Combination of Lyapunov and Density Functions for Stability of Rotational Motion

José Fernandes Vasconcelos, Anders Rantzer, *Fellow, IEEE*, Carlos Silvestre, *Member, IEEE*, and Paulo Jorge Oliveira, *Member, IEEE*

***Abstract*—Lyapunov methods and density functions provide dual characterizations of the solutions of a nonlinear dynamic system. This work exploits the idea of combining both techniques, to yield stability results that are valid for almost all the solutions of the system. Based on the combination of Lyapunov and density functions, analysis methods are proposed for the derivation of almost input-to-state stability, and of almost global stability in nonlinear systems. The techniques are illustrated for an inertial attitude observer, where angular velocity readings are corrupted by non-idealities.**

***Index Terms*—Asymptotic stability, density functions, input-to-state stability, Lyapunov methods.**

NOMENCLATURE

The notation adopted is fairly standard. The set of $n \times m$ matrices with real entries is denoted as $\mathrm{M}(n,m)$ and $\mathrm{M}(n) := \mathrm{M}(n,n)$. The set of special orthogonal matrices is denoted as $\mathrm{SO}(n) := \{\mathbf{R} \in \mathrm{M}(n) : \mathbf{R}'\mathbf{R} = \mathbf{I}, \det(\mathbf{R}) = 1\}$. The nominal, the measured, and the estimated quantity $\mathbf{s}$ are denoted by $\bar{\mathbf{s}}$, $\mathbf{s}_r$ and $\hat{\mathbf{s}}$, respectively, and $\|\mathbf{s}\|$ denotes the Frobenius norm. The supremum norm [17] of a time function $\mathbf{s}(t)$ is denoted by $\|\mathbf{s}\|_\infty$. The operator $(\mathbf{s})_\times$ produces the skew symmetric matrix defined by the vector $\mathbf{s} \in \mathbb{R}^3$ such that $(\mathbf{s})_\times \mathbf{b} = \mathbf{s} \times \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^3$, and $(\cdot)_\otimes$ is the unskew operator such that $((\mathbf{s})_\times)_\otimes = \mathbf{s}$. The vectorization operator, denoted as $\mathrm{vec}(\cdot)$, returns a vector in $\mathbb{R}^{n\times m}$ by stacking the columns of a $n \times m$ matrix from left to right. The divergence operator [7], [15] is denoted by $\mathrm{div}(\cdot)$. The time dependence of the variables will be omitted in general, but otherwise denoted for the sake of clarity.

I. INTRODUCTION

GLOBAL stability is usually a highly desirable property in control and estimation algorithms. However, topological obstacles to continuous global stabilization arise in many dynamic systems, due to the fact that no smooth vector field can have a global attractor, unless the state space on which it is defined is homeomorphic to $\mathbb{R}^n$ [5]. As a consequence, controllers and observers designed using continuous state feedback on smooth manifolds, will always produce some trajectories that do not converge to the origin [2], [10]. Due to the presence of unstable manifolds, stability analysis using Lyapunov's second theorem is more complex.

New analysis tools have been introduced, by adopting the milder notion of almost global stability [1], [13]. In this framework, an equilibrium is "almost globally stable" in the sense that for all initial states, except for a set of zero measure, the dynamics converge to the equilibrium. A dual to the Lyapunov second method for analysis of almost global stability is developed in [13], [14], based on density functions, that represent the stationary density of a substance that flows along the system trajectories [11]–[13]. Almost global stability is obtained by verifying that, for a time-invariant density function, particles are generated almost everywhere and hence must flow to a sink, located at the origin.

A similar approach has been adopted for the analysis of input-to-state stability (ISS). The ISS paradigm has been extensively developed in recent years, as presented in the comprehensive survey of ISS notions and results found in [17], and in the list of references contained therein. The limitations to global stability on non-Euclidean spaces, and the fact that global stability is a necessary condition for ISS, motivate the relaxation to almost ISS, proposed in [1] and considered in [8]. The notions of robust and weakly almost ISS are proposed, and results for these properties using density functions are investigated. More important, it is suggested that a combination of Lyapunov methods with density function results, may be the right technique for proving almost ISS in general. Surprisingly enough, this enriching insight seems to have gone unnoticed in the subsequent literature.

This work develops the idea of combining Lyapunov and density functions, for the stability analysis of nonlinear systems. In the proposed analysis techniques, the Lyapunov function is adopted to characterize the system trajectories, however the Lyapunov stability analysis is limited by the existence of unstable manifolds. Density functions are used to resolve for the regions where the Lyapunov method is inconclusive, yielding sufficient conditions for instability of undesirable equilibrium points, and for convergence of almost all solutions to the region where stability is guaranteed by the Lyapunov function.

The first result proposed in this work formalizes an analysis method for almost ISS. It is evidenced how Lyapunov and density functions can be adopted to analyze local and weakly almost ISS, respectively, and it is shown that the combination of these

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J. F. Vasconcelos, C. Silvestre, and P. J. Oliveira are with the Institute for Systems and Robotics, Instituto Superior Técnico, Lisbon, Portugal (e-mail: {jf-vasconcelos, cjs, pjcro}@isr.ist.utl.pt).

A. Rantzer is with the Department of Automatic Control, Faculty of Engineering, Lund University, 221 00 Lund, Sweden (e-mail: rantzer@control.lth.se).

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two ISS concepts may yield the desired almost ISS property. The second major contribution yields a new tool for local stability analysis of isolated points, using density functions. Combined with LaSalle's invariance principle, it is evidenced that the proposed analysis result can be used to attain almost global stability of the origin.

The techniques are illustrated for the error dynamics of an attitude observer defined on SO(3), where angular velocity readings are corrupted by non-idealities, such as unknown bias and bounded measurement noise. The proposed analysis results are adopted to study almost ISS of the estimation error in the presence of sensor noise, and almost GAS of a reduced-order error dynamics in the presence of sensor bias. The presented almost GAS derivation of the attitude observer is an alternative to the analysis methods based on Hartman-Grobman theorem, and the almost ISS property of the attitude observer is novel to the best of the author's knowledge.

