Advanced Signal Processing Aiding Techniques for Inertial Navigation Systems

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Abstract— **This paper describes the development of an USBL (Ultra-Short Baseline) aided Inertial Navigation Systems (INS) Acoustic Positioning and Localization system, for applications in underwater missions intended to help marine biologists on their experiences with marine fauna.**

After the problem formulation and the characterization of the underwater scenario, an algebraic solution is suggested for a specific situation of stationarity, which relies exclusively on direction and Doppler velocity measurements. Then filtering techniques based on the principle of Kalman filter are also proposed. The main objectives include position and velocity estimation of the parties in the underwater operation. The observability of the dynamic system, which assembles on its equations all the variables of the existing problem, is also studied.

Briefly, an hardware architecture is disclosed that supports the USBL structure and the acquisition and sampling of acoustic signals. To these acoustic signals, after an appropriate processing, information concerning directions and distances is extracted. The architecture was tested successfully in underwater environments, thence the obtained results allow to confirm its correct operation.

I. INTRODUCTION

In what may concern underwater acoustics, during the last decades, efforts have often been concentrated in large scale observation of marine species, as reported in [1] and [2]. In order to attain satisfiable localization results, whether absolute or relative, the species under study should spend most of their time at the surface, which is a non-controllable and quite unpredictable behavior of the animals. Emerging is a necessary condition so that RF communications can happen. Moreover, marine animal studies require an innate need for human approach, i.e. consolidated strategies that allow the biologists to work underwater and to be provided with knowledge of the surroundings. This leads to a situation where it is desired to determine the position of the species under study, with no restrictions regarding to the level of depth or distance, as long as the physical system does not implode and the acoustic signal preserves a decent signal to noise ratio, sufficient for successful detection.

The problem of source localization may be even complemented with a navigation scenario including path following, where the path is outlined by a surface vehicle and the follower is the agent. Assuming the agent is an Autonomous Underwater Vehicle (AUV), recently in [3] the authors proposed a solution for the navigation problem, whereas in [4] a new method for simultaneous source localization and navigation is presented. The previous solutions were shown to be globally asymptotically stable (GAS) under a persistent excitation condition.

In the field of available tools for acoustic signals processing, Ultra-Short Baseline (USBL) acoustic positioning systems have become reliable tools, even if with lower quality performances when compared with RF transmissions, though with satisfiable accuracy results, which can be improved in the aftermath of the application of Kalman filters and Inertial Navigation Systems (INS) tightly-coupled integration techniques [5]. Further, with proper processing of acoustic signals, one can extract more information besides the position, for instance an unambiguous tag identification or relative velocity. Recent developments in acoustic transmitters, mainly the de facto standard from VEMCO©, have even allowed to incorporate, through time delay codification coding, additional characteristics of the environment like temperature and pressure, with respect to the location of the source. These miniature transmitters, with long periods of autonomy, are an affordable and effective solution for undersea fields of research.

Recently, under the scope of the Fundação para a Ciência e Tecnologia (FCT) funded MAST/AM project, an apparatus was designed and built that presents a convenient solution for maritime biologists (the agents) to carry into the underwater environment. The full set of equipment includes a Surface Robotic Tool (SRT), presented in [6], and late progresses allowed to reach a final stage on the Portable Underwater Robotic Tool (PURT) construction. The latter is essentially an extension of the body of the diver, providing a high degree of maneuverability in order to track moving sources (commonly fishes) equipped with low power consumption acoustic signal transmitters, commercially available and with various possible configurations concerning the emitted signal properties, e.g. periodicity, voltage amplitude, signal natural frequency, etc..

It should be noted that despite the underwater environment to not be homogeneous, one can assume it is without compromising the future results, thus avoiding a complex and extensive exercise of characterize the submerged channel.

This paper is organized as follows. Section II describes the framework of the problem and outlines the system dynamics. It also includes a proposed algebraic solution for the stationarity paradigm. The filter design along with the proof of observability are presented in Section III, together with simulation and further discussions. Conclusions and

This work was supported by project FCT PEst-OE/EEI/LA0009/2013 and by project FCT MAST/AM - PTFC/EEA - CRO/111107/2009

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future work are reported in Section V.

A. Notation

Throughout the paper, a bold symbol stands for a multi-dimensional variable, the symbol $\mathbf{0}$ denotes a matrix of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as diag($\mathbf{A}_1$,...,$\mathbf{A}_n$) and the set of unit vectors on $\mathbb{R}^3$ is denoted by $S(2)$. $\delta(t)$ represents the Dirac delta function. A positive-definite matrix $\mathbf{M}$ is identified as $\mathbf{M} \succ \mathbf{0}$. The superscript T, as in $[\ \ ]^T$, denotes the transpose operator. Likewise, an inverse generic matrix $\mathbf{M}$ is expressed as $\mathbf{M}^{-1}$. $\mathbf{S}(\cdot)$ is the skew matrix of a given vector and $E(\cdot)$ is an expected value.

II. PROBLEM STATEMENT

The analysis of the underwater scenery unfolds in a dual paradigm, depending on the assumption of stationarity of the agent. Notwithstanding, whichever be the consideration in terms of stationarity of the involved agents and sources, the geometry of the problem is unique. A general inertial frame, named $\mathcal{I}$, serves as a global reference for the underwater domain, and typically it will be solidary with a fixed and known point at the surface of the Earth, commonly referred to as the *Zero of the mission*. Hence, $\mathcal{I}$ is a traditional North/East/Down (NED) frame, and it does not rotate relatively to the Earth.

In addition, there is an auxiliary buoy at the surface of the water, subject to the movement of the waves, which produce small velocities and accelerations. This buoy holds a USBL aided INS acoustic positioning system, therefore it may be denominated as a Surface Robotic Tool (SRT). Furthermore, it determines its own current position expressed in $\mathcal{I}$ and is ready to receive interrogations from any of the involved agents beneath the surface. After the interrogation and the proper deductions, the SRT transmits to the agents their respective positions in frame $\mathcal{I}$.

An agent, carrying a PURT, with inertial coordinates expressed by vector ${}^{\mathcal{I}}\mathbf{p} \in \mathbb{R}^3$, adopts for himself a frame designated as $\mathcal{B}$, which stands for *Body-Fixed Frame* (BFF). The position of this agent will be coincident with the origin of the aforementioned frame. A source, commonly a fish equipped with one of the VEMCO© tags, roams indefinitely within a virtual unlimited space. Its inertial coordinates are expressed by ${}^{\mathcal{I}}\mathbf{s} \in \mathbb{R}^3$ while its coordinates in $\mathcal{B}$ are detailed by ${}^{\mathcal{B}}\mathbf{s} \in \mathbb{R}^3$.

The aim of this paper is essentially to solve the problem of estimating the position of the source in a real-time operation. This goal is justified by the need of marine biologists to better understand the behavior of some species, namely migrations, as well as resuming interrupted treatment routines.

As one may immediately conclude, there are always two different geometric interpretations of localization.

The first is a relative localization, i.e. a non geo-referenced problem, which means the agent is merely interested in knowing the position of the source according to its own position. Although a quite simplified version of the problem, this is, as a matter of fact, the purpose of the MAST/AM project, since biologists will be focused on *where to move* and not *where are we?*. However, that does not preclude more complex approaches.

The second geometric interpretation is a full characterization of the problem, aiming for the achievement of higher levels of information. Considering the fact that $\mathcal{B}$ is able to rotate, as a direct consequence of the diver orientation, one must not forget the rotation matrix from $\mathcal{B}$ to $\mathcal{I}$ when establishing equalities that host variables from both frames. Therefore, the mathematical bond between frames is enunciated by

$$ {}^{\mathcal{I}}\mathbf{s} = {}^{\mathcal{I}}\mathbf{p} + {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\ {}^{\mathcal{B}}\mathbf{s}, \quad (1) $$

where ${}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}$ is the rotation matrix from $\mathcal{B}$ to $\mathcal{I}$, whose entries follow from the conventional Euler angles.

