J_1 and verifies $J_1 < \gamma^2$ is

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t)B^T(t) \\ C^T(t)C(t) & -A^T(t) \end{bmatrix} \begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\lambda}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ C(t) \end{bmatrix} y(t)$$

with $\hat{x}(0) = 0$, $\lambda(t) = 0$.

Remarks

- Proof is omitted, see [2] for details.
- The H_∞ smoother structure is equal to the H_2!
- Smoothers for all 4 cases are well known.
- Much more recent results than the H_2 solutions
- Other functionals have already been solved, e.g. mixed H_2/H_∞ Also, solutions for nonlinear cases available
- Now a couples of examples from [5] are included to document some of the results outlined

Examples, from [5]

Example 1: In this example, we demonstrate the reduced peak-error-level of an H_∞-filter, and its inherent robustness. We apply H_∞-optimal and L_2-optimal filters on the following second order resonant system

$$\dot{x} = \begin{bmatrix} 0 & w_n \\ -w_n & -2\xi w_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \quad y = [0 \;\; 1]\, x + n, \quad z = [1 \;\; 0]\, x$$

where w_n and ξ are not certain. The filters were designed for a nominal system with $w_n = 11$ and $\xi = 0.1$. Figure 4.1 depicts the Bode magnitude plot of T_{rd} of the H_∞ and L_2 filters, for the nominal case, and an envelope of T_{rd}, for w_n varying in the range [8.2-13.7] and ξ varying in the range [0.075-0.125].

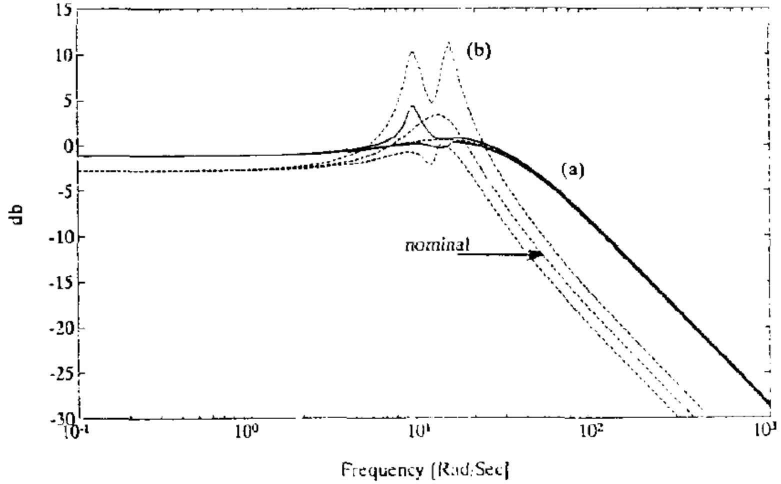


Fig. 4.1: Sensitivity comparison between: (a) the H_∞-filter, and (b) the L_2-filter.

Examples, from [5]

Example 2 (Deconvolution): In this example we demonstrate the tradeoff that exists between the L_2 and H_∞ performance in a continuous-time, steady-state filter design. In the deconvolution problem of Fig. 4.2, we use the noise corrupted measurement of the output of a system, to estimate a regularized version of its input. The regularizing filter is required to make the deconvolution problem well-posed. We look for a filter that achieves $\|T_{rd}\|_\infty < \gamma$ for the following systems:

$$G_s(s)=\frac{100}{s^2+0.4s+100}\ ,\ G_r(s)=\frac{10^4}{s^2+130s+10^4}\ ,\ \mathrm{SNR}=100$$

Recalling that $\gamma \to \infty$ leads to L_2-estimation, we are motivated to check few values of γ. The transfer function T_{rd} for central filters that were designed with different values for γ is depicted in Fig. 4.3. The effect of the design parameter γ on the performance of the above deconvolutor is further emphasized in Fig. 4.4., where the H_∞-norm that is actually achieved is related to the design parameter γ, and the corresponding L_2-norm of T_{rd}. In this typical example, we see that γ is an effective design parameter for values that are near γ_0, where a significant improvement in the L_2 performance can be gained by slightly compromising the H_∞ performance.

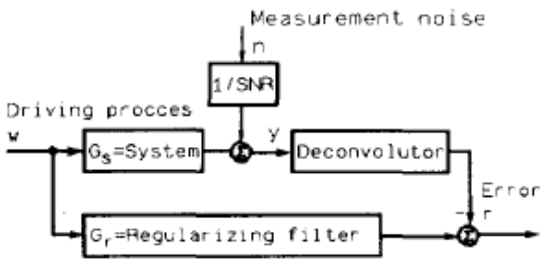


Fig 4.2 The deconvolution scheme

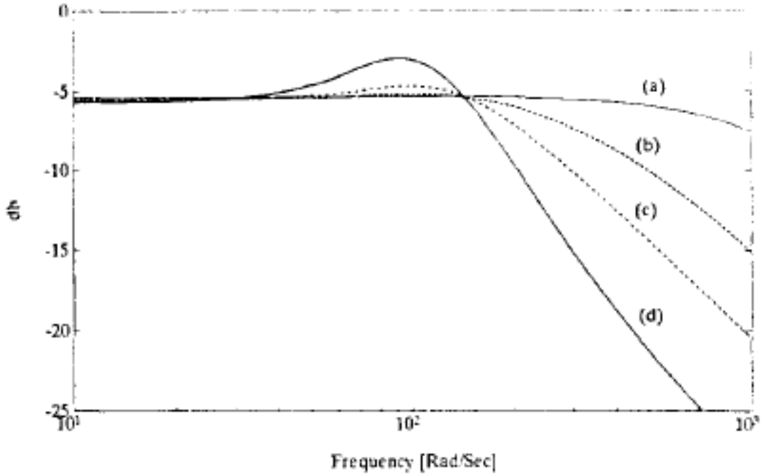


Fig 4.3: The Bode plot of T_{rd} for: (a) $\gamma=\gamma_0$; (*b*) $\gamma=1.02\gamma_0$; (*c*)$\gamma=1.1\gamma_0$; (*d*)$\gamma \to \infty$.

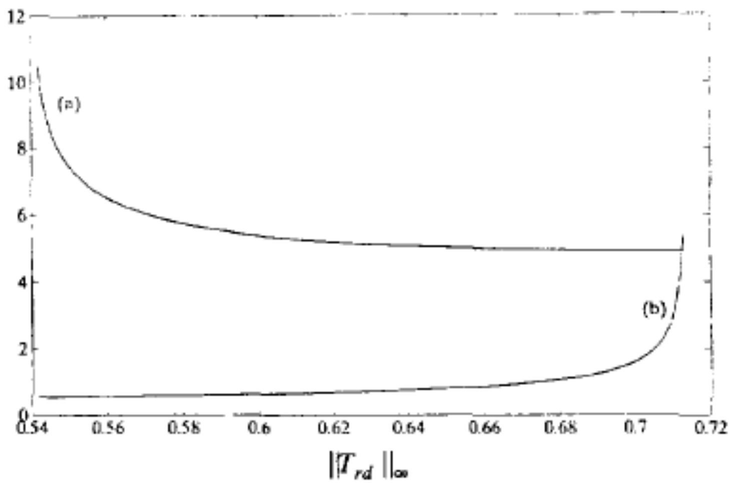


Fig 4.4: The tradeoff between L_2 and H_∞ performance: (a) $\|T_{rd}\|_2$; (*b*) γ.