This work is organized as follows. Section II derives the stability analysis results based on the combination of Lyapunov and density functions. In Section III, the proposed analysis results are applied to an attitude observer with non-ideal velocity measurements. Concluding remarks and guidelines for future work are discussed in Section IV.

II. Stability Analysis Using Lyapunov and Density Functions

This section derives new stability analysis results based on the combination of Lyapunov and density functions. The stability analysis methods are designed for ISS analysis of nonlinear systems with unknown inputs, and for local stability analysis of equilibrium points other than the origin. A method to analyze almost GAS is also proposed, by combining the local analysis results with LaSalle's invariance principle.

A. Almost Input-to-State Stability Analysis

The analysis of input-to-state stability, using the combination of Lyapunov and density function techniques, is considered for systems in the form

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in M$ is the state, M is a smooth manifold, and $f : M \times U \to TM$, is a locally Lipschitz manifold map which satisfies $f(x, u) \in T_x M$, for all $x \in M$ and all $u \in U \subset \mathbb{R}^m$. The notion of ISS is classically defined using comparison functions [17]. However, the limitations to global stability on non-Euclidean spaces motivate the relaxation proposed in [1], formulated as follows.

Definition 1 (Almost ISS, [1]): The system (1) is almost ISS with respect to the origin, denoted as 0_M, if 0_M is locally asymptotically stable and

$$\forall u,\ \forall \text{a.a. } x(t_0) \in M,\ \limsup_{t\to\infty} |x(t)| \leq \gamma\left(\|u\|_\infty\right) \tag{2}$$

where γ is a class $\mathcal{K}$ function, $|\cdot|$ is the distance to the origin, and "∀a.a." abbreviates the quantifier "for almost all".

Expressed in words, almost ISS is verified when, for each valid input, all initial conditions outside a set of zero measure converge to a neighborhood of the origin, whose radius grows monotonically with the bound on the input. The density functions framework relaxes the concept of ISS, by considering that a zero measure set of trajectories can be effectively destabilized by the input, but that almost all trajectories converge to a neighborhood of the origin. Note that the quantifiers in (2) are not commutable in general, because the set of converging initial conditions is a function of the input u.

In this work, a method to derive almost ISS is obtained by combining the properties of Lyapunov and density functions. The adopted methodology has been sketched in [1] by means of examples, however it seems to have been unnoticed in subsequent literature. This section provides a contribution to the concept of combining Lyapunov and density functions, by formulating the technique in explicit mathematical statements, that characterize the result as the combination of two ISS properties. These two ISS concepts are introduced in the following.

Definition 2 (Local ISS): A system (1) is locally input-to-state stable with respect to 0_M, if 0_M is locally asymptotically stable and there exists $r > 0$ such that

$$\forall u,\ \forall\, |x(t_0)| \leq r, \limsup_{t\to\infty} |x(t)| \leq \gamma_1\left(\|u\|_\infty\right) \tag{3}$$

where γ_1 is a class $\mathcal{K}$ function.

Definition 3 (Weakly Almost ISS, [1]): A system (1) is weakly almost ISS with respect to 0_M, if 0_M is locally asymptotically stable and

$$\forall u,\ \forall \text{a.a. } x(t_0) \in M,\ \liminf_{t\to\infty} |x(t)| \leq \gamma_2\left(\|u\|_\infty\right) \tag{4}$$

where γ_2 is a class $\mathcal{K}$ function.

Provided that these ISS properties are verified, the main result of this section shows that almost ISS is attained.

Lemma 1 (Almost ISS): Assume that the system (1) is locally ISS and weakly almost ISS, then, for all inputs such that $\gamma_2(\|u\|_\infty) < r$, the system is almost ISS with $\gamma = \gamma_1$.

Proof: Weakly almost ISS, expressed in (4), implies that, by the continuity of the solutions of (1), almost every solution satisfies $|x(t)| \leq \gamma_2(\|u\|_\infty) < r$ for some t, thus entering the region where the trajectories eventually satisfy the lim sup condition expressed in (3), yielding (2). ∎

The proposed ISS analysis technique is based on Lemma 1, which shows that almost ISS can be obtained by combining local ISS with weakly almost ISS, for sufficiently small inputs. Lyapunov methods can be used to derive local ISS [4], while weakly almost ISS is associated with density functions [1].

The stability analysis technique is illustrated in Fig. 1. Lyapunov techniques yield local ISS based on ultimate boundedness and/or ISS results [4, Theorems 4.18 and 4.19]. As shown in Fig. 1(a), Lyapunov methods find a region $\{x \in M : \gamma_1(\|u\|_\infty) < |x| < r\}$ where the Lyapunov function V decreases along the system trajectories ($\dot{V} < 0$). The level sets of V are analyzed to show that the trajectories converge to a level set that is positively invariant and that is a subset of $\{x \in M : |x| < \gamma_1(\|u\|_\infty)\}$, hence guaranteeing that the