Another way of addressing the problem of localization is via the deconstruction of the positions vectors as a product of ranges and directions, where ranges are the norm of those same positions vectors. In addition, one knows that the agent will be interested in gathering information about the source relatively to himself.

Let ${}^{\mathcal{I}}\mathbf{d} \in \mathbb{R}^3$ and ${}^{\mathcal{B}}\mathbf{d} \in \mathbb{R}^3$ be the direction vectors of the source relatively to the agent expressed in $\mathcal{I}$ and $\mathcal{B}$, respectively, given by

$$ {}^{\mathcal{I}}\mathbf{d} = \frac{{}^{\mathcal{I}}\mathbf{s} - {}^{\mathcal{I}}\mathbf{p}}{\| {}^{\mathcal{I}}\mathbf{s} - {}^{\mathcal{I}}\mathbf{p}\|} \quad (2) $$

and

$$ {}^{\mathcal{B}}\mathbf{d} = \frac{{}^{\mathcal{B}}\mathbf{s}}{\| {}^{\mathcal{B}}\mathbf{s}\|}. \quad (3) $$

Expressing now the mathematical equivalent of (1) in terms of directions, it results

$$ \left\|{}^{\mathcal{I}}\mathbf{s} - {}^{\mathcal{I}}\mathbf{p}\right\|\ {}^{\mathcal{I}}\mathbf{d} = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\ {}^{\mathcal{B}}\mathbf{d}\left\|{}^{\mathcal{B}}\mathbf{s}\right\|. \quad (4) $$

As the distance between the agent and the source is independent from the frame where it is considered, it can be written $\left\|{}^{\mathcal{I}}\mathbf{s} - {}^{\mathcal{I}}\mathbf{p}\right\| = \left\|{}^{\mathcal{B}}\mathbf{s}\right\|$. Thereby, equations (2) and (3) relate to each other through

$$ {}^{\mathcal{I}}\mathbf{d} = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\ {}^{\mathcal{B}}\mathbf{d}. \quad (5) $$

Using the formalisms of the Doppler effect, i.e. the change in frequency of the transmitted signals, one can obtain the Doppler velocity of the source relatively to the agent, v^d, which consists in the projection of the source velocity vector, $\mathbf{v}$, over the direction vector ${}^{\mathcal{B}}\mathbf{d}$. Mathematically speaking, it is possible to write

$$ \mathbf{v} \cdot {}^{\mathcal{B}}\mathbf{d} = v^d. \quad (6) $$

With a determined sample rate, the agent will be given three different measurements: his current position in the inertial frame, ${}^{\mathcal{I}}\mathbf{p}$, the Doppler velocity, v^d, from the source relatively to himself and, at last, the direction vector between the source position and its own position, expressed in $\mathcal{B}$, i.e. ${}^{\mathcal{B}}\mathbf{d}$.

In the real-time domain, the agent is interested only in putting together information about the source expressed in its own frame, i.e. the agent aims to estimate ${}^{\mathcal{B}}\mathbf{s}$.

A. Algebraic Solution

The main objective of this subsection is to prove that, given three direction measurements between an agent and a source and three Doppler velocity measurements belonging to the same source, it is possible to determine the three ranges that allow to fully characterize the movement of the source in the course of time. Hence, the computational algorithm performs a cycle after each three time steps.

The following assumptions are considered in this subsection:

1. The position of the agent is constant;

2. The source velocity vector is constant along each trio of measurements.

Recalling the problem statement introduced before, and with the previous assumptions in mind, it can be stipulated that ${}^{\mathcal{I}}\mathbf{p} \in \mathbb{R}^3$ is unknown, as all the following computations are expressed in $\mathcal{B}$. Wherefore, one might get rid of subscripts, using $\mathbf{s} \in \mathbb{R}^3$ alone as the generic position of the source.

The designated frame could be reasoned as an overlap of both $\mathcal{B}$ and $\mathcal{I}$ frames, resulting a frame that is solidary with the USBL array. Consequently, from now on until the end of this chapter the rotation matrix is not necessary.

Assuming that one has r measurements available, then sampling proceeds over $i = 1,2,...,r$. Let $v_i^d \in \mathbb{R}$ be the Doppler velocity on moment $t_i \in \mathbb{R}$ and $\mathbf{v}_i \in \mathbb{R}^3$ be the vectorial speed of the source on the same moment. For each time instant there is a known direction represented by $\mathbf{d}_i \in \mathbb{R}^3$. Hence the first step is to write the equation that establishes the relation amidst the Doppler velocities and the directions for each time step i. It follows that

$$\mathbf{v}_i^T \mathbf{d}_i = v_i^d. \tag{7}$$

Considering the stipulated assumptions, $\mathbf{v}_i$ can be interpreted as a mean speed over a distance from point $\mathbf{s}_i$ to point $\mathbf{s}_{i+2}$. It is also known that $\mathbf{s}_i = R_i \mathbf{d}_i$, where $R_i \in \mathbb{R}$ is the distance from the source to the agent at the time instant t_i. The time between measures, i.e. $\Delta t_{i,i+1} := t_{i+1} - t_i$, does not need to be strictly constant, besides it is dependent on acquisition times defined by VEMCO© tags, which are here used as emitters. Considering assumption **2**, there is a linear proportion between the traveled distance and velocity, which means

$$\mathbf{s}_{i+1} - \mathbf{s}_i = \mathbf{v}\Delta t_{i,i+1}. \tag{8}$$

Therefore, combining (7) (8) results in

$$[R_{i+1}\mathbf{d}_{i+1} - R_i\mathbf{d}_i]^T \mathbf{d}_i = v_i^d \Delta t_{i,i+1}. \tag{9}$$

Unit vector $\mathbf{d}_i$ has an important property: $\mathbf{d}_i^T\mathbf{d}_i = \|\mathbf{d}_i\| = 1$. From here it can be written

$$R_i = R_{i+1}(\mathbf{d}_{i+1}^T\mathbf{d}_i) - v_i^d \Delta t_{i,i+1}, \tag{10}$$

where (10) holds two variables: R_i and R_{i+1}.

Equation (10) is still valid when considering the next time step, meaning that it is also possible to write

$$R_{i+2} = \frac{R_{i+1} + v_{i+1}^d \Delta t_{i+1,i+2}}{\mathbf{d}_{i+1}^T \mathbf{d}_{i+2}}. \tag{11}$$

Again, based on assumption **2**, it is true that the distance traveled from point to point will depend on the time from one measure to another and also on the velocity during that transition. Said assumption restricts the range of possible paths between each trio of points. It follows

$$\frac{\|\mathbf{s}_{i+1} - \mathbf{s}_i\|}{\Delta t_{i,i+1}} = \frac{\|\mathbf{s}_{i+2} - \mathbf{s}_{i+1}\|}{\Delta t_{i+1,i+2}}, \tag{12}$$

from where it is possible to write

$$\|R_{i+1}\mathbf{d}_{i+1} - R_i\mathbf{d}_i\| = \frac{\Delta t_{i,i+1}}{\Delta t_{i+1,i+2}}\|R_{i+2}\mathbf{d}_{i+2} - R_{i+1}\mathbf{d}_{i+1}\|. \tag{13}$$

The necessary conditions to solve the problem and determine the three positions are now gathered. Both equations (10) and (11) were written with the common dependence on the right side. Substituting them in (13) gives

$$\begin{aligned} &\|R_{i+1}\mathbf{d}_{i+1} - R_{i+1}(\mathbf{d}_{i+1}^T\mathbf{d}_i)\mathbf{d}_i + v_i^d\Delta t_{i,i+1}\mathbf{d}_i\| = \\ &= \frac{\Delta t_{i,i+1}}{\Delta t_{i+1,i+2}} \left\| \left(\frac{R_{i+1} + v_{i+1}^d \Delta t_{i+1,i+2}}{\mathbf{d}_{i+1}^T \mathbf{d}_{i+2}} \right) \mathbf{d}_{i+2} - R_{i+1}\mathbf{d}_{i+1} \right\|. \end{aligned} \tag{14}$$