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New Methods for State Estimation

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DEEC/IST and ISR

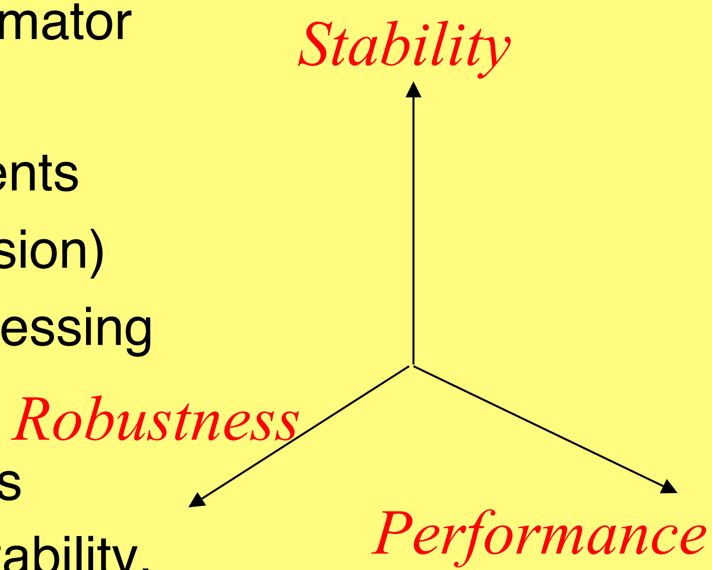
Last Revised: December 14, 2007

Ref. No. NLO#2

Key Challenges in Estimation

Characteristics of the envisioned Estimator

- Reduced computational requirements
- Causal (to be used during the mission)
- Possible to be refined in post-processing



In the linear case, all relevant features are obtained together: exponential stability, optimal performance and robustness (gain and phase margins).

In the nonlinear case no optimal common solution is available.

(e.g. EKF is the performance tentative solution).

Theme

- Stochastic H_2 filtering, prediction, and smoothing problems are only optimal for linear time-varying systems under Gaussian disturbance assumptions with known power spectral densities
- H_∞ allows to lift the noise assumptions for LTV systems
- Real world systems are nonlinear!
- In general, EKF does not guarantee stability, performance, nor robustness
- Nonlinear observers can outperform linear or linearized versions of observers (EKF / SOF), both for structured and unstructured disturbances [1, 2]

Exponential Observers for Linear Systems

Consider the linear system

$$\Sigma_{\mathcal{L}}: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \qquad \begin{matrix} x(t) \in R^n \\ y(t) \in R^p \\ u(t) \in R^m \end{matrix}$$

where the pair (A, C) is observable.

The Luenberger observer, in a deterministic setup, is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K\left(y(t) - C\hat{x}(t)\right)$$

Exponential stability can be proven resorting to the Lyapunov equation

$$(A - KC)^T P + P(A - KC) = -2Q$$

That is, for a positive definite matrix Q there exists a unique positive definite P, such that the above equation is verified.

Observers for Nonlinear Systems

Consider the class of affine nonlinear systems

$$\Sigma_{\mathcal{N}} : \begin{cases} \dot{x}(t) = f(x) + g(x)u(t) \\ y(t) = h(x) \end{cases} \qquad \begin{array}{l} x(t) \in R^n \\ y(t) \in R^p \\ u(t) \in R^m \end{array}$$

where $f(.)$, $g(.)$, and $h(.)$ *are known.*

The suggested Luenberger-like nonlinear observer would be

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + g(\hat{x}(t))u(t) + \mathcal{K}(y(t) - h(\hat{x}(t)))$$

What fails in the stability proof for this nonlinear observer?

Lipschitz Nonlinear Observers

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Last Revised: December 14, 2007

Ref. No. NLO#2

Thau's Observers @ 1973

$$\Sigma_{\mathcal{G}} : \begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) \\ y(t) = Cx(t) \end{cases} \qquad x(t) \in R^n \quad y(t) \in R^p \qquad (1)$$

$A, C,$ and $f(.)$ are known; the pair (A,C) is observable, u is a deterministic input, and $f(.)$ is a Lipschitz time-invariant function, i.e.

$$\| f(x(t)) - f(\hat{x}(t)) \| \leq L \| x(t) - \hat{x}(t) \|$$

Proposed observer (motivated by Luenberger's and Kalman's work)

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + K(y(t) - C\hat{x}(t)) \qquad (2)$$

Thau's Observers @ 1973

Proposition : Given the observability condition for systems of class (1) for any positive definite matrix Q, there exists a unique positive definite matrix P, such that the Lyapunov equation

$$A_0^T P + P A_0 = -2Q, \qquad (3)$$

is verified, where $A_0 = A - KC$.

Main result:

Theorem[2] : For the class of systems (1), if the gain K is selected such that it can make the solution of the Lyapunov equation to satisfy

$$\frac{\lambda_{\min}(Q)}{\|P\|} > L \qquad (4)$$

for all x then the Thau observer (2) is assymptotically stable.

Elements of Proof (I)

Defining $\tilde{x}(t) = x(t) - \hat{x}(t)$, and for the Lyapunov candidate function $\mathrm{V}(\mathrm{t}) = \tilde{x}(t)^T P \tilde{x}(t)$, where $P > 0$ is a symmetric constant matrix, lets apply Lyapunov's second method

$$\dot{V}(t) = \frac{d}{dt}\left[\tilde{x}(t)^T P \tilde{x}(t)\right] = \dot{\tilde{x}}(t)^T P \tilde{x}(t) + \tilde{x}(t)^T P \dot{\tilde{x}}(t)$$

The error dynamics is given by

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x} + f(x(t)) - f(\hat{x}(t)) = A_0 \tilde{x} + f(x(t)) - f(\hat{x}(t)),$$

and

$$\dot{V}(t) = \tilde{x}(t)^T \left(A_0^T P + P A_0\right)\tilde{x}(t) + 2\tilde{x}(t)^T P\left[f(x(t)) - f(\hat{x}(t))\right]$$

Elements of Proof (II)

From the Lyapunov equation (3) one can write

$$\dot{V}(t) = -2\tilde{x}(t)^T Q\tilde{x}(t) + 2\tilde{x}(t)^T P\left[f(x(t)) - f(\hat{x}(t))\right]$$

From the Lipschitz condition

$$\dot{V}(t) \le -2\tilde{x}(t)^T Q\tilde{x}(t) + 2L\|\tilde{x}(t)\|\|P\|\|\tilde{x}(t)\|$$