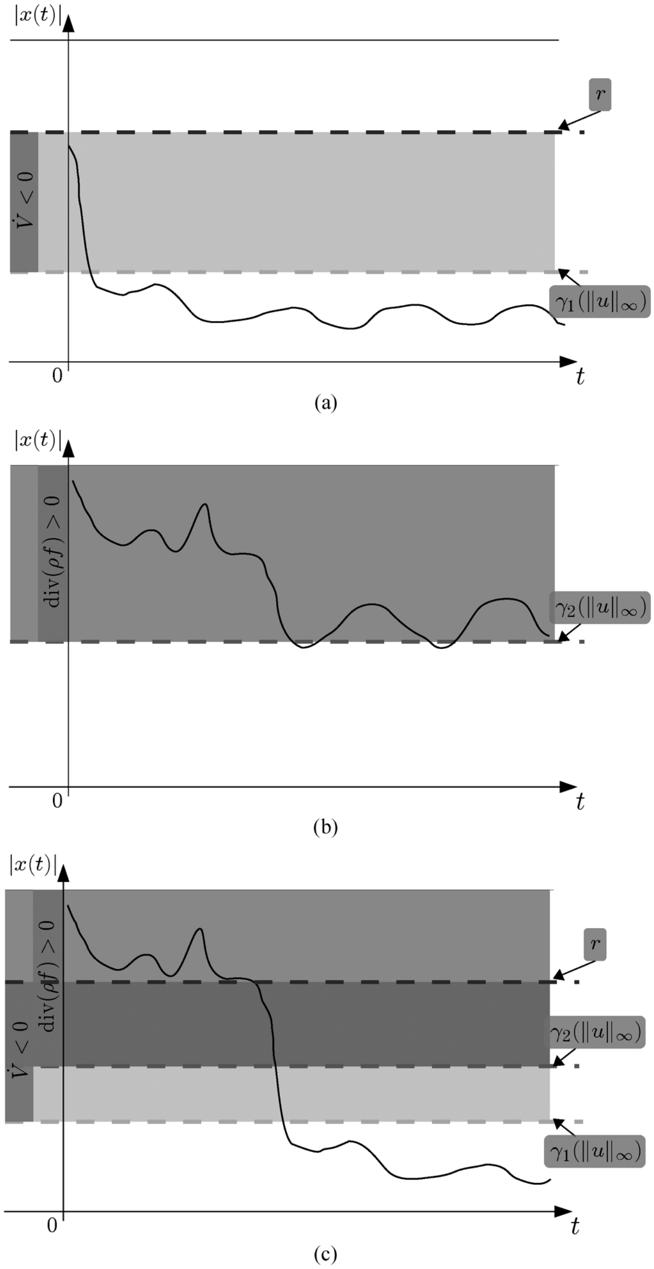

Fig. 1. Combination of Lyapunov and density functions for almost ISS of the origin. (a) Local ISS using Lyapunov function analysis: solutions emanating below the bound r converge to the region bounded by $\gamma_1(\|u\|_\infty)$. (b) Weakly almost ISS using density function analysis: the lim inf property of almost all solutions satisfies the bound $\gamma_2(\|u\|_\infty)$. (c) Almost ISS using Lyapunov and density functions analysis: by the lim inf property, almost all trajectories enter the region below the bound r, and converge to the region bounded by $\gamma_1(\|u\|_\infty)$.

solutions of the system are ultimately bounded with bound $\gamma_1(\|u\|_\infty)$.

However, the behavior of the solutions for $|x(t)| \geq r$ is undetermined by the Lyapunov function analysis, and density functions techniques are adopted to guarantee that almost all solutions enter $\{x \in M : |x| < r\}$ for some time instant. This is obtained by finding a density function ρ such that $\mathrm{div}(\rho f) > 0$ in the region $\{x \in M : |x| > \gamma_2(\|u\|_\infty)\}$, which yields weakly almost ISS by [1, Theorem 4]. Hence, the trajectories of the system are endowed with the lim inf characteristic depicted in Fig. 1(b), and enter the region $\{x \in M : \gamma_1(\|u\|_\infty) < |x| < r\}$ in finite time, as shown in Fig. 1(c). Consequently, almost ISS is obtained.

B. Local Stability and Almost Global Stability Analysis

The proposed stability analysis results are derived for autonomous nonlinear systems of the form

$$\dot{x} = f(x) \tag{5}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, and the associated flow $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\phi_t(x_0) = x(t, x_0)$, where $x(t, x_0)$ denotes the solution of the system at time t with initial condition x_0. In the remainder of this work, it is assumed that ϕ_t is well defined [16, Chapter 7].

Assumption 1: The flow ϕ_t is unique, continuous, and exists for all non-negative t.

To formulate the stability results in the presence of multiple equilibrium points, some concepts are introduced, for more details the reader is referred to [16]. The values at time t of the trajectories starting in the set $\mathcal{A}$ are denoted by $\phi_t(\mathcal{A}) = \{x \in \mathbb{R}^n : \exists x_0 \in \mathcal{A}, x = \phi_t(x_0)\}$. The local inset of x_w is the set of all initial conditions inside a neighborhood W of x_w that converge to x_w without leaving W, i.e.

$$\mathcal{Z}_W(x_w) = \{x \in W : \forall \epsilon, \exists T, \forall t > T,\ |\phi_t(x) - x_w| < \epsilon \text{ and } \forall t > 0,\ \phi_t(x) \in W\}. \tag{6}$$

The global inset of x_w, denoted as $\mathcal{W}(x_w)$, is defined by taking (6) with $W = \mathbb{R}^n$.

The following theorem is a new result in density function methodologies, and provides sufficient conditions to show that an equilibrium point is not stable, given a suitable density function. This property is of interest to exclude the stability of equilibrium points other than the origin.

Theorem 2: Let $x_w \in \mathbb{R}^n \setminus \{0\}$, and suppose there exists a non-negative $\rho \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$, integrable in a neighborhood W of x_w, and with $\mathrm{div}(\rho f) > 0$ in W. Then, the global inset of x_w has zero measure.

Proof: First, it is shown that the local inset, denoted as $\mathcal{Z}_W$ with a slight abuse of notation, has zero measure. By Lemma 13 presented in Appendix A, the local inset $\mathcal{Z}_W$ is measurable. Using [13, Lemma A.1] with $D = W$ produces

$$0 \geq \int_{\phi_t(\mathcal{Z}_W)} \rho(x)dx - \int_{\mathcal{Z}_W} \rho(z)dz = \int_0^t \int_{\phi_\tau(\mathcal{Z}_W)} [\mathrm{div}(\rho f)](x)dxd\tau.$$

Since $\mathrm{div}(\rho f) > 0$ in W, then $\phi_t(\mathcal{Z}_W) \subset \mathcal{Z}_W \subset W$ has zero measure. The flow ϕ_t is a diffeomorphism and hence $\mathcal{Z}_W$ has zero measure, for results on set measure results see Appendix A. By Lemma 14 presented in the appendix, the zero measure of $\mathcal{Z}_W$ yields that $\mathcal{W}(x_w)$ has zero measure, which concludes the proof. ∎

The combination of the density function results presented in Theorem 2 with LaSalle's invariance principle can be used to provide almost global stability of the origin. The technique is based on using LaSalle's invariance principle to show that the trajectories approach a candidate set M in the sense discussed in [4]; and then using the $\mathrm{div}(\rho f) > 0$ property for $M \setminus \{0\}$, to show that the set of trajectories converging to $M \setminus \{0\}$ is of zero measure, and hence that the origin is almost globally asymptotically stable.