The argument of the norm on the left half of (14) can be written as

$$R_{i+1}\mathbf{d}_{i+1} - R_{i+1}\mathbf{d}_i\mathbf{d}_i^T\mathbf{d}_{i+1} + v_i^d\Delta t_{i,i+1}\mathbf{d}_i. \tag{15}$$

Putting $\mathbf{d}_{i+1}$ and R_{i+1} in evidence it results

$$R_{i+1}[\mathbf{I} - \mathbf{d}_i\mathbf{d}_i^T]\mathbf{d}_{i+1} + v_i^d\Delta t_{i,i+1}\mathbf{d}_i. \tag{16}$$

The term $[\mathbf{I} - \mathbf{d}_i\mathbf{d}_i^T]$ can be obtained from a product between skew matrices. A skew matrix of a generic vector $\mathbf{a} \in \mathbb{R}^3$, denoted as $\mathbf{S}(\mathbf{a})$, is given by

$$\mathbf{S}(\mathbf{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}. \tag{17}$$

Considering another generic vector $\mathbf{b} \in \mathbb{R}^3$, an interesting property is associated to skew matrices, which establishes that

$$\mathbf{S}(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b}. \tag{18}$$

If $\mathbf{a}$ is a unit vector then $\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{a}) = \mathbf{a}\mathbf{a}^T - \mathbf{I}$. Since the direction is in fact a unit vector, (16) can thus be rewritten as

$$-R_{i+1}\mathbf{S}(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{d}_{i+1} + v_i^d \Delta t_{i,i+1}\mathbf{d}_i. \tag{19}$$

The result of the external product is a vector that is perpendicular to the two vectors involved in the operation. Hence, in (19), the result of the first term of the sum is orthogonal to the second one, since $\mathbf{S}(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{d}_{i+1} \perp \mathbf{d}_i$, therefore the norm of (19) to the second power is the sum of the norms of both parcels, each elevated to the second power, resulting in

$$R_{i+1}^2 \|\mathbf{S}(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{d}_{i+1}\|^2 + \|v_i^d \Delta t_{i,i+1}\mathbf{d}_i\|^2. \tag{20}$$

Next, it follows that

$$\begin{aligned}\|\mathbf{S}(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{d}_{i+1}\|^2 &= [\mathbf{S}(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{d}_{i+1}]^T \mathbf{S}(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{d}_{i+1} \\ &= \mathbf{d}_{i+1}^T \mathbf{S}^T(\mathbf{d}_i)\mathbf{S}^T(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{d}_{i+1}\end{aligned} \tag{21}$$

and, considering that $\mathbf{S}^T(\mathbf{a}) = -\mathbf{S}(\mathbf{a})$, results

$$\begin{aligned}\|\mathbf{S}(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{d}_{i+1}\|^2 &= \mathbf{d}_{i+1}^T \mathbf{S}^4(\mathbf{d}_i)\mathbf{d}_{i+1} \\ &= \mathbf{d}_{i+1}^T \mathbf{S}(\mathbf{d}_i)\mathbf{S}^3(\mathbf{d}_i)\mathbf{d}_{i+1}.\end{aligned} \tag{22}$$

The skew matrix has two other special properties which states that $\mathbf{S}^3(\mathbf{a}) = -(\mathbf{a}^T\mathbf{a})\mathbf{S}(\mathbf{a})$ and $\mathbf{S}^2(\mathbf{a}) = \mathbf{a}\mathbf{a}^T - (\mathbf{a}^T\mathbf{a})\mathbf{I}$. Both properties are generic, thus valid for any vector $\mathbf{a}$. From that, one can write

$$\begin{aligned}\|\mathbf{S}(\mathbf{d}_i)\mathbf{S}(\mathbf{d}_i)\mathbf{d}_{i+1}\|^2 &= -\mathbf{d}_{i+1}^T \mathbf{S}^2(\mathbf{d}_i)\mathbf{d}_{i+1} \\ &= -\mathbf{d}_{i+1}^T [\mathbf{d}_i\mathbf{d}_i^T - \mathbf{d}_i^T\mathbf{d}_i\mathbf{I}]\mathbf{d}_{i+1} \\ &= -\mathbf{d}_{i+1}^T [\mathbf{d}_i(\mathbf{d}_{i+1}^T\mathbf{d}_i) - \mathbf{d}_{i+1}(\mathbf{d}_i^T\mathbf{d}_i)] \\ &= -(\mathbf{d}_{i+1}^T\mathbf{d}_i)^2 + 1 \\ &= 1 - (\mathbf{d}_i^T\mathbf{d}_{i+1})^2 \\ &= 1 - \cos^2(\alpha_{i,i+1}) = \sin^2(\alpha_{i,i+1}),\end{aligned} \tag{23}$$

where $\alpha_{i,i+1}$ is the angle between the direction vectors $\mathbf{d}_i$ and $\mathbf{d}_{i+1}$.

The next step is to apply the same kind of procedure to the right half of (14). Again,

$$\begin{aligned}&\left\| \left(\frac{R_{i+1} + v_{i+1}^d \Delta t_{i+1,i+2}}{\mathbf{d}_{i+1}^T \mathbf{d}_{i+2}} \right) \mathbf{d}_{i+2} - R_{i+1}\mathbf{d}_{i+1} \right\|^2 = \\ &\left\| \left(\frac{R_{i+1} + v_{i+1}^d \Delta t_{i+1,i+2}}{\mathbf{d}_{i+1}^T \mathbf{d}_{i+2}} \right) \mathbf{d}_{i+2} \right\|^2 + \|R_{i+1}\mathbf{d}_{i+1}\|^2 + \\ &-2 \left[\left(\frac{R_{i+1} + v_{i+1}^d \Delta t_{i+1,i+2}}{\mathbf{d}_{i+1}^T \mathbf{d}_{i+2}} \right) \mathbf{d}_{i+2} \right]^T R_{i+1}\mathbf{d}_{i+1}.\end{aligned} \tag{24}$$

After simplifications, the right half of (24) follows as

$$\begin{aligned}&\left(\frac{R_{i+1} + v_{i+1}^d \Delta t_{i+1,i+2}}{\mathbf{d}_{i+1}^T \mathbf{d}_{i+2}} \right)^2 + R_{i+1}^2 + \\ &-2(R_{i+1} + v_{i+1}^d \Delta t_{i+1,i+2})R_{i+1} = \\ &= R_{i+1}^2 \left(\frac{1}{(\mathbf{d}_{i+1}^T\mathbf{d}_{i+2})^2} - 1 \right) + \frac{(v_{i+1}^d \Delta t_{i+1,i+2})^2}{(\mathbf{d}_{i+1}^T\mathbf{d}_{i+2})^2} + \\ &+ R_{i+1} \left(\frac{2v_{i+1}^d \Delta t_{i+1,i+2}}{(\mathbf{d}_{i+1}^T\mathbf{d}_{i+2})^2} - 2v_{i+1}^d \Delta t_{i+1,i+2} \right) \\ &= R_{i+1}^2 \tan^2(\alpha_{i+1,i+2}) + \\ &+ R_{i+1} 2v_{i+1}^d \Delta t_{i+1,i+2} \tan^2(\alpha_{i+1,i+2}) + \frac{(v_{i+1}^d \Delta t_{i+1,i+2})^2}{\cos^2(\alpha_{i+1,i+2})},\end{aligned} \tag{25}$$

where, as seen before, $\alpha_{i+1,i+2}$ is the angle between the direction vectors $\mathbf{d}_{i+1}$ and $\mathbf{d}_{i+2}$.