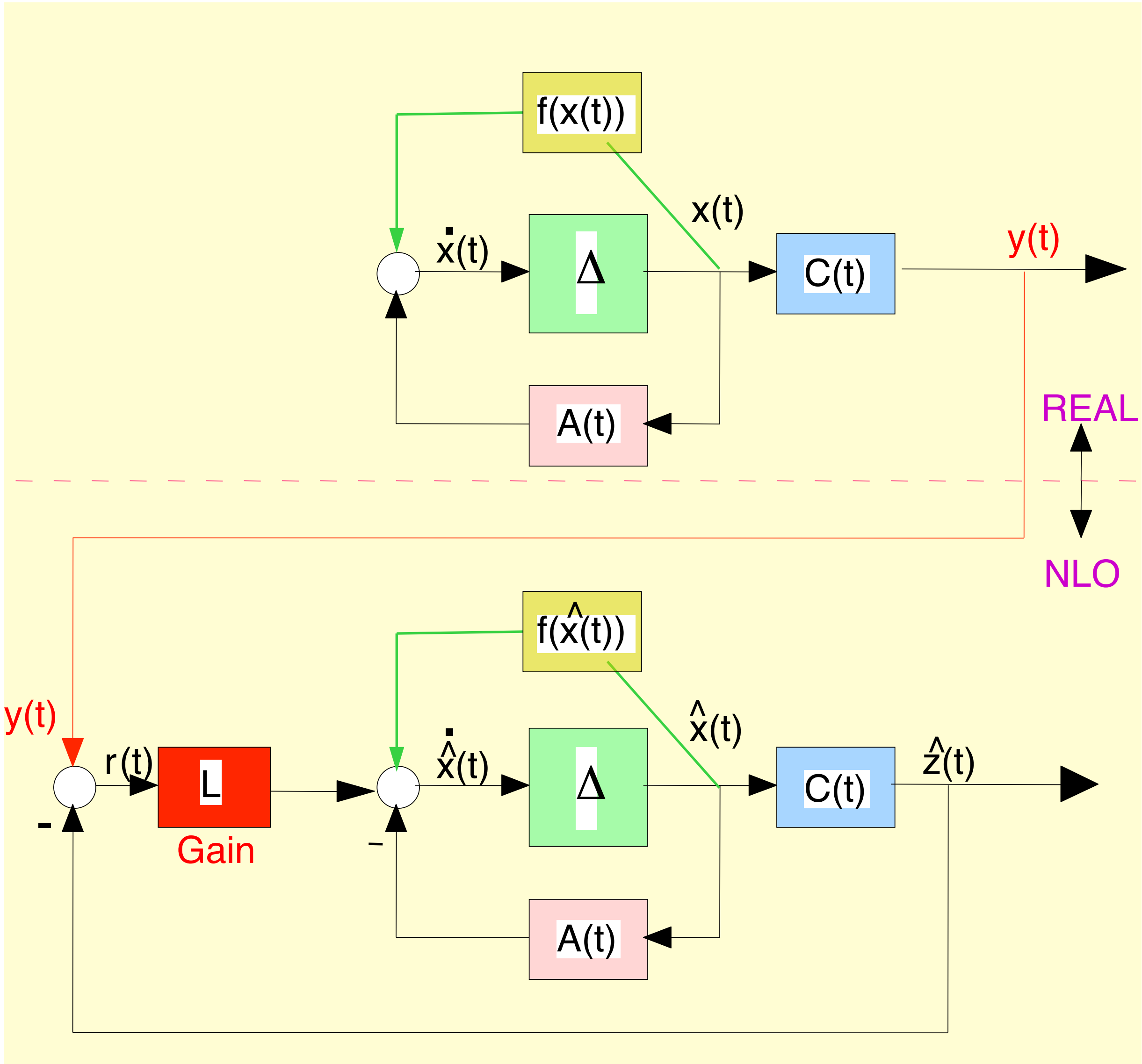
$$\le -2\lambda_{\min}(Q)\|\tilde{x}(t)\|^2 + 2L\|\tilde{x}(t)\|\|P\|\|\tilde{x}(t)\|$$

$$\le -2\left[\lambda_{\min}(Q) - L\|P\|\right]\|\tilde{x}(t)\|^2$$

if $\lambda_{\min}(Q) > L\|P\|$ then $\dot{V}(t) < 0$, i.e. it is enough that (4) be verified.

The choice of K impacts indirectely on P and constitutes a trial and error process. Moreover, it can lead to very conservative results.

Visualization of the Filter



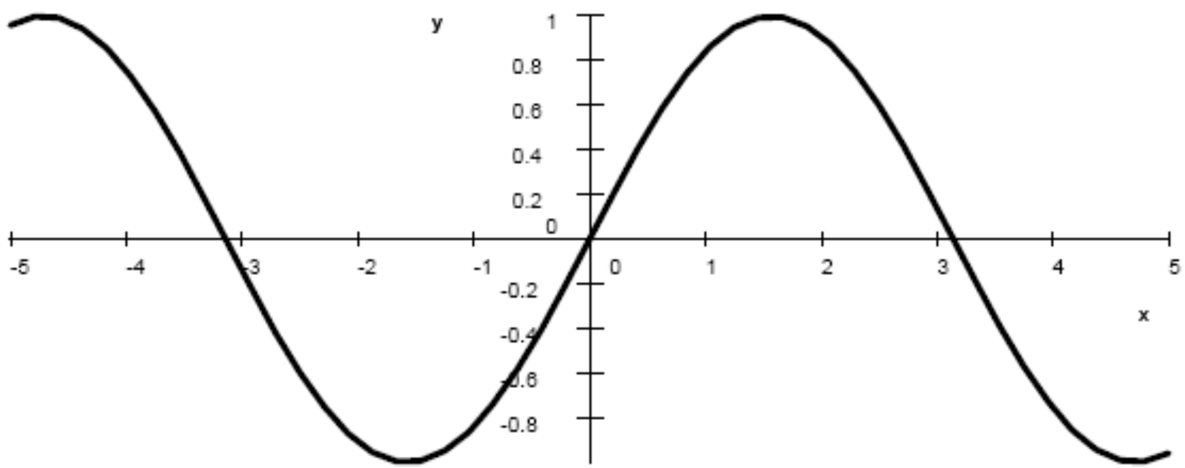
Example with difficulties [4]

Example 1 *nonlinear system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$y = [1, 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Lipschitz constant for $f(x) = \sin x_1$ *is* $L = 1$. $y = \sin x$



Example with difficulties [4]

The minimum occurs when Q=I.

$$\frac{1}{\|P\|} > 1$$

$$\|P\|^2 = \sum_{i,j=|}^{n} p_{ij}^2 = tr(P^T P)$$

we select $P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{3} \end{bmatrix}$, $P = P^T > 0$, $\|P\| = 0.68 < 1$, *but* A_0 *no exists. If we select*

The method is not effective, however given *L* and *P*, is of some use to proove stability of the observer.

$\lambda_{\max}(P) = 1.809 > 1$, *the condition is not satisfied. So the method is difficult.*

Exponential Observers @ 1975

Consider the class of nonlinear systems

$$\Sigma_{\mathcal{K}} : \begin{cases} \dot{x}(t) = f(x(t)) & x(t) \in R^n \\ y(t) = h(x(t)) & y(t) \in R^p \end{cases} \qquad (5)$$

where $f(.)$ and $h(.)$ are continuously differentiable.

The proposed observer has the structure

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + g(y(t), h(\hat{x}(t))) \qquad (6)$$

where $g(.)$ is also continuously differentiable.

An observer should verify in general that

$$g(y(t), h(\hat{x}(t))) = 0 \quad \text{if} \quad h(x(t)) = h(\hat{x}(t)) \qquad (7)$$

however it is impossible to guarantee that $\hat{x}(0) = x(0)$.

Exponential Observers @ 1975

Theorem[3]: Given the nonlinear autonomous systems (5) and (6) and some function *g(.)* satisfying (7), if there exists a scalar function $V(\tilde{x})$, where $\tilde{x} = x - \hat{x}$ and $\rho > 1$ such that

$$\text{a) } V(\tilde{x}) \geq c\|\tilde{x}\|, \text{ for all } \tilde{x} \in R^n, V(0) = 0$$

$$b) \dot{V}(\tilde{x}) \leq -\lambda V(\tilde{x}), \text{ for all } \tilde{x} \in R^n \text{ and for some } \lambda > 0.$$

then the observer (6) is exponentially stable and

$$\|x(t) - \hat{x}(t)\| \leq ke^{-\lambda t}$$

The generic proof for autonomous systems is an immediate consequence of the Lyapunov second method.