Lemma 3: Let M be a countable union of isolated points containing the origin, and assume that the trajectories of the system (5) approach M as $t \to \infty$. If there is a density function that satisfies the conditions of Theorem 2 for all $x_w \in M \setminus \{0\}$, then the origin of (5) is almost GAS.

Proof: Since the trajectories of (5) approach M as $t \to \infty$, then, by definition [4], [16], the trajectories satisfy $\forall x_0$, $\forall \varepsilon$, $\exists T > 0$, $\forall t > T$, $\inf_{y \in M} \|\phi_t(x_0) - y\| < \varepsilon$. By the continuity of $\phi_t(x)$, choosing $\varepsilon < \min_{x,y \in M} \|x - y\|$ shows that each solution of (5) must converge to an isolated point $x_w \in M$. By Theorem 2, the condition $\mathrm{div}(\rho f) > 0$ for a neighborhood W of every $x_w \in M \setminus \{0\}$ guarantees that the global inset of x_w has zero measure. The set of initial conditions that converge to $M \setminus \{0\}$, given by $\cup_{x_w \in M \setminus \{0\}} \mathcal{W}(x_w)$, is a countable union of zero measured sets and hence has zero measure. Consequently, almost all solutions converge to the origin, and hence the origin is almost globally asymptotically stable. ∎

Remark 1: As argued in [1], density function methods may provide an alternative approach to the stability analysis based on Hartman-Grobman theorem [16]. The result of Theorem 2, based on density function theory, allows for local stability analysis without assuming hyperbolicity of the equilibrium point.

III. Stability Analysis of a Nonlinear Attitude Observer

In this section, the proposed analysis methods are applied to study the stability of an inertial attitude observer in the presence of non-ideal velocity readings. The considered attitude observer, summarized in Appendix B, is based on the work presented in [18] and similar observers can be found in [6], [9]. The almost ISS analysis technique is demonstrated for the case of sensor readings corrupted by a bounded disturbance, and the almost GAS analysis is applied for the case of a simplified attitude observer with biased velocity measurements.

A. Stability of the Nonlinear Observer in the Presence of Unmodeled Disturbance

The combination of Lyapunov techniques for almost ISS analysis is illustrated for the attitude estimation error kinematics in the presence of unmodeled bounded sensor noise. Using the observer kinematics formulated in [18] and described in Appendix B, the attitude error kinematics are given by

$$\dot{\mathcal{R}} = -k_\omega \mathcal{R}(\mathcal{R} - \mathcal{R}') - \mathcal{R} \cdot (\mathbf{u}_\omega)_\times \tag{7}$$

where $\mathcal{R} := \hat{\mathcal{R}}'\bar{\mathcal{R}}$ is the attitude estimation error, $\hat{\mathcal{R}}, \bar{\mathcal{R}} \in \mathrm{SO}(3)$ denote the estimated and the nominal attitude matrices, respectively, $k_\omega \in \mathbb{R}^+$ is a feedback gain, $\mathbf{u}_\omega \in \mathbb{R}^3$ is the velocity sensor disturbance, and the considered set of valid disturbances is given by $\{\mathbf{u} \in \mathbb{R}^3 : \|\mathbf{u}\|_\infty \leq u_{\max}\}$, $u_{\max} \in \mathbb{R}^+$. The trajectories of the kinematics (7) satisfy $\mathcal{R}(t) \in \mathrm{SO}(3)$ for all t, even in the presence of the velocity disturbance, and hence the system is well defined.

Almost ISS of the origin is obtained by applying the combination of Lyapunov techniques described in Section II-A, namely *i*) Lyapunov methods to attain local ISS, *ii*) density function techniques to yield weakly almost ISS, and *iii*) Lemma 1 to attain almost ISS. The next proposition bears local ISS by showing uniform ultimate boundedness [4], i.e., the trajectories emanating from initials conditions in a known neighborhood of the origin converge to a neighborhood of the origin in finite time, independently of t_0.

Theorem 4: Let $k_\omega > (u_{\max}/2)$, then for any initial condition

$$\mathcal{R}(t_0) \in \left\{\mathcal{R} \in \mathrm{SO}(3) : \|\mathbf{I} - \mathcal{R}\|^2 < r\left(\|\mathbf{u}_\omega\|_\infty\right)\right\} \tag{8a}$$

where $r(u) = 4\left(1 + \sqrt{1 - \frac{u^2}{4k_\omega^2}}\right)$, there exists T, independent of t_0, such that for all bounded inputs $\|\mathbf{u}_\omega\|_\infty \leq u_{\max}$, the trajectory of the system (7) satisfies

$$\mathcal{R}(t) \in \left\{\mathcal{R} \in \mathrm{SO}(3) : \|\mathbf{I} - \mathcal{R}\|^2 < \gamma_1\left(\|\mathbf{u}_\omega\|_\infty\right)\right\} \tag{8b}$$

for all $t \geq t_0 + T$, where $\gamma_1(u) = 4\left(1 - \sqrt{1 - \frac{u^2}{4k_\omega^2}}\right)$.