Finally, using simplifications (23) and (25) on each side of (14), respectively, results

$$\begin{aligned}&R_{i+1}^2 \sin^2(\alpha_{i,i+1}) + (v_i^d \Delta t_{i,i+1})^2 = \\ &= \frac{\Delta t_{i,i+1}}{\Delta t_{i+1,i+2}} \Bigg(R_{i+1}^2 \tan^2(\alpha_{i+1,i+2}) + \\ &+ R_{i+1} 2v_{i+1}^d \Delta t_{i+1,i+2} \tan^2(\alpha_{i+1,i+2}) + \frac{(v_{i+1}^d \Delta t_{i+1,i+2})^2}{\cos^2(\alpha_{i+1,i+2})} \Bigg).\end{aligned} \tag{26}$$

Solving for R_{i+1}, the solution is obtained by settling the right coefficients in the quadratic equation $ax^2 + bx + c = 0$, with coefficients a, b and c given, respectively, by

$$\begin{aligned}a &= \sin^2(\alpha_{i,i+1}) - \frac{\Delta t_{i,i+1}}{\Delta t_{i+1,i+2}} \tan^2(\alpha_{i+1,i+2}) \\ b &= -2\frac{\Delta t_{i,i+1}}{\Delta t_{i+1,i+2}} v_{i+1}^d \Delta t_{i+1,i+2} \tan^2(\alpha_{i+1,i+2}) \\ c &= (v_i^d \Delta t_{i,i+1})^2 - \frac{\Delta t_{i,i+1}}{\Delta t_{i+1,i+2}} \frac{(v_{i+1}^d \Delta t_{i+1,i+2})^2}{\cos^2(\alpha_{i+1,i+2})}.\end{aligned} \tag{27}$$

If, for instance, a sampling time of 1s is chosen, all Δt variables may be excluded from the above equations, resulting a more compact way of representing the coefficients a, b and c. It follows

$$\begin{aligned}a &= \sin^2(\alpha_{i,i+1}) - \tan^2(\alpha_{i+1,i+2}) \\ b &= -2v_{i+1}^d \tan^2(\alpha_{i+1,i+2}) \\ c &= (v_i^d)^2 - \frac{(v_{i+1}^d)^2}{\cos^2(\alpha_{i+1,i+2})}.\end{aligned} \tag{28}$$

From one of the assumptions of the problem, it is possible to infer that the source is assumed to move linearly along a trio of measurements. That statement implies that only one of the three measured Doppler velocities can be zero. Consequently, situations like a zero solution are avoided *a priori*, thus excluding from the problem circular trajectories around the (stationary) agent.

Also, $\alpha_{i+1,i+2}$ should never be zero, i.e. the direction vector must not remain constant. This leads to a need for a persistent excitation condition, which is addressed in Section III-B.

Unbounded values in $\tan^2(\alpha_{i+1,i+2})$ ought to be avoided, but that requirement is always accomplished in a practical sense, e.g. the source is never too close neither it travels at great speed.

III. LOCALIZATION FILTER DESIGN

A. Kalman Filter

Recalling the aforementioned problem statement from Section II, suppose that there is a Surface Robotic Tool (SRT) at the surface, and that the SRT provides information regarding the existence of an inertial frame. Furthermore, the PURT carried by the agent has the ability to communicate with that same SRT.

Let $\mathcal{B}$ be the identification of the moving frame and $\mathcal{I}$ the identification of the stationary one. ${}^{\mathcal{I}}\mathbf{s}(t) \in \mathbb{R}^3$ is the position of the source relatively to the inertial frame, while ${}^{\mathcal{B}}\mathbf{s}(t) \in \mathbb{R}^3$ is that same position expressed in $\mathcal{B}$. The SRT transmits to the agent its own current position in the inertial framework, consequently, besides the direction and Doppler velocities measurements, the new additional measure is ${}^{\mathcal{I}}\mathbf{p}(t) \in \mathbb{R}^3$. The velocity of the agent, which is unknown, is thus given by ${}^{\mathcal{I}}\mathbf{v}_p(t) = {}^{\mathcal{I}}\dot{\mathbf{p}}(t)$. Like for the source, this velocity is assumed to be constant, hence ${}^{\mathcal{I}}\dot{\mathbf{v}}_p(t) = \mathbf{0}$ and ${}^{\mathcal{I}}\dot{\mathbf{v}}_s(t) = \mathbf{0}$.

The two frames relate to each other through

$$ {}^{\mathcal{I}}\mathbf{s}(t) = {}^{\mathcal{I}}\mathbf{p}(t) + {}^{\mathcal{I}}\mathbf{d}(t)\|{}^{\mathcal{B}}\mathbf{s}(t)\|. \tag{29} $$

For the sake of simplicity, $\mathcal{I}$ will be omitted in the notation, since this is the designated frame of work. Henceforth, direction vector $\mathbf{d}(t)$ comes as

$$ \mathbf{d}(t) = \frac{\mathbf{s}(t) - \mathbf{p}(t)}{\|\mathbf{s}(t) - \mathbf{p}(t)\|} = ({}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R})\frac{{}^{\mathcal{B}}\mathbf{s}(t)}{\|{}^{\mathcal{B}}\mathbf{s}(t)\|}, \tag{30} $$

where $({}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R})$ is the rotation matrix, which is used to translate the direction vector obtained during the real-time operation, thus expressed in $\mathcal{B}$, into the inertial frame. This is a consequence of the random underwater space orientation of the agent.

The variables to be determined, i.e. the states considered for the filter design, are the position of the source and the velocity of both agent and source. However, a fourth state should also be part of the vector of estimates: the position of the agent. The agent velocity is the time derivative of its own position. Unless the agent position is added to the states vector, the velocity could not be estimated correctly, even if the first is already a known variable.

As a matter of fact, without the mentioned fourth state, $\mathbf{v}_p(t)$ can not be successfully estimated as a consequence of lack of related information.

To avoid a non-linear situation, one might add a fifth state to the states vector, it being the range between the source and the agent, resulting in

$$ \mathbf{x}(t) = [\mathbf{s}^T(t) \quad \mathbf{p}^T(t) \quad \mathbf{v}_s^T(t) \quad \mathbf{v}_p^T(t) \quad \|\mathbf{r}(t)\|]^T \quad \in \mathbb{R}^{13}, \tag{31} $$

where $\mathbf{r}(t) := \mathbf{s}(t) - \mathbf{p}(t)$, $\mathbf{r}(t) \in \mathbb{R}^3$. Next, the range derivative is given by

$$ \frac{d}{dt}\|\mathbf{r}(t)\| = (\mathbf{v}_\mathbf{s}(t) - \mathbf{v}_\mathbf{p}(t))^T\mathbf{d}(t) = v^d, \tag{32} $$

which comes as no surprise, since the distance between the source and the agent is expected to remain constant if $v^d = 0$. Still, this does not necessarily mean that both agent and source are at rest; one of them may be stopped while the other travels around the first in perfect circles, i.e. the internal product between the direction and the velocity of the moving target returns zero, such as they can also be both moving with the same speed and orientation. These situations must be avoided, leading to a need for a persistent excitation condition.