Exponential Observers @ 1975

Main result:

Theorem[3] : Given the nonlinear autonomous system (5) and the proposed observer (6) with $g\left(y(t), h\left(\hat{x}(t)\right)\right) = K\left(y(t) - h\left(\hat{x}(t)\right)\right)$, if there exists an n x m gain matrix K, that, given $Q > 0$ there exits $P > 0$ such that

$$\left(\nabla f(x) - K\nabla g(x)\right)^T P + P\left(\nabla f(x) - K\nabla g(x)\right) = -2Q \qquad (8)$$

From any $\hat{x}(t_0)$, an exponential observer (6) is obtained verifying

$$\left\|x(t) - \hat{x}(t)\right\| \le \left(\frac{q_2}{q_1}\right)^{1/2} e^{-\lambda(t - t_0)} \left\|x(t_0) - \hat{x}(t_0)\right\|$$

for all $t > t_0$, where q_1 and q_2 are the smallest and largest eigenvalues of Q, respectively.

Elements of Proof (I)

Defining $\tilde{x}(t) = x(t) - \hat{x}(t)$, the error dynamics is given by

$$\dot{\tilde{x}}(t) = f(x(t)) - f(\hat{x}(t)) - K(y(t) - h(\hat{x}(t)))$$

For the Lyapunov candidate function $\mathrm{V}(\mathrm{t}) = \tilde{x}(t)^T P \tilde{x}(t)$,

where P > 0 is a symmetric constant matrix (2nd method)

$$\dot{V}(t) = \frac{d}{dt}\tilde{x}(t)^T P\tilde{x}(t) = \dot{\tilde{x}}(t)^T P\tilde{x}(t) + \tilde{x}(t)^T P\dot{\tilde{x}}(t)$$

$$= (f(x(t)) - f(\hat{x}) - K(h(x) - h(\hat{x}))^T P\tilde{x}$$

$$+ \tilde{x}^T P(f(x) - f(\hat{x}) - K(h(x) - h(\hat{x}))$$

* Explicit dependence on *t* will be omitted.

Elements of Proof (II)

Resorting to the fundamental theorem of integral calculus

$$f(x) - f(\hat{x}) - K\big(h(x) - h(\hat{x})\big) = \int_0^1 \big(\nabla f(w_s) - K\nabla h(w_s)\big)\tilde{x}\, ds$$

for $w_s = sx + (1-s)\hat{x}$. Therefore

$$\dot{V}(t) = \tilde{x}^T \int_0^1 \big(\nabla f(w_s) - K\nabla h(w_s)\big)^T P + P\big(\nabla f(w_s) - K\nabla h(w_s)\big) ds\, \tilde{x}$$

Using (8) and the fact that $\tilde{x}^T P \tilde{x} > \varepsilon \|\tilde{x}\|^2$ results

$$\dot{V}(t) < -2\varepsilon \tilde{x}^T Q \tilde{x} < -2\frac{q_2}{q_1}\varepsilon V(t), \qquad \text{for } \varepsilon > 0.$$

Example, from [3]

EXAMPLE 1. Consider the following nonlinear system:

$$\dot{x}_1 = x_1, \qquad \dot{x}_2 = x_1 - 2x_2 + e^{-x_2}, \qquad y = x_1 + x_2$$

then the gradients of **f** and **h** are

$$\nabla \mathbf{f} = \begin{bmatrix} 1 & 0 \\ 1 & -2 - e^{-x_2} \end{bmatrix} \quad \text{and} \quad \nabla h = [1 \;\; 1].$$

Let B be a 2×1 constant matrix with elements b_1, b_2 to be determined, then

$$\nabla \mathbf{f} - B\nabla \mathbf{h} = \begin{bmatrix} 1 - b_1 & -b_1 \\ 1 - b_2 & -2 - e^{-x_2} - b_2 \end{bmatrix}.$$

The symmetric part of $\nabla \mathbf{f} - B\nabla h$ is

$$(\nabla \mathbf{f} - B\nabla h)_{sym} = \begin{bmatrix} 1 - b_1 & \frac{1}{2}(1 - b_2 - b_1) \\ \frac{1}{2}(1 - b_2 - b_1) & -2 - e^{-x_2} - b_2 \end{bmatrix}.$$

Thus, if the matrix is selected to be $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ then

$$(\nabla \mathbf{f} - B\nabla h)_{sym} = \begin{bmatrix} -1 & 0 \\ 0 & -1 - e^{-x_2} \end{bmatrix}. \qquad (16)$$

In (16), the two eigenvalues are -1 and $-1 - e^{-x_2}$, so the maximum eigenvalue is -1, i.e.,

$$\mathbf{w}^T(\nabla \mathbf{f} - B\nabla h)\mathbf{w} = \mathbf{w}^T(\nabla \mathbf{f} - B\nabla h)_{sym}\mathbf{w} \leqslant (-1) \cdot \| \mathbf{w} \|^2.$$

Now by Theorem 2 we have that

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) + \begin{bmatrix} 2 \\ -1 \end{bmatrix} (\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{z}))$$

or

$$\dot{z}_1 = z_1 + 2[y - (z_1 + z_2)]$$
$$\dot{z}_2 = z_1 - 2z_2 + e^{-z_2} - [y - (z_1 + z_2)]$$

is an exponential observer with $z_1(0)$, $z_2(0)$ arbitrarily given for the system of the example.

Thau's Observers @ 1975

Consider the class of autonomous nonlinear systems

$$\Sigma_{\mathcal{G}} : \begin{cases} \dot{x} = Ax + \phi(x, y, \dot{y}) & x : x(t) \in R^n \\ y = Cx & y : y(t) \in R^p \end{cases} \quad (9)$$

where A and C are known and $\phi(.)$ is Lipschitz in its arguments and verifies

$$\phi(x, y, \dot{y}) = \phi_1(y) + \nabla\phi_2(y)\dot{y} + \phi_3(x) \quad (10)$$

where $\phi_1(.), \phi_3(.) \in C^1, \phi_2(.) \in C^2$ and such that

$$C\nabla\phi_2(y)\dot{y} = 0.$$

Exponential Observers @ 1975

Theorem[3] : For the nonlinear autonomous systems (9), verifying (10) if

a) the pair *(A,C)* is observable

b) there exist $P > 0,\ Q > 0$, and a gain vector K such that

$$(A - KC)^T P + P(A - KC) = -2Q$$

and

$$\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > \|\nabla \phi_3\|_\infty \qquad (11)$$

then there exist an exponential observer for system (9).

Elements of Proof (I)

Define the new variable $w = x - \phi_2(\mathrm{y})$, and note that

$$\dot{\phi}_2 = \nabla\phi_2(y)\dot{y}$$

The derivative, relative to time, of this new signal verifies

$$\dot{w} = Ax + \phi_1(y) + \nabla\phi_2(y)\dot{y} + \phi_3(x) - \nabla\phi_2(y)\dot{y}$$

That can be simplified to

$$\dot{w} = Aw + \phi_1(y) + A\phi_2(y) + \phi_3(w + \phi_2)$$

Considering that $y_1 = Cw = y - C\phi_2(y)$, the observer

$$\dot{\hat{w}} = A\hat{w} + \phi_1(y) + A\phi_2(y) + \phi_3(\hat{w} + \phi_2) - K(y_1 - C\hat{w})$$

is proposed.