Proof: The proof is based on the derivation of boundedness for nonlinear systems presented in [4, Theorem 4.18], using Lyapunov methods. The time derivative of the Lyapunov function $V = \|\mathbf{I} - \mathcal{R}\|^2/2$ along the system trajectories is given by $\dot{V} = -k_\omega \|\mathbf{I} - \mathcal{R}^2\|^2/2 + \mathrm{tr}((\mathcal{R} - \mathcal{R}')/2 \cdot (\mathbf{u}_\omega)_\times)$. Algebraic manipulation of $\dot{V}$ produces

$$\begin{aligned}\dot{V} &\leq -k_\omega \frac{\|\mathbf{I} - \mathcal{R}^2\|^2}{2} + \frac{1}{2}\|\mathcal{R} - \mathcal{R}'\| \left\|(\mathbf{u}_\omega)_\times\right\| \\ &= -k_\omega \|\mathbf{I} - \mathcal{R}^2\| \left(\frac{1}{2}\|\mathbf{I} - \mathcal{R}^2\| - \frac{\|\mathbf{u}_\omega\|}{k_\omega\sqrt{2}}\right)\end{aligned}$$

where $\mathrm{tr}(\mathbf{A}'\mathbf{B}) < \sqrt{\mathrm{tr}(\mathbf{A}'\mathbf{A})\mathrm{tr}(\mathbf{B}'\mathbf{B})}$ and $2\mathrm{tr}(\mathbf{I} - \mathcal{R}^2) = \|\mathbf{I} - \mathcal{R}^2\|^2 = \|\mathcal{R} - \mathcal{R}'\|^2$ were used. It is immediate that

$$\frac{1}{2}\|\mathbf{I} - \mathcal{R}^2\| > \frac{\|\mathbf{u}_\omega\|_\infty}{k_\omega\sqrt{2}} \Rightarrow \dot{V} < 0.$$

Using $\|\mathbf{I} - \mathcal{R}^2\|^2 = (8 - \|\mathbf{I} - \mathcal{R}\|^2)\|\mathbf{I} - \mathcal{R}\|^2/2$ produces

$$\frac{\|\mathbf{I} - \mathcal{R}^2\|}{2} > \frac{\|\mathbf{u}_\omega\|_\infty}{k_\omega\sqrt{2}} \Leftrightarrow \gamma_1\left(\|\mathbf{u}_\omega\|_\infty\right) \leq V \leq r\left(\|\mathbf{u}_\omega\|_\infty\right)$$

where the conditions $\|\mathbf{u}_\omega\|_\infty \leq u_{\max}$ and $k_\omega > (u_{\max}/2)$ guarantee that $(1 - (u^2/4k_\omega^2)) > 0$ and hence that $r(\|\mathbf{u}_\omega\|_\infty)$ and $\gamma_1(\|\mathbf{u}_\omega\|_\infty)$ are well defined.

Consider the level sets defined by the Lyapunov function

$$\begin{aligned}\underline{\Omega}_t &= \left\{\mathcal{R} \in \mathrm{SO}(3) : V(\mathcal{R}) \leq \gamma_1\left(\|\mathbf{u}_\omega\|_\infty\right)\right\} \\ \bar{\Omega}_t &= \left\{\mathcal{R} \in \mathrm{SO}(3) : V(\mathcal{R}) \leq r\left(\|\mathbf{u}_\omega\|_\infty\right)\right\}\end{aligned}$$

then $\mathcal{R} \in (\bar{\Omega}_t \setminus \underline{\Omega}_t)$ implies that $\dot{V} < 0$. Hence, $\bar{\Omega}_t$ is a positively invariant set, the trajectories of the system starting in $\bar{\Omega}_t$ enter $\underline{\Omega}_t$ in finite time, see [4, Section 4.8] for a motivation of the level sets involved, and any solution starting in $\underline{\Omega}_t$ will remain in the set since $\dot{V} < 0$ in the corresponding boundary. The

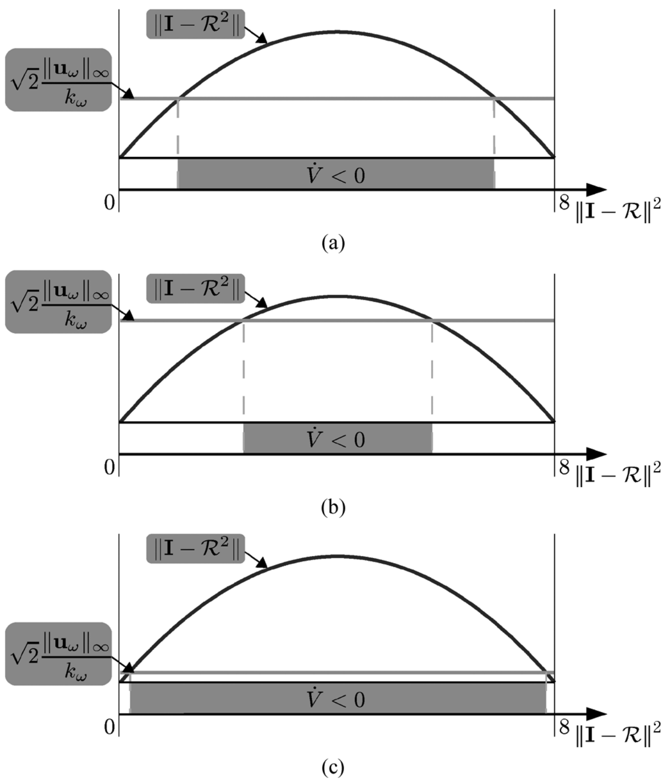

Fig. 2. Region of convergence as a function of the noise to gain ratio $\|\mathbf{u}_\omega\|_\infty/k_\omega$. (a) Balanced noise bound and gain. (b) Larger noise to gain ratio reduces the region where $\dot{V} < 0$. (c) Smaller noise to gain ratio increases the region where $\dot{V} < 0$.

initial conditions given by (8a) satisfy $\mathcal{R}(t_0) \in \bar{\Omega}_t$; any $\mathcal{R} \in \underline{\Omega}_t$ satisfies (8b), which concludes the proof. ■

The results stated in Theorem 4 are obtained using Lyapunov stability theory, and guarantee that any trajectory emanating from (8a) converges to a bounded region, as shown in Fig. 1(a). Fig. 2 portrays the regions (8a) and (8b) as a function of the noise to gain ratio $\|\mathbf{u}_\omega\|_\infty/k_\omega$. The region (8a) is smaller for large noise/small gain configuration, as illustrated in Figs. 2(a) and (b), and, conversely, is larger for small noise/large gain configuration, as illustrated in Figs. 2(a) and 2(c).