The final states vector derivative, $\dot{\mathbf{x}}(t) \in \mathbb{R}^{13}$, is given by

$$ \dot{\mathbf{x}}(t) = [\mathbf{v}_s^T(t) \quad \mathbf{v}_p^T(t) \quad \mathbf{0} \quad \mathbf{0} \quad (\mathbf{v}_\mathbf{s}(t) - \mathbf{v}_\mathbf{p}(t))^T\mathbf{d}(t)]^T, \tag{33} $$

thus matrix $\mathbf{A}$ becomes

$$ \mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{d}^T(t) & -\mathbf{d}^T(t) & 0 \end{bmatrix} \in \mathbb{R}^{13\times 13}. \tag{34} $$

From (30) it can be written

$$ \mathbf{0} = \mathbf{s}(t) - \mathbf{p}(t) - \|\mathbf{s}(t) - \mathbf{p}(t)\|\mathbf{d}(t), \tag{35} $$

hence matrix $\mathbf{C}$ follows as

$$ \mathbf{C}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & -\mathbf{d}(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{d}^T(t) & -\mathbf{d}^T(t) & 0 \end{bmatrix} \in \mathbb{R}^{7\times 13}. \tag{36} $$

The final measurements vector, $\mathbf{y}(t) \in \mathbb{R}^{7\times 12}$, naturally follows as

$$ \mathbf{y}(t) = \begin{bmatrix} \mathbf{p}(t) \\ \mathbf{0} \\ v^d \end{bmatrix} = \begin{bmatrix} \mathbf{s}(t) - \mathbf{d}(t)\|\mathbf{s}(t) - \mathbf{p}(t)\| \\ \mathbf{s}(t) - \mathbf{p}(t) - \|\mathbf{s}(t) - \mathbf{p}(t)\|\mathbf{d}(t) \\ (\mathbf{v}_s(t) - \mathbf{v}_p(t))^T\mathbf{d}(t) \end{bmatrix}. \tag{37} $$

B. Observability

If through the external outputs of a system one cannot infer how its internal states evolved then the system is said to be not observable. Therefore, the objective proposed by this section is to prove the observability of the Linear Time Varying (LTV) system expressed by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t), \tag{38}$$

where $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are given by (34) and (36), respectively. The state vector is still given by (31).

The following proposition [Proposition 4.2, [7]] is useful in the sequel.

Proposition: Let $\mathbf{f}(t) : [t_0, t_f] \subset \mathbb{R} \to \mathbb{R}^n$ be a continuous and i-times continuously differentiable function on $\mathcal{T} := [t_0, t_f], T := t_f - t_0 > 0$, and such that

$$\mathbf{f}(t_0) = \dot{\mathbf{f}}(t_0) = ... = \mathbf{f}^{(i-1)}(t_0) = \mathbf{0}. \tag{39}$$

Further assume that there exists a nonnegative constant C such that $\|\mathbf{f}^{(i+1)}(t)\| \leq C$ for all $t \in \mathcal{T}$. If there exist $\alpha > 0$ and $t_1 \in \mathcal{T}$ such that $\|\mathbf{f}^{(i)}(t_1)\| \geq \alpha$ then there exist $0 < \delta \leq T$ and $\beta > 0$ such that $\|\mathbf{f}(t_0 + \delta)\| \geq \beta$.

To prove the observability is to guarantee the outputs are enough to determine the correspondent current state. That said, a generic response $\mathbf{y}(t)$ must determine uniquely an initial state $\mathbf{x}(t_0)$.

From the Peano-Baker series it follows the transition matrix associated to $\mathbf{A}$, denoted by $\boldsymbol{\phi}(t,t_0) \in \mathbb{R}^{13\times13}$ and given by

$$\boldsymbol{\phi}(t,t_0) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & (t-t_0)\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & (t-t_0)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \int_{t_0}^{t} \mathbf{d}^T(\sigma)d\sigma & -\int_{t_0}^{t} \mathbf{d}^T(\sigma)d\sigma & 1 \end{bmatrix}. \tag{40}$$

The transition matrix is such that

$$\mathbf{x}(t) = \boldsymbol{\phi}(t,t_0)\mathbf{x}(t_0). \tag{41}$$

The one condition to guarantee the observability of the LTV expressed in (38) is to ensure that the Observability Gramian is invertible, the latter being defined as

$$\boldsymbol{\mathcal{W}}(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\phi}^T(\tau, t_0)\mathbf{C}^T(\tau)\mathbf{C}(\tau)\boldsymbol{\phi}(\tau, t_0)d\tau. \tag{42}$$

Theorem: The LTV system (38) is observable on $\mathcal{T}$ if and only if (Necessity and Sufficiency) the direction vector $\mathbf{d}(t)$ does not remain constant between two successive moments. Mathematically speaking,

$$\exists_{t_1 \in \mathcal{T}} : \mathbf{d}^T(t_0)\mathbf{d}(t_1) < 1. \tag{43}$$

Proof: Obviously, if the direction remained constant the internal product between the two vectors would wind up to be the norm of $\mathbf{d}$ raised to second power, i.e. 1.

Starting with an example where it is shown that (43) is a necessary condition, let $\mathbf{c} = [\mathbf{c}_1^T\ \mathbf{c}_2^T\ \mathbf{c}_3^T\ \mathbf{c}_4^T\ c_5]^T \in \mathbb{R}^{13}$, with $\mathbf{c}_i \in \mathbb{R}^3$, for $i = 1,2,3,4$, and $c_5 \in \mathbb{R}$, be a unit vector. Since it must be $\|\mathbf{c}\| = 1$ for all $\mathbf{c}$,

$$\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0, t_f)\mathbf{c} \neq 0 \tag{44}$$

in order to the LTV system be observable. Then,

$$\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0, t_f)\mathbf{c} = \int_{t_0}^{t_f} \|\mathbf{f}(\tau)\|^2 d\tau, \tag{45}$$

where

$$\mathbf{C}(\tau)\boldsymbol{\phi}(\tau, t_0) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & (\tau - t_0)\mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} & \mathbf{C}_{23} & \mathbf{C}_{32} & -\mathbf{d}(\tau) \\ \mathbf{0} & \mathbf{0} & \mathbf{d}^T(\tau) & -\mathbf{d}^T(\tau) & 0 \end{bmatrix} \tag{46}$$

with $\mathbf{C}_{23} = (\tau - t_0)\mathbf{I} - \mathbf{d}(\tau)\int_{t_0}^{\tau} \mathbf{d}^T(\sigma)d\sigma = -\mathbf{C}_{32}$ and, consequently,

$$\mathbf{f}(\tau) = \begin{bmatrix} \mathbf{f}_1(\tau) \\ \mathbf{f}_2(\tau) \\ f_3(\tau) \end{bmatrix} \quad \in \mathbb{R}^7, \tau \in \mathcal{T}, \tag{47}$$

with

$$\begin{aligned} \mathbf{f}_1(\tau) &= \mathbf{c}_2 + (\tau - t_0)\mathbf{c}_4 \quad \in \mathbb{R}^3, \\ \mathbf{f}_2(\tau) &= \mathbf{c}_1 - \mathbf{c}_2 + \\ &\quad + \left[(\tau - t_0)\mathbf{I} - \mathbf{d}(\tau)\int_{t_0}^{\tau} \mathbf{d}^T(\sigma)d\sigma\right](\mathbf{c}_3 - \mathbf{c}_4) + \\ &\quad - \mathbf{d}(\tau)c_5 \quad \in \mathbb{R}^3, \\ f_3(\tau) &= \mathbf{d}^T(\tau)(\mathbf{c}_3 - \mathbf{c}_4) \quad \in \mathbb{R}. \end{aligned}$$

As for the first derivative of $\mathbf{f}(\tau)$ in order to τ, it is given by

$$\frac{d\mathbf{f}(\tau)}{d\tau} = \begin{bmatrix} \mathbf{c}_4 \\ \frac{d\mathbf{f}_2(\tau)}{d\tau} \\ \dot{\mathbf{d}}^T(\tau)(\mathbf{c}_3 - \mathbf{c}_4) \end{bmatrix} \quad \in \mathbb{R}^7, \tag{48}$$

with $d\mathbf{f}_2(\tau)/d\tau$ given by

$$\frac{d\mathbf{f}_2(\tau)}{d\tau} = \left[\mathbf{I} - \dot{\mathbf{d}}(\tau)\int_{t_0}^{\tau} \mathbf{d}^T(\sigma)d\sigma - \mathbf{d}(\tau)\mathbf{d}^T(\tau)\right](\mathbf{c}_3 - \mathbf{c}_4) + \\ - \dot{\mathbf{d}}(\tau)c_5.$$

Assume then that the condition in (43) is not verified, i.e. the direction vector remain constant and equal to $\mathbf{d}(t_0)$ for all $t \in \mathcal{T}$, and let $\mathbf{c}_2 = \mathbf{c}_3 = \mathbf{c}_4 = \mathbf{0}$. It results

$$\mathbf{f}_2(\tau) = \mathbf{c}_1 - \mathbf{d}(\tau)c_5. \tag{49}$$

Since $\mathbf{c}$ is a unit vector one can define $\mathbf{c}_1 = \frac{\sqrt{2}}{2}\mathbf{d}(t_0)$ and $c_5 = \frac{\sqrt{2}}{2}$, therefore

$$\mathbf{f}_2(\tau) = \frac{\sqrt{2}}{2}\mathbf{d}(t_0) - \mathbf{d}(\tau)\frac{\sqrt{2}}{2} = \mathbf{0} \quad \forall t \in \mathcal{T}. \tag{50}$$

Consequently, the Observability Gramian $\boldsymbol{\mathcal{W}}$ is not invertible and the LTV system is not observable. As one may conclude, the condition in (43) has to be true on $\mathcal{T}$ for the LTV system to be observable also on $\mathcal{T}$, i.e. (43) is a necessary condition.