Elements of Proof (II)

Defining $e = w - \hat{w}$, the error dynamics is given by

$$\dot{e} = (A - KC)e + \phi_3(w + \phi_2) - \phi_3(\hat{w} + \phi_2)$$

For the Lyapunov candidate function $\mathrm{V}(\mathrm{t}) = e^T Pe$,

where P > 0 is a symmetric constant matrix (2nd method)

$$\dot{V}(t) = \dot{e}^T Pe + e^T P\dot{e}$$

$$= -2e^T Qe +$$

$$(\phi_3(w + \phi_2) - \phi_3(\hat{w} + \phi_2))^T Pe + e^T P(\phi_3(w + \phi_2) - \phi_3(\hat{w} + \phi_2))$$

Elements of Proof (III)

$$\dot{V} \leq -2\lambda_{\min}(Q) + \left(\int_0^1 \nabla \phi_3 (w_s + \phi_2(y)) ds e \right)^T Pe$$

$$+ e^T P \left(\int_0^1 \nabla \phi_3 (w_s + \phi_2(y)) ds e \right)$$

$$\leq \left[-2\lambda_{\min}(Q) + 2\|P\| \|\nabla \phi_3\|_\infty \right] \|e\|^2$$

From this relation (11) is immediate.

Example, from [3]

EXAMPLE 3. We consider a simple pendulum with viscous damping and without driving torque.

$$\ddot{x} + a_2\dot{x} + a_3 \sin x = 0, \qquad y = x, \tag{28}$$

where a_2, a_3 are constants.

Let us rewrite Eq. (28) as a vector differential equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -a_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -a_3 \sin x_1 \end{bmatrix},$$

$$y = [1 \;\; 0]\,\mathbf{x}.$$

Now the linear part of this system is observable and we denote the nonlinear part by

$$\phi(\mathbf{x}) = \begin{bmatrix} 0 \\ -a_3 \sin x_1 \end{bmatrix} \quad \text{so} \quad \nabla\phi(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ -a_3 \cdot \cos x_1 & 0 \end{bmatrix}.$$

If $a_2 = \frac{3}{4}$, $a_3 = \frac{1}{8}$ then the following matrices

$$B = \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{29}$$

satisfy Condition (b) of Theorem 3. So the observer (23) with B given by (29) and $\phi_1 = 0$, $\phi_2 = 0$ is an exponential observer and the ϵ of (27) is $\frac{1}{[illegible]}(5 - (5)^{1/2}) > 0$.

Lipschitz Observers @ 1998

Consider the class of non-autonomous nonlinear systems

$$\Sigma_{\mathcal{G}} : \begin{cases} \dot{x}(t) = Ax(t) + \phi(x(t), u(t)) \\ y(t) = Cx(t) \end{cases} \qquad \begin{matrix} x(t) \in R^n \\ y(t) \in R^p \\ u(t) \in R^m \end{matrix} \tag{12}$$

A, C, and $f(.)$ are known; (A, C) is observable, u is a deterministic input, and $\phi(.)$ is a Lipschitz time-invariant function, i.e.

$$\left\| \phi(x(t), u(t)) - \phi(\hat{x}(t), u(t)) \right\| \leq \gamma \left\| x(t) - \hat{x}(t) \right\| \tag{13}$$

Proposed observer (motivated by Luenberger's and Kalman's work)

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \phi(\hat{x}(t), u(t)) + K(y(t) - C\hat{x}(t)) \tag{14}$$

Lipschitz Observers @ 1998

The estimation error dynamics is

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) + [\phi(x(t), u(t)) - \phi(\hat{x}(t), u(t))]$$

and the following major result holds:

Theorem [5] : For the nonlinear non - autonomous systems (12), verifying (13), if the observer given by (14) satisfies

a) the pair *(A,C)* is observable

b) the gain *K* can be chosen such that as to ensure

$$\min_{\omega \in R^+} \sigma_{\min}(A - KC - Ij\omega) > \gamma \qquad (15)$$

then it is asymptotically stable.

Elements of Proof (I)

The proof is done in three parts (see [5] for details):

i) If $\min_{\omega \in R^+} \sigma_{\min}\left(A - KC - j\omega I\right) > \gamma$, then there exits $\varepsilon > 0$ such that the matrix

$$H = \begin{bmatrix} \left(A\text{-}KC\right) & \gamma^2 I \\ -I - \varepsilon I & -\left(A\text{-}KC\right)^T \end{bmatrix}$$

has no imaginary eigenvalues.

ii) If $\left(A - KC\right)$ is stable then there exists a $P > 0$ such that there exits a solution to the equation

$$\left(A - KC\right)^T P + P\left(A - KC\right) + \gamma^2 PP + I + \varepsilon I = 0$$

Elements of Proof (II)

cont...

iii) Defining $\tilde{x}(t) = x(t) - \hat{x}(t)$, the error dynamics is given by

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) + \left[\phi(x(t), u(t)) - \phi(\hat{x}(t), u(t))\right]$$

For the Lyapunov candidate function $\mathrm{V}(\mathrm{t}) = \tilde{x}(t)^T P \tilde{x}(t)$,

where $\mathrm{P} > 0$ is a symmetric constant matrix (2nd method)

$$\dot{V}(t) = \frac{d}{dt}\tilde{x}(t)^T P \tilde{x}(t) = \dot{\tilde{x}}(t)^T P \tilde{x}(t) + \tilde{x}(t)^T P \dot{\tilde{x}}(t)$$

$$= (A - KC)^T P \tilde{x} + \tilde{x}^T P (A - KC) +$$

$$2\tilde{x}^T P\left[\phi(x(t), u(t)) - \phi(\hat{x}(t), u(t))\right]$$

Using the properties introduced before, results:

$$\dot{V}(t) \leq \tilde{x}^T \left[(A - KC)^T P + P(A - KC) + \gamma^2 PP + I\right]\tilde{x}$$

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Nonlinear Observers with Linearizable Error Dynamics

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Ref. No. NLO#3

Theme

Consider the class of non-autonomous nonlinear systems

$$\Sigma_{\mathcal{G}} : \begin{cases} \dot{x} &= f(x,u) \\ y &= h(x) \end{cases}$$

where $x(t) \in R^n$ is the vector of system states, $u(t) \in R^m$ are the inputs, and $y(t) \in R^p$ are the system outputs expressed as a column vector and abb. as x, u, and y.

The search for a "very special" property...

Given a nonlinear system,with nonlinear measurements of the state available, find a coordinate transformation that renders the dynamics and the output linear on the new coordinates!!!

(except for a nonlinear output injection term)

Theme

- Challenge for the control problem set at IFAC 1978 (Helsinki) by Roger Brockett to Arthur Krener [1]

- Control problem well understood (during the 80s), see [1, 2] for a survey on the new techniques: feedback linearization, input-output linearization, backstepping , zero dynamics, …

- Harder to be solved for nonlinear observers

Relevant questions:

- Conditions for the existence of such transformation

- Synthesis methods (complexity)

- Robustness relative to unmodelled dynamics...

Krener and Isidori @ 1983

$$z = \varphi(x)$$
$$w = \gamma(y)$$

1) Find $\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \rightarrow \begin{cases} \dot{z} = Az + Bu + \alpha(y,u) \\ w = Cz \end{cases}$!