Following the proposed technique, a density function is adopted to show that almost all trajectories of the system (7) satisfy a lim inf condition, whose bound guarantees that the solutions enter the set (8a) in finite time.

Theorem 5: The system (7) is weakly almost ISS with respect to $\mathbf{I}$. Namely, the solutions verify

$$\forall \mathbf{u}_\omega,\ \forall \text{a.a. } \mathcal{R}(t_0) \in \mathrm{SO}(3),\ \liminf_{t\to\infty} \|\mathbf{I} - \mathcal{R}(t)\|^2 \le \gamma_2\left(\|\mathbf{u}_\omega\|_\infty\right) \tag{9}$$

where $\gamma_2(u) = 8(u/k_\omega)^2/(1 + (u/k_\omega)^2)$.

Proof: The result is obtained by satisfying the conditions of [1, Theorem 4], with the density function

$$\rho(\mathcal{R}) = \frac{1}{\mathrm{tr}^2(\mathbf{I} - \mathcal{R})}. \tag{10}$$

From the local ISS property obtained in Theorem 4, it is immediate that $\mathcal{R} = \mathbf{I}$ is a locally stable equilibrium point for $\mathbf{u}_\omega = 0$.

The function $f := \mathrm{vec}(k\mathcal{R}(\mathcal{R}' - \mathcal{R}) - \mathcal{R} \cdot (\mathbf{u}_\omega)_\times)$ is locally Lipschitz over SO(3) and C^1 over SO(3) $\setminus \{\mathbf{I}\}$. The density function $\rho(\mathcal{R})$ is of class C^1 over SO(3) $\setminus \{\mathbf{I}\}$ and, given that SO(3) is compact, verifies $\int_{\mathrm{SO}(3)\setminus U} \rho(\mathcal{R}) d\mathcal{R} < +\infty$, for all open neighborhoods U of 0_M.

The divergence is given by

$$\mathrm{div}(\rho f) = \frac{k_\omega}{\mathrm{tr}^3(\mathbf{I} - \mathcal{R})} \left(\|\mathbf{I} - \mathcal{R}\|^2 + \frac{2}{k_\omega} (\mathcal{R} - \mathcal{R}')'_\otimes \mathbf{u}_\omega \right)$$

where $\mathrm{div}(\rho f) = \rho \mathrm{div}(f) + \nabla(\rho)' f$, $\mathrm{div}(f) = -2k_\omega \mathrm{tr}(\mathcal{R})$ and $\nabla(\rho) = 2\mathrm{tr}(\mathbf{I} - \mathcal{R})^{-3}\mathrm{vec}(\mathbf{I})$, for more details on the computations of divergence and integrals in SO(3) see [3]. To attain the "density propagation inequality" [1, Theorem 4], given by

$$\forall u,\ \forall x \in M, |x| \ge \gamma_2\left(|u|\right) \Rightarrow \mathrm{div}\left[\rho(x) f(x,u)\right] \ge Q(x) \tag{11}$$

with $Q(x) > 0$ for almost all $x \in M$, the sufficient condition

$$\|\mathbf{I} - \mathcal{R}\|^2 + \frac{2}{k_\omega} (\mathcal{R} - \mathcal{R}')'_\otimes \mathbf{u}_\omega \ge \xi \|\mathbf{I} - \mathcal{R}\|^2 \quad 0 < \xi < 1$$

is analyzed, such that (11) is verified with $Q(\mathcal{R}) = \xi k_\omega \|\mathbf{I} - \mathcal{R}\|^2/\mathrm{tr}^3(\mathbf{I} - \mathcal{R}) > 0$ for almost all $\mathcal{R} \in \mathrm{SO}(3)$. The inequality is satisfied if

$$\|\mathbf{I} - \mathcal{R}\|^2 \ge 4 \frac{\|\mathbf{u}_\omega\|}{(1-\xi)k_\omega} \left\| \left(\frac{\mathcal{R} - \mathcal{R}'}{2} \right)_\otimes \right\|$$

$$\Leftrightarrow \|\mathbf{I} - \mathcal{R}\|^2 \ge \sqrt{2} \frac{\|\mathbf{u}_\omega\|}{(1-\xi)k_\omega} \|\mathbf{I} - \mathcal{R}\|^2$$

$$\Leftrightarrow \|\mathbf{I} - \mathcal{R}\|^2 \ge \frac{8 \left\| \frac{\mathbf{u}_\omega}{(1-\xi)k_\omega} \right\|^2}{1 + \left\| \frac{\mathbf{u}_\omega}{(1-\xi)k_\omega} \right\|^2} = \gamma_2 \left(\frac{\|\mathbf{u}_\omega\|}{1-\xi} \right)$$

Since $\gamma_2(u)$ is a class $\mathcal{K}$ function ($\gamma_2(0) = 0$ and $(\mathrm{d}\gamma_2(u)/\mathrm{d}u) > 0$), the inequality (11) is verified with

$$\forall \mathbf{u}_\omega,\ \forall \mathcal{R} \in \mathrm{SO}(3),\ \|\mathbf{I} - \mathcal{R}\|^2 \ge \gamma_2 \left(\frac{\|\mathbf{u}_\omega}{1-\xi} \right)$$
$$\Rightarrow \mathrm{div}(\rho f) \ge \frac{k_\omega \|\mathbf{I} - \mathcal{R}\|^2}{\mathrm{tr}^3(\mathbf{I} - \mathcal{R})} \xi$$

and hence $\forall \mathbf{u}_\omega$, $\forall \text{a.a.} \mathcal{R}(t_0) \in \mathrm{SO}(3)$, $\liminf_{t\to\infty} \|\mathbf{I} - \mathcal{R}(t)\|^2 \le \gamma_2(\|\mathbf{u}_\omega\|_\infty/(1-\xi))$. The constant ξ is arbitrarily small, and hence taking $\xi \to 0$ yields the result expressed in (9). ■

Using the result expressed in Lemma 1, almost ISS is obtained from the local ISS property derived in Theorem 4, and the weakly almost ISS analyzed in Theorem 5.