However, another procedure may reveal the condition in (43) to be sufficient for the LTV system become observable. This means that if $\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0,t_f)\mathbf{c}=0$ then it does not exists a $t_1\in\mathcal{T}$ such that (43) is verified. In other words, the objective is to prove that for any unit vector $\mathbf{c}\neq\mathbf{0}$, $\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0,t_f)\mathbf{c}>0$ under the condition in (43).

Hence, start by evaluating $\mathbf{f}(\tau)$ at t_0, yielding

$$\mathbf{f}(t_0)=\begin{bmatrix}\mathbf{c}_2\\ \mathbf{c}_1-\mathbf{c}_2-\mathbf{d}(t_0)c_5\\ \mathbf{d}^T(t_0)(\mathbf{c}_3-\mathbf{c}_4)\end{bmatrix}. \tag{51}$$

Firstly, two cases may be considered:

i) If $\mathbf{c}_2\neq\mathbf{0}$, then $\|\mathbf{f}(t_0)\|>0$, and it follows from *Proposition* that $\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0,t_f)\mathbf{c}>0$;

ii) If $\mathbf{c}_2=\mathbf{0}$ and $\mathbf{c}_1\neq\mathbf{d}(t_0)c_5$ it leads again to $\|\mathbf{f}(t_0)\|>0$, succeeding $\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0,t_f)\mathbf{c}>0$.

Now, if $\mathbf{c}_3\neq\mathbf{c}_4$, $\mathbf{f}_3(t_0)\neq\mathbf{0}\Rightarrow\|\mathbf{f}(t_0)\|>0$, resulting $\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0,t_f)\mathbf{c}>0$.

For further analysis on combinations of possible values of $\mathbf{c}$, one needs the evaluation of $d\mathbf{f}(\tau)/d\tau$ at $\tau=t_0$, which is given by

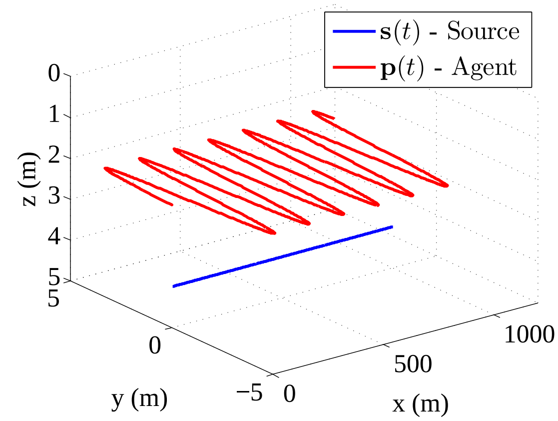


Fig. 1: Simulated trajectory.

$$\left.\frac{d\mathbf{f}(\tau)}{d\tau}\right|_{\tau=t_0}=\begin{bmatrix}\mathbf{c}_4\\ [\mathbf{I}-\mathbf{d}(t_0)\mathbf{d}^T(t_0)](\mathbf{c}_3-\mathbf{c}_4)-\dot{\mathbf{d}}(t_0)c_5\\ \dot{\mathbf{d}}^T(t_0)(\mathbf{c}_3-\mathbf{c}_4)\end{bmatrix}. \tag{52}$$

Suppose now that $\mathbf{c}_2=\mathbf{0}$, $\mathbf{c}_1=\mathbf{d}(t_0)c_5$, for any scalar c_5, and $\mathbf{c}_3=\mathbf{c}_4$ so that $\mathbf{f}(t_0)=\mathbf{0}$.

iii) If $\mathbf{c}_4\neq\mathbf{0}$, $d\mathbf{f}_1/d\tau\neq\mathbf{0}\Rightarrow\left\|\left.\frac{d\mathbf{f}(\tau)}{d\tau}\right|_{\tau=t_0}\right\|>0$ and it follows from using twice the Proposition introduced in the beginning of this section that $\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0,t_f)\mathbf{c}>0$.

iv) In opposition, if $\mathbf{c}_3=\mathbf{0}\Rightarrow\mathbf{c}_4=\mathbf{0}$ one obtains

$$\left.\frac{d\mathbf{f}(\tau)}{d\tau}\right|_{\tau=t_0}=\begin{bmatrix}\mathbf{0}\\ -\dot{\mathbf{d}}(t_0)c_5\\ 0\end{bmatrix}, \tag{53}$$

and

$$\left.\frac{d^{(i)}\mathbf{f}(\tau)}{d\tau^{(i)}}\right|_{\tau=t_0}=\begin{bmatrix}\mathbf{0}\\ -\mathbf{d}^{(i)}(t_0)c_5\\ 0\end{bmatrix}, \tag{54}$$

where the superscript i represents the i-th derivative. Since $\|\mathbf{c}\|=1$, and if $\mathbf{c}_1$ had been $\mathbf{0}$ before, then it must be $c_5=1$ to hold the fact that $\|\mathbf{c}\|$ is a unit vector, however proceed considering $\mathbf{c}_1=\mathbf{d}(t_0)c_5$.

If $\dot{\mathbf{d}}(t_0)\neq\mathbf{0}$, $\|\,d\mathbf{f}(\tau)/d\tau|_{\tau=t_0}\,\|>\mathbf{0}$ and, using *Proposition* twice, $\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0,t_f)\mathbf{c}>0$. The direction is under a continuous-time function and according to (43) it is true that at least there is a moment, suppose t_1, when the direction derivative can not be zero, i.e. $\dot{\mathbf{d}}(t_1)\neq\mathbf{0}$, resulting $\|\,d\mathbf{f}(\tau)/d\tau|_{\tau=t_1}\,\|>\mathbf{0}$. Using *Proposition* twice, $\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0,t_f)\mathbf{c}>0$.

The proof is thus concluded as it is shown that $\mathbf{c}^T\boldsymbol{\mathcal{W}}(t_0,t_f)\mathbf{c}>0$ for all $\|\mathbf{c}\|=1$, which means that the observability Gramian is invertible and as such (38) is observable. ■

C. Simulation Results

In the simulations, the source and the agent trajectories are those depicted in Fig. 1. Clearly, the persistent excitation condition (43) is satisfied, which allows to apply the solutions proposed in the paper.

The source and agent initial positions are $\mathbf{s}(0)=[5\ 0\ 2]^T$ m and $\mathbf{p}(0)=[5\ 0\ 2]^T$ m, respectively.