2) Design an observer of the form

$$\dot{\hat{z}} = A\hat{z} + Bu + \alpha(y,u) + K(w - C\hat{z}).$$

3) Error dynamics $\tilde{z} = \hat{z} - z$ (assume observability of *(A,C)*)

$$\dot{\tilde{z}} = (A - KC)\tilde{z}$$

- First systematic approach [3] that resorts to a nonlinear state transformation to linearize the original system up to an additional output injection term

Krener and Isidori @ 1983

The proposed solution proposed is composed of three steps (see [1, 3] for details):

1) A set of partial differential equations (PDE) must be solved to find $\gamma(y)$

2) The integrability of conditions for this PDE involve the vanishing of a pseudo-curvature

3) A coordinate transformation $z=\phi(x)$ can be obtained after a set of PDEs is solved, resorting to conditions on the Lie derivatives of the outputs

"The process is more complicated then feedback linearization and even less likely to be successful..." in [1]

Kazantzis and Kravaris @ 1997

Slitghly different objective:

Given a nonlinear system, with nonlinear measurements of its state available, find a nonlinear state transformation that renders the observer error dynamics linear!!!

(except for a nonlinear output injection term)

Consider the class of autonomous nonlinear systems

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases} \quad (2)$$

where $f : R^n \to R^n$, $h : R^n \to R^m$ are analytic vector fields.

The origin $x = 0$ is an equilibrium point, $f(0) = 0$, and $h(0) = 0$.

Kazantzis and Kravaris @ 1997

Motivated by Luenberger's original ideas on the linear observer design problem, the proposed approach will try to reconstruct a nonlinear invertible function $z = \theta(x)$.

with time derivative that verifies

$$\dot{z} = \frac{\partial\theta}{\partial x}\frac{dx}{dt} = \frac{\partial\theta}{\partial x}\dot{x} = Az - \beta(y)$$

Using the definition of the system (2) and for the intended dynamics, the following PDE must be verified

$$\frac{\partial\theta}{\partial x}(x)f(x) = A\theta(x) - \beta(h(x)) = Az - \beta(y), \qquad (3)$$

Kazantzis and Kravaris @ 1997

Assumption A1 : The Jacobian F of the vector field $f(x)$ evaluated at $x = 0$ has eigenvalues k_i, $i = 1,\ldots, n$ with

$$0 \notin ConvexHull\{k_1,\ldots,k_n\}$$

Assumption A2 : Denoting the m x n matrix $H = \left[\frac{\partial h_1}{\partial x}^T(0) \quad \ldots \quad \frac{\partial h_m}{\partial x}^T(0)\right]$ it is assumed that the m x n matrix

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$ has rank n.

A2 essecially states that (3) is locally stable.

Kazantzis and Kravaris @ 1997

Lyapunov's Auxiliary Theorem: Consider the first-order system of quasi-linear differential equations

$$\frac{\partial w}{\partial x}\varphi(x,w)=\psi(x,w) \quad (4)$$

with $\varphi(0,0)=0$, $\psi(0,0)=0$, and $\frac{\partial \varphi}{\partial w}(0,0)=0$, where w is the unknown. Under assumptions ***A1***, ***A2***, and independence of the eigenvalues of $\frac{\partial \varphi}{\partial x}(0,0)$ relative to the ones of $\frac{\partial \psi}{\partial w}(0,0)$, then the above system of PDEs admits a unique analytic solution, in the neighborhood of $x=0$.

The novelty in [4] was the use of this result app to (3) to guarantee the existence and uniqueness of solutions.

Kazantzis and Kravaris @ 1997

Solution for the system of PDEs $\frac{\partial \theta}{\partial x}(x)f(x) = A\theta(x) - \beta(y)$

Linear Method : *For* $\frac{\partial w}{\partial x}\varphi(x,w) = \psi(x,w)$, consider the linear case

$$\varphi(x,w) = Fx$$

$$\psi(x,w) = Aw - BHx$$

with $F, A, B = \frac{\partial \beta}{\partial x}(0)$, and H constant matrices. Then the unique solution of (3) is $w = Tx$, where T is the solution of

$$TF + AT = BH. \qquad (5)$$

Unique solution when F and A do not have common eigenvalues.

Kazantzis and Kravaris @ 1997

Theorem: Consider that for the dynamic system (2) ***A1*** and ***A2*** hold and the n - th order dynamic system of the form

$$\dot{z} = Az - \beta(y)$$

where A is Hurwitz, $B = \frac{\partial \beta}{\partial x}(0)$, and (A,B) is controlable.

Then there exists a locally invertible nonlinear map $z = \theta(x)$ that makes the dynamic system above a full order observer.

Why is this method or structure acttractive?...

Kazantzis and Kravaris @ 1997

Theorem: Let $z = \theta(x)$ be an invertible solution of (3). The system

$$\dot{\hat{x}} = f(\hat{x}) - \left[\frac{\partial\theta}{\partial\hat{x}}(\hat{x})\right]^{-1}\left(\beta(y) - \beta(h(\hat{x}))\right) \qquad (3)$$

is an asymptotic full-order observer for (2) such that

$$\frac{d}{dt}(\hat{z} - z) = \frac{d}{dt}\left(\theta(\hat{x}) - \theta(x)\right) = A\left(\theta(\hat{x}) - \theta(x)\right) = A(\hat{z} - z).$$

Proof (brief) :

$$\frac{d}{dt}\left(\theta(\hat{x}) - \theta(x)\right) = \frac{\partial\theta}{\partial\hat{x}}\dot{\hat{x}} - \frac{\partial\theta}{\partial x}\dot{x} = \frac{\partial\theta}{\partial\hat{x}}\left(f(\hat{x}) - \left[\frac{\partial\theta}{\partial\hat{x}}(\hat{x})\right]^{-1}\left(\beta(y) - \beta(h(\hat{x}))\right)\right) - \frac{\partial\theta}{\partial x}f(x) =$$

$$A\theta(\hat{x}) - \beta\left(h(\hat{x})\right) - \left(\beta(y) - \beta\left(h(\hat{x})\right)\right) - A\theta(x) + \beta(y) = A\left(\theta(\hat{x}) - \theta(x)\right)$$

Kazantzis and Kravaris @ 1997

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases}$$

Observer design
In original coordinates

$$\dot{\hat{x}} = f(\hat{x}) - \left[\frac{\partial \theta}{\partial \hat{x}}(\hat{x})\right]^{-1} \left(\beta(y) - \beta(h(\hat{x}))\right)$$