Theorem 6: Let $k_\omega > (u_{\max}/\sqrt{3})$. Then, the trajectories of the system (7) satisfy

$$\forall \mathbf{u}_\omega,\ \forall \text{a.a. } \mathcal{R}(t_0) \in \mathrm{SO}(3),$$
$$\limsup_{t\to\infty} \|\mathbf{I} - \mathcal{R}(t)\|^2 < \gamma\left(\|\mathbf{u}_\omega\|_\infty\right) \tag{12}$$

where $\gamma(u) = \gamma_1(u) = 4\left(1 - \sqrt{1 - \frac{u^2}{4k_\omega^2}}\right)$, i.e., the attitude observer is almost ISS with respect to $\mathbf{I}$.

Proof: The proof is immediate from Lemma 1. However, it is re-derived to illustrate the combination of Lyapunov

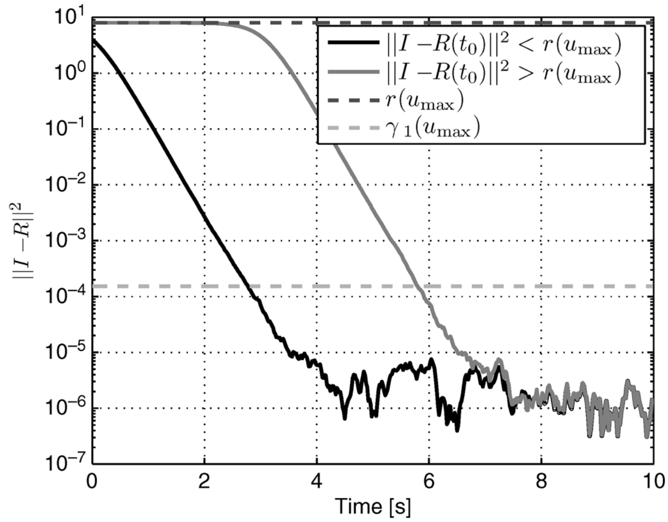

Fig. 3. Simulation results of the attitude observer, illustrating almost ISS. For $\|\mathbf{I} - \mathcal{R}(t_0)\|^2 < r(u_{\max})$, local ISS is guaranteed by Lyapunov methods (Theorem 4); for $\|\mathbf{I} - \mathcal{R}(t_0)\|^2 \geq r(u_{\max})$, density function analysis shows weakly almost ISS (Theorem 5). The combination of both ISS results (Lemma 1) produces the almost ISS property expressed in Theorem 6 and depicted in the figure.

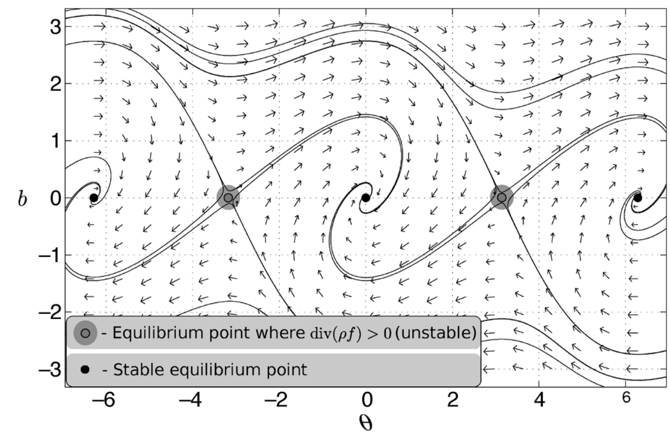

Fig. 4. Phase portrait of the reduced order attitude observer. Using the density function property $\mathrm{div}(\rho f) > 0$ in a neighborhood of the equilibrium points $(\theta, b) = (\pi + 2\pi k, 0)$, $k \in \mathbb{Z}$, shows that these are unstable.

and density function techniques for the present attitude observer. Using the lim inf condition (9), the trajectories of the system enter the positively invariant set (8a) for some t, if $\gamma_2(\|\mathbf{u}_\omega\|_\infty) < r(\|\mathbf{u}_\omega\|_\infty)$, which is equivalent to $2(\|\mathbf{u}_\omega\|_\infty/k_\omega)^2/(1 + (\|\mathbf{u}_\omega\|_\infty/k_\omega)^2) < 1 + \sqrt{1 - (\frac{\|\mathbf{u}_\omega\|_\infty}{2k_\omega})^2}$, that is satisfied for $\frac{\|\mathbf{u}_\omega\|_\infty}{k_\omega} < \sqrt{3}$. Consequently, almost all solutions enter the set defined in (8a) in finite time, and thus verify the lim sup condition of Theorem 4, yielding almost ISS. ■

Simulation results of the observer estimation error are depicted in Fig. 3. Note that the exponential convergence for $\gamma_1(u_{\max}) < \|\mathbf{I} - \mathcal{R}(t)\|^2 < r(u_{\max})$ is justified by the fact that $\dot{V} < -\alpha V$ in that region, for some $\alpha \in \mathbb{R}^+$. According to the proposed ISS derivation technique, based on Lemma 1, almost ISS of the attitude observer, formulated in (12), was obtained by combining the weakly almost ISS property (9), given by density function techniques, with the local ISS property (8), derived using Lyapunov techniques.

B. Stability of the Nonlinear Observer in the Presence of Biased Sensor Readings

In this section, the proposed stability analysis is illustrated for the attitude observer, in a case where the sensor disturbance is a constant bias, i.e., $\dot{\mathbf{u}}_\omega = 0$. Although the ISS results of Section III-A can be applied by considering the bias as an unmodeled disturbance, the bias dynamics are known, as detailed in [18] and summarized in Appendix B. In this case, the observer is augmented to dynamically estimate and compensate for the sensor bias, yielding stronger stability properties.