The system dynamics, including additive system disturbances and sensor noise, read as

$$\begin{cases}\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)+\mathbf{w}(t)\\ \mathbf{y}(t)=\mathbf{C}(t)\mathbf{x}(t)+\mathbf{n}(t)\end{cases}, \tag{55}$$

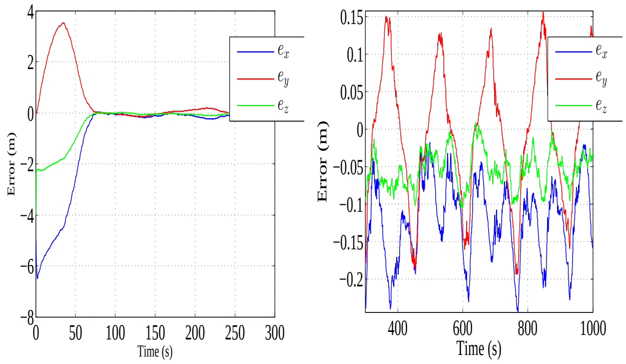
where $\mathbf{w}(t)\in\mathbb{R}^{13}$ is zero-mean white Gaussian noise, with $E[\mathbf{w}(t)\mathbf{w}^T(t-\tau)]=\boldsymbol{\Xi}\delta(\tau)$, $\boldsymbol{\Xi}\succ\mathbf{0}$, $\mathbf{n}(t)\in\mathbb{R}^7$ is zero-mean white Gaussian noise, with $E[\mathbf{n}(t)\mathbf{n}^T(t-\tau)]=\boldsymbol{\Theta}\delta(\tau)$ and $\boldsymbol{\Theta}\succ\mathbf{0}$. The noises are uncorrelated, therefore $E[\mathbf{w}(t)\mathbf{n}^T(t-\tau)]=\mathbf{0}$. Finally, the case considered here is the common one, where the noises are additive, however, that might not be the reality and thus the proposed solution is not an optimal one.

Regarding the covariance matrices, they were defined as

$$\boldsymbol{\Theta}=\mathrm{diag}(\mathbf{10^{-4}},\mathbf{10},10^{-4})$$
$$\boldsymbol{\Xi}=\mathrm{diag}(\mathbf{10^{-5}},\mathbf{10^{-5}},\mathbf{10^{-5}},\mathbf{0.1},10^{-3}).$$

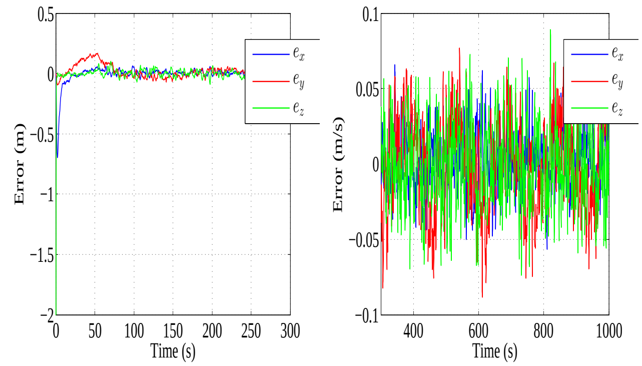
Noise was considered with 0.01m/s for Doppler velocity entries, 0.1m for the agent position and one degree of standard deviation for direction angles. Initial estimates were considered to be unknown, therefore they were set to $\hat{\mathbf{x}}(0)=\mathbf{0}$.

The final results are presented in Fig. [2 — 5].



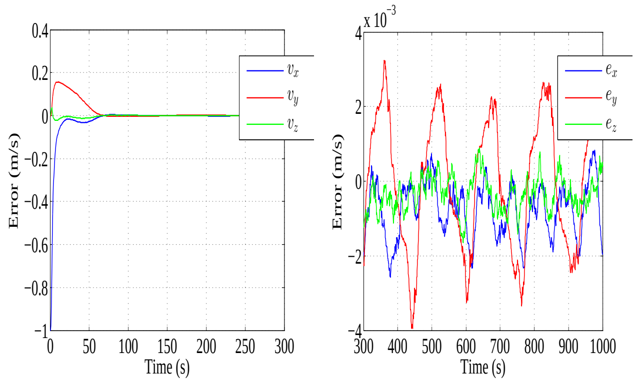
(a) Estimation error. (b) Detailed estimation error.

Fig. 2: Source Position.



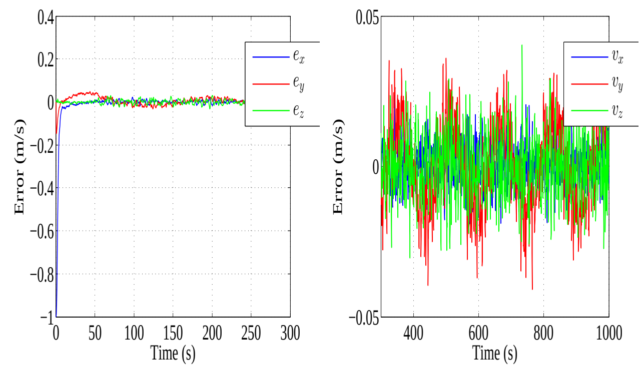
(a) Estimation error. (b) Detailed estimation error.

Fig. 4: Agent position.



(a) Estimation error. (b) Detailed estimation error.

Fig. 3: Source velocity.



(a) Estimation error. (b) Detailed estimation error.

Fig. 5: Agent velocity.

Even if the convergence of the estimation error associated to the source position is moderately slow, that is only due to the unknown initial estimate. Still, the overall results are quite reasonable, confirming this option as a reliable one.

IV. Acoustic Positioning Scheme

The PURT has a set of four hydrophones arranged in a specific mechanical configuration. Each one of them is the start of an internal communication channel that guides the acoustic signal through an Analog to Digital Converter (ADC). Before due conversion, the signal passes through an amplification stage, which limits the signal amplitude to the range $[-2.5; 2.5]$ V. This imposed (analog input voltage) range is a necessary requirement due to the limited voltage input span of the ADC devices. The digital signals are then sampled at a rate of 250 kHz and sent, via an Ethernet communication, to a MATLAB® application, where data is processed.

The path of Conversion - Acquisition - Dispatch is closely regulated by a *D.Module.C6713* Digital Signal Processor (DSP). This module interleaves the operations of Conversion and Acquisition, both executed on a *D.Module.ADDA16* Analog/Digital and Digital/Analog converter board, with Dispatch, the latter performed by a *D.Module.91C111* networking daughter card, featuring a 100 Mbps Ethernet Media Access Controller (EMAC). The three modules, which form the Core Unit, are products from a German manufacturer D.SignT© Digital Signalprocessing Technology.

During the first attempts to obtain experimental results, the utilization of VEMCO® tags revealed itself to be a ineffective decision. Although the sinusoidal signal is easily processed, i.e. detecting the presence of those signals is quite simple, their high frequency of transmission (> 63kHz) have damaged the subsequent processing, mainly due to weak (cross) correlation results, largely affected by multipath and noise corruptions. Therefore, direction results were never reasonable, and in some cases they could not have been calculated due to impossible physical interpretations.

Commercially available solutions are mostly based on pure sinusoidal signals, whereby they do not stray from the VEMCO® tags. The alternative to the latter consisted on using a built-from-scratch (by the DSOR research team) Acoustic Transmitter Case that provides a set of spread spectrum (SS) signals. Besides a pure sinusoid with a natural frequency of 25 kHz, the set also includes a Direct Sequence Spread Spectrum (DSSS) signal and a Frequency Hoping Spread Spectrum (FHSS) signal, both SS signals with improved quality through a closed-loop design methodology for underwater transducers pulse-shaping [8]. In terms of primary hardware components, the self-contained and waterproof case, henceforth called BlackBox, includes a microcontroller, a GPS and a power supply stack; a Power amplifier and impedance matching circuit; and lastly, a Pack of batteries. The transmitted signal is emitted on water through an omnidirectional ITC-1042 transducer.

Section III-A mentioned a scheme of interrogations in order to determine distances. Recalling the Problem Statement in Section II, the Surface Robotic Tool (SRT) is intrinsically linked to the calculation of distances, since it provides a position, i.e. a distance along with a direction. In what may concern the computational core behind the SRT, the BlackBox offers interesting similarities, hence this is a good opportunity to validate the range measurement operation.