$$z = \theta(x).$$

$$\hat{z} = \theta(\hat{x}).$$

Observer in new coordinates

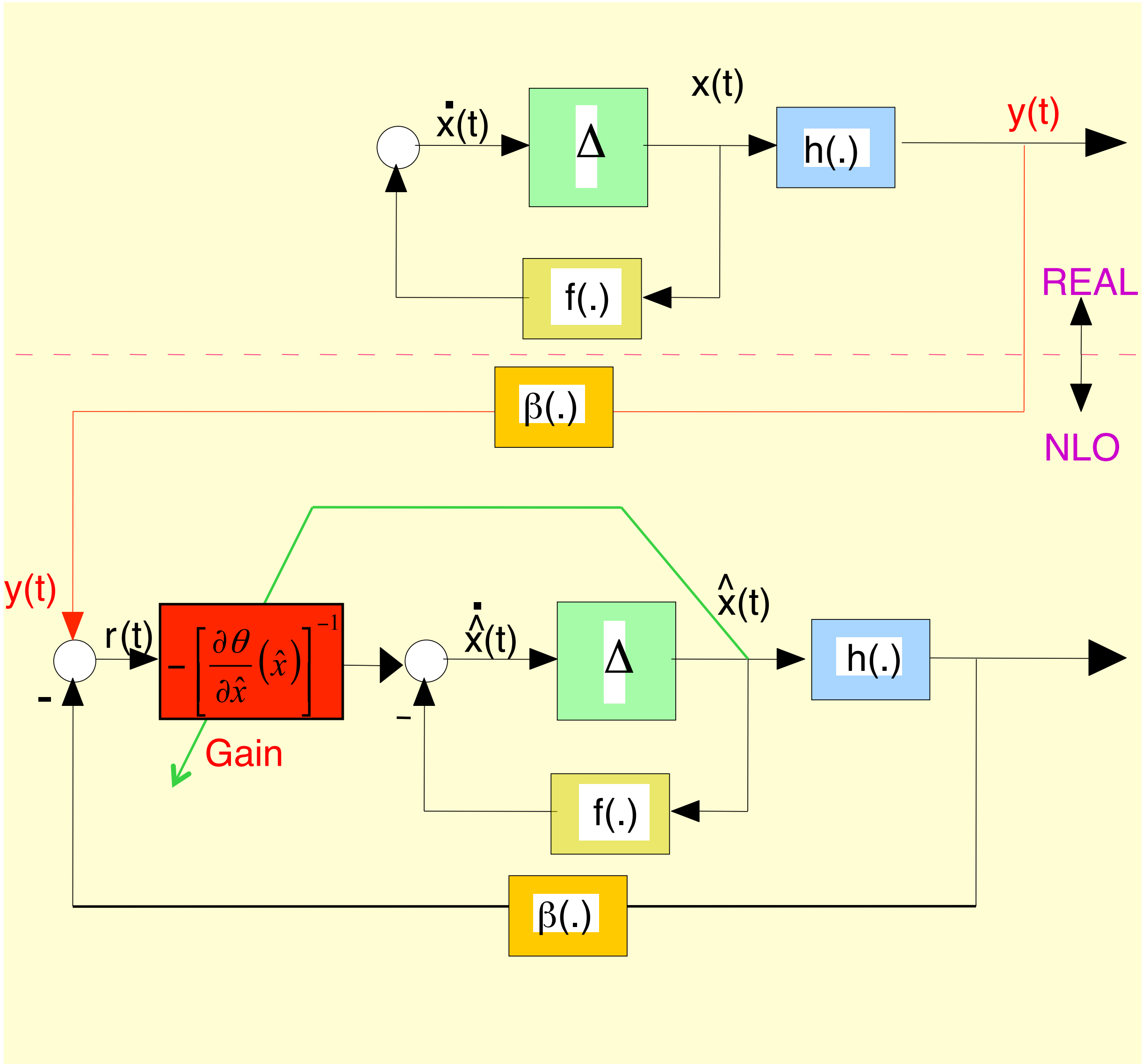
$$\dot{z} = Az - \beta(y)$$

$$\dot{\hat{z}} = A\hat{z} - \beta(y)$$

$$\dot{\tilde{z}} = A\tilde{z}$$

Linear error dynamics

Visualization of the Filter



Krener and Xiao @ 2002 [5]

Converse Theorem: Consider the class of nonlinear systems

$$\dot{z} = g(z)$$
$$y = h(z)$$

where g and h are continuous vector fields and $g(0) = h(0) = 0$.

If there exists a nonlinear observer $\dot{\hat{z}} = \hat{g}(\hat{z}, y)$ such that the error dynamics $\tilde{z} = z\text{-}\hat{z}$ is linear, i.e. $\dot{\tilde{z}} = A\tilde{z}$, then there exists a continuous vector field $\beta: R^p \to R^n$ such that

$$g(z) = Az - \beta(h(z))$$
$$\hat{g}(\hat{z}) = A\hat{z} - \beta(y)$$

Example I from [5]

3. Examples. As discussed in the introduction, there are distinct advantages to considering *nonlinear output injection* $\beta(y)$. It is desirable that θ be a diffeomorphism over as large a range as possible, for this is the domain of convergence of the observer. Nonlinear output injection can make θ a global diffeomorphism.

To illustrate this, we consider a Duffing oscillator

$$\ddot{x} = x - x^3,$$
$$y = x,$$

which is equivalent to the planar system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1^3 \end{bmatrix},$$
$$y = x_1.$$

Example I from [5]

This system is trivially transformed into a linear system with output injection (1.2)

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \begin{bmatrix} -2y \\ -3y + y^3 \end{bmatrix}$$

by

$$\theta(x) = x,$$
$$\beta(y) = \begin{bmatrix} -2y \\ -3y + y^3 \end{bmatrix}.$$

Notice that β is nonlinear and θ is trivially a global diffeomorphism. The observer (1.4) is

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} - \begin{bmatrix} -2y \\ -3y + y^3 \end{bmatrix},$$

and the error dynamics

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

is linear and exponentially stable with poles at $-1 \pm i$.

Example I from [5]

The example is trivial but illustrates two important facts. The first is the advantage of allowing nonlinear β. We could take it to be linear,

$$\beta(y) = \begin{bmatrix} -2 \\ -3 \end{bmatrix} y,$$

and still solve the PDE (1.3) for θ. But the solution might be hard to find, it could have an infinite power series expansion, and it might not be a global diffeomorphism.

The second point is that the Duffing oscillator is truly nonlinear; it has three equilibria and two homoclinic orbits, and the rest of the trajectories are limit cycles. Yet it is possible to build a globally convergent error with linear error dynamics.

Run demo!

Example II from [5]

Next we consider a Van der Pol oscillator,

$$\ddot{x} = -(x^2 - 1)\dot{x} - x,$$
$$y = x,$$

which is equivalent to the planar system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ x_1^2 x_2 \end{bmatrix},$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Now we have

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + x_2 - x_1^2 x_2 \end{bmatrix}, \qquad h(x) = x_1,$$
$$F = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Example II from [5]

We look for a nonlinear coordinate transformation $z = \theta(x)$ such that in the new coordinates z, the system can be described in the form

$$\dot{z} = Az - \beta(y).$$

Let us choose A and β to be

$$A = \begin{bmatrix} b_1 & 1 \\ b_2 - 1 & 1 \end{bmatrix}, \qquad \beta(y) = \begin{bmatrix} b_1 y + \frac{y^3}{3} \\ b_2 y + \frac{y^3}{3} \end{bmatrix},$$

where b_1, b_2 are constants such that $1 + b_1 < 0$, $b_1 - b_2 + 1 > 0$. Clearly, A is stable since $\text{trace}(A) = 1 + b_1 < 0$ and $\det(A) = b_1 - b_2 + 1 > 0$. Moreover $A = F + BH$ with $B = [b_1, b_2]'$. The solution of (1.3) in this case is given by

$$\theta(x) = \begin{bmatrix} x_1 \\ x_2 + \frac{x_1^3}{3} \end{bmatrix}.$$

Note that θ is polynomial and *globally invertible* on $\mathbf{R}^2$. This is because we chose a nonlinear β. The resulting observer is again globally convergent with exponentially stable linear error dynamics in $\tilde{z}$ coordinates despite the nonlinearities of the Van der Pol oscillator. See Figure 1.

Example II from [5]

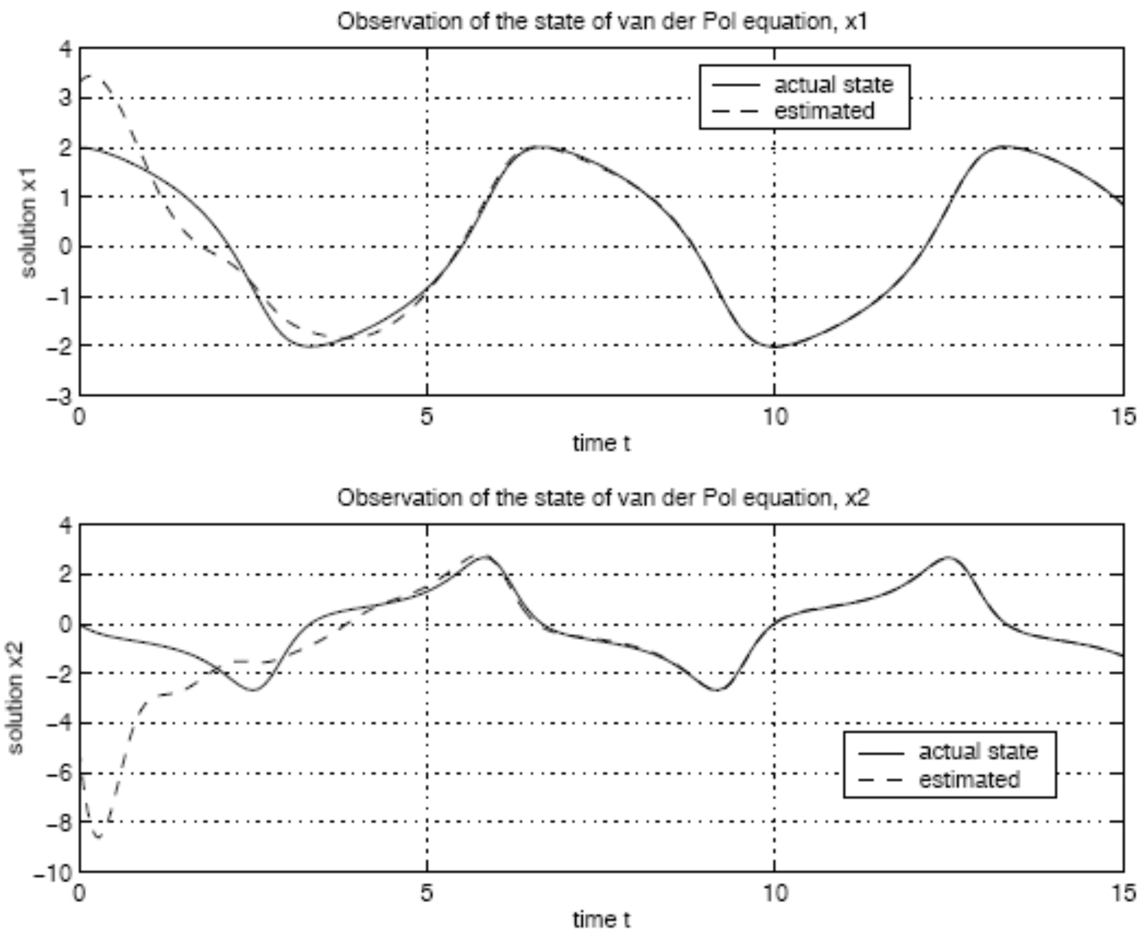


FIG. 1. *Observation of Van der Pol oscillator.*

Krener and Xiao @ 2002 [5]

Consider the class of non - autonomous nonlinear systems

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$$

where $f : R^n \rightarrow R^n$, $h : R^n \rightarrow R^m$ are analytic vector fields.

The origin $x = 0$ is an equilibrium point, $f(0) = 0$, and $h(0) = 0$.

Assume that the following relations are verified

$$\begin{cases} f(x,u) = f_0(x) + f_1(x,u) \\ h(x,u) = h_0(x) + h_1(x,u) \end{cases}$$

where $f_1(x,0) = 0, h_1(x,0) = 0$ and $f_0(0) = h_0(0) = 0$.

Let $F = \dfrac{\partial f_0}{\partial x}(0), H = \dfrac{\partial h_0}{\partial x}(0)$, and $B = \dfrac{\partial \beta}{\partial x}(0)$.

Krener and Xiao @ 2002 [5]

Applying the previous results, under the same technical conditions to the pair f_0, h_0, and for the nonlinear coodinate transformation $z = \varphi(x)$, $i.e$

$$\frac{\partial \varphi}{\partial x}(x) f_0(x) = A\varphi(x) - \beta(h_0(x))$$

requires the solution of the equation

$$TF = TA - BHT.$$

The following nonlinear observer is obtained

$$\dot{\hat{x}} = f(\hat{x}, u) - \left[\frac{\partial \varphi}{\partial \hat{x}}\right]^{-1} \left(\beta(y) - \beta(h(\hat{x}, u))\right)$$

Krener and Xiao @ 2002 [5]

Let $e = \varphi(\hat{x}) - \varphi(x)$. Then e verifies the differential equation

$$\dot{e} = \frac{\partial\varphi}{\partial\hat{x}} f(\hat{x},u) - \left(\beta(y) - \beta(h(\hat{x},u))\right) - \frac{\partial\varphi}{\partial x} f(x,u)$$

$$= \frac{\partial\varphi}{\partial\hat{x}}\left(f_0(\hat{x}) + f_1(\hat{x},u)\right) - \left(\beta(y) - \beta(h(\hat{x},u))\right) - \frac{\partial\varphi}{\partial x}\left(f_0(x) + f_1(x,u)\right)$$

Given the relations verified for the pairs f_0, h_0 in PDE form, i.e

$$\frac{\partial\phi}{\partial x}(x) f_0(x) = A\phi(x) - \beta(h_0(x)), \qquad \frac{\partial\phi}{\partial x}(\hat{x}) f_0(\hat{x}) = A\phi(\hat{x}) - \beta(h_0(\hat{x}))$$

$$\dot{e} = Ae + N(\hat{x},u) - N(x,u)$$

where $N(x_i,u) = \frac{\partial\varphi}{\partial x}(x_i) f_1(x_i,u) + \beta(h(x_i,u)) - \beta(h_0(x_i))$.

Krener and Xiao @ 2002 [5]

If we further assume that f_1 is locally Lipschitz then

$$\left\|N(x_1,u)-N(x_2,u)\right\| \leq L(u)\left\|x_1-x_2\right\|$$

A design similar to the ones introduced in the previous lesson is possible, i.e. for A Hurwitz, then for any $Q \geq 0$ then there exist a $P \geq 0$ such that

$$A^T P + PA = -2Q$$

And for the Lyapunov candidate function $V(e) = e^T Pe$ we have that

$$\dot{V}(e) \leq \left(-2\lambda_{\min}(Q) + 2L(u)\lambda_{\max}(P)\right)\|e\|$$

Hence if
$$\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > L(u)$$

then $e = 0$ is locally asymptotically stable.

Krener and Xiao @ 2002 [5]

- Other methods to solve the PDE could be used [5]
- Design method easier to be accomplished than [3]
- The authors of [4] claim “to be able to do so for all linearly observable, real analytic systems whose spectrum of the linear part lies wholly in the right half complex plane”.
- Krener and Xiao extended the method to arbitrary specta [5] (the Siegel domain) and showed that the sufficient conditions were also necessary.
- Discrete time [6] and state and disturbance estimation design [7] versions became available

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