Mostly, the computations are based on Time of Flight (TOF) measurements, which presuppose the synchronization between the signal transmitter and the receiver. To synchronize them, somehow they ought to be over the same timebase. Traditionally, the timebase is initialized with an interrupt that sets a new zero and agrees with the beginning of the transmitted signal. The goal is to feed both the transmitter and the receiver with that same interruption such that when the PURT receives the signal, automatically it is able to determine the distance based on the multiplication of the time of arrival with the speed of sound on water.

Slight modifications, particularly in terms of code level, were made in both the BlackBox and in the DSP; in addition, two GPS antennas were added to the final set, one connected to the BlackBox and the other to the DSP unit. In turn, the GPS yields a kind of *Dirac Signal*: a very-low width rectangular pulse that provides a synchronized interruption.

A. *Results of the Acoustic Tests*

The tests were conducted in Docas de Belém, a location with close to ideal conditions, i.e. sufficient depth and wide sea area, thus ensuring a more fragile presence of multipath and noise interferences.

The USBL array was hold into the same position. The variable herein considered is the orientation of the source according to the Body Frame $\mathcal{B}$. 0^o and $\pm 90^o$ headings were assessed, in other words, the USBL array was either facing the transducer or placed perpendicular to it. Thus the direction is expressed only in the (2-dimensional) horizontal plane[1], herein expressed as an angle of arrival θ while the distance is denoted by R. The transducer was placed in three different positions with the same depth as the array's. Fig. 6 provides an aerial panoramic view of Docas de Belém with marked Emission (red) and Reception (yellow) positions.

Let Δ_{TDOA} be the number of delayed samples associated to a Time Difference of Arrival (TDOA). If c is the speed of sound on water, f_s the sampling frequency and L the distance between the pair of hydrophones, θ results in

[1]Due to the absence of a vertical plan, the USBL array only needs a pair of hydrophones to determine directions.

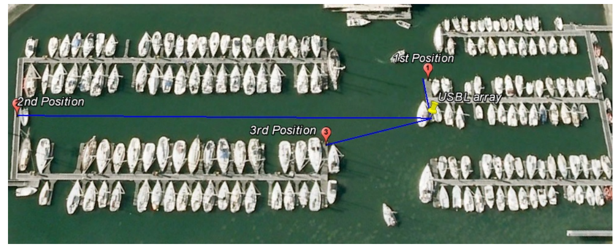


Fig. 6: Tests site. - Created with Google Earth©.

$$\theta = \arcsin\left(\frac{c\Delta_{TDOA}}{Lf_s}\right). \quad (56)$$

Given the non-linearity of (56), it might be interesting to analyze the sensitivity S of θ with respect to Δ_{TDOA} measurements, which follows as

$$S(\theta, \Delta^i_{TDOA}) = \left|\frac{\partial\theta}{\partial\Delta_{TDOA}}\right|_{\Delta_{TDOA}=\Delta^i_{TDOA}}, \quad (57)$$

where Δ^i_{TDOA} is the measurement of delayed samples on moment i. Observing Fig. 7, it is possible to conclude that the determination of θ is more sensitive the higher the amount of delayed samples is, therefore when using relative headings of $\pm 90^o$ it is expected to obtain a response in time with bigger angle of arrival hops.

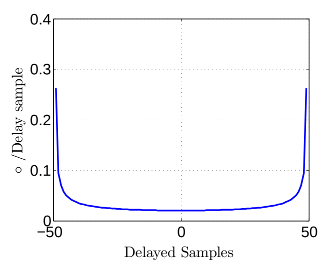
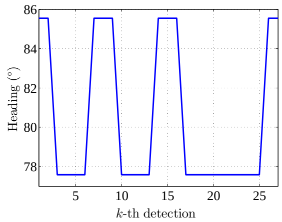
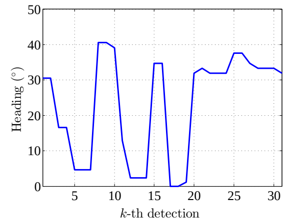


Fig. 7: Sensitivity of θ wrt Δ_{TDOA}.

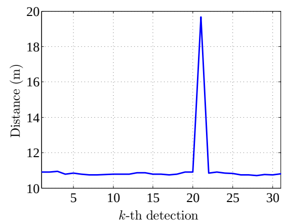
The first results suffered from a short range between the source and the agent and a boat was moored near the pier. Therefore, not only the planar wave approximation was poor but also it could not be avoided multipath caused by reflections on the boat. As a consequence, the results concerning the angle of arrival were more affected, as it can be seen in Fig. 8, specially when the source is 0^o oriented in $\mathcal{B}$. Also note the angle hops when the heading is 90^o; this situation leads to a theoretical maximum of delayed samples, thus having a great angle variation when delay samples differ by one, which is happening in Fig. 8a. Apart from one outlier, the range determination is consistent and coherent when compared with the expected value. The outlier might have been caused by a multipath detection instead of the original signal, virtually delaying the acquisition and thus yielding a greater range.



(a) Heading - Source oriented at 90°.

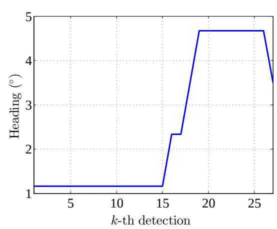


(b) Heading - Source oriented at 0°.

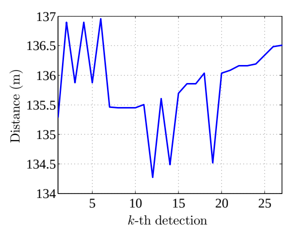


(c) Distance - Source oriented at 0°.

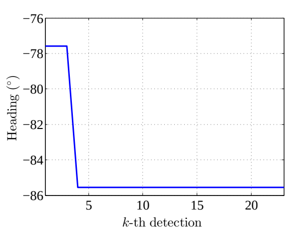
Fig. 8: Results with BlackBox in 1st Position.



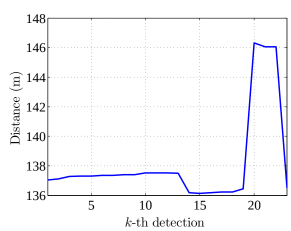
(a) Heading - Array oriented at 0°.



(b) Distance - Array oriented at 0°.



(c) Heading - Array oriented at −90°.



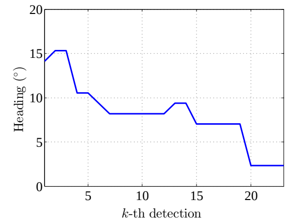
(d) Distance - Array oriented at −90°.

Fig. 9: Results with BlackBox in 2nd Position.

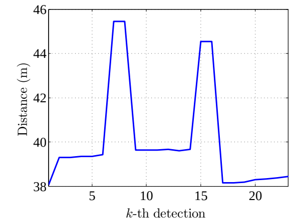
Although far, hence damaging the quality of the received signals by degrading the signal-to-noise ratio, the second position was amongst the duo of localizations that offered the best conditions due to cleared space between the transmitter and the receiver. The small deviations can be explained by the fact that the array was being hold with bare hands (no permanent installation was possible), thence not having the same heading.

The third and last position also yielded good results, similar to the ones from second position.

The three tests were not isolated, i.e. boats were often crossing the region of transmission. On the other hand,



(a) Heading - Array oriented at 0°.



(b) Distance - Array oriented at 0°.

Fig. 10: Results with BlackBox in 3rd Position.

field incursions on open sea are expected to return better values. Nevertheless, these results are sufficient to validate the acoustic positioning system, which confirms the PURT as a reliable solution for marine tracking.

V. CONCLUSIONS

This paper presented a novel time-varying Kalman filter with globally asymptotically stable error dynamics for the problem of localization based on direction and Doppler measurements to a single source. The observability of the system was characterized, which allowed to conclude about the asymptotic stability of the Kalman filter. Simulations results were presented that illustrate the good performance achieved by the proposed solution. Sea tests were conducted that validate the USBL acoustic positioning system. Future work includes further tests, therein with the proposed solution.

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