The closed loop error kinematics of the augmented observer are given by

$$\dot{\mathcal{R}} = \mathcal{R}\left[k_\omega(\mathcal{R}' - \mathcal{R}) + (\mathrm{b}_\omega)_\times\right] \quad \dot{\mathbf{b}}_\omega = k_{b_\omega}(\mathcal{R}' - \mathcal{R})_\otimes \quad (13)$$

where $k_\omega, k_{b_\omega} \in \mathbb{R}^+$ are feedback gains, $\mathbf{b}_\omega = \hat{\mathbf{b}}_\omega - \mathbf{u}_\omega$ is the bias estimation error, $\hat{\mathbf{b}}_\omega$ is the estimate of the velocity reading bias, and $\dot{\mathbf{u}}_\omega = 0$. The stability analysis technique is illustrated for the case where initial bias and attitude estimation errors exist along the z-axis, i.e., $\mathbf{b}(t_0) = [0\ 0\ b_0]'$, $b_0 \in \mathbb{R}$, and $\mathcal{R}(t_0) = \exp(\theta_0(\boldsymbol{\lambda}_0)_\times)$, $\theta_0 \in \mathbb{R}$, $\boldsymbol{\lambda}_0 = [0\ 0\ 1]'$. In this case, the trajectories of (13) are characterized by $\mathcal{R}(t) = \exp(\theta(t)(\boldsymbol{\lambda}_0)_\times)$, $\mathbf{b}(t) = [0\ 0\ b(t)]'$, and the dynamics can be reduced to

$$\dot{\theta} = -\sin(\theta) + b \quad \dot{b} = -\sin(\theta) \quad (14)$$

with initial conditions $\theta(t_0) = \theta_0$, $b(t_0) = b_0$.

The stability of the second order system (14) is analyzed using Lemma 3. Namely, choosing a suitable Lyapunov function and appealing to LaSalle's invariance principle, an invariant set M is derived. Then, a density function and the results expressed in Theorem 2 are used to show that almost all solutions approach a subset of M as $t \to \infty$.

Proposition 7: The trajectories of the system (14) approach $M = \{(\theta, b) \in \mathbb{R}^2 : \theta = \pi k, k \in \mathbb{Z}, b = 0\}$ as $t \to \infty$.

Proof: The result is obtained by considering the Lyapunov function $V = 2(1 - \cos(\theta)) + b^2$, with $\dot{V} = -2\sin^2(\theta)$. If the trajectories of (14) are bounded, the convergence result is immediate from LaSalle's invariance principle [16]. To show that the trajectories are bounded, the properties of the Lyapunov function are analyzed. Denoting the time index explicitly, $\dot{V}(t) \leq 0$ and $V(t) \geq 0$ imply that $V(t)$ converges to a limit and that $b(t)$ is bounded. The boundedness of θ is shown by contradiction. An unbounded θ implies the existence of a sequence $\theta(t_1) < \theta(t_2) < \ldots < \theta(t_k)$ such that $\sin(\theta(t_k)) = (-1)^k$ for all $k \in \mathbb{N}$. Because $|\dot{\theta}(t_k)|$ is bounded by $1 + \max_{t \in \mathbb{R}^+} |b(t)|$, there is a fixed C for all k such that $\sin^2(\theta(t)) > 1/4$ for $t \in [t_k\ t_k + C]$. Hence, $V(t_{k+1}) < V(t_k) - C/2$ and V becomes negative for large k, which is a contradiction. ■

The phase portrait of the system, depicted in Fig. 4, suggests that the equilibrium points in the set $E = \{(\theta, b) \in \mathbb{R}^2 : \theta = 2\pi k + \pi, k \in \mathbb{Z}, b = 0\} \subset M$ are unstable, and that almost all trajectories approach $M \setminus E = \{(\theta, b) \in \mathbb{R}^2 : \theta = 2\pi k, k \in \mathbb{Z}, b = 0\}$ as $t \to \infty$. This is demonstrated by combining Proposition 7 with the results based on density functions.

Proposition 8: Almost all trajectories of the system (14) approach the set $\{(\theta, b) \in \mathbb{R}^2 : \theta = 2\pi k, k \in \mathbb{Z}, b = 0\}$ as $t \to \infty$.

Proof: To exclude the points in the set E, we use the density function $\rho = 1/(2(1-\cos(\theta)) + b^2)$. The divergence is

$$\operatorname{div}(\rho f) = \frac{2\,(1-\cos(\theta)) - \cos(\theta) b^2}{\left(2(1-\cos(\theta)) + b^2\right)^2}.$$

There is a neighborhood of every point in E where ρ is integrable, and $\operatorname{div}(\rho f) > 0$. By Theorem 2, the set of initial conditions converging to each point in E has zero measure. The set E is a countable union of points, and hence the set of initial conditions that approach E has zero measure. Consequently, almost all the trajectories approach $M \setminus E$. ■

IV. Conclusion

This work addressed the combination of Lyapunov and density functions, for stability analysis of nonlinear systems. Almost ISS of the origin was formulated as the combination of local ISS and weakly almost ISS, that can be derived using the properties of Lyapunov and density functions, respectively. For the case of autonomous systems, it was shown that global stability of the origin can be obtained by combining LaSalle's invariance principle, with a density function that excludes the stability of undesirable equilibrium points. The proposed techniques were illustrated for the stability analysis of an attitude observer with non-ideal angular velocity readings.

The stability results for unforced systems were derived and illustrated for the case where the invariant set was a countable union of isolated points. Given that most of the tools adopted in the proof are valid for generic sets, future work will address the rigorous extension of the proposed stability results to sets. Also, the illustration of the technique for non-hyperbolic equilibria will be studied in future work.

Appendix A
Set Measure Results

This section presents some set measure results adopted in the paper, that are presented for the sake of completeness.

Proposition 9 ([16]): If f is smooth then ϕ_t is a diffeomorphism for each t.

Theorem 10 ([15]): Every Borel set is measurable. In particular, each open set and each closed set is measurable. The collection of measurable set is σ-algebra; that is the complement of a measurable set is measurable and the union (and intersection) of a countable collection of measurable sets is measurable.

Lemma 11 ([7]): Suppose $\mathcal{A} \in \mathbb{R}^n$ has measure zero and $F : \mathcal{A} \to \mathbb{R}^n$ is a smooth map. Then $F(\mathcal{A})$ has measure zero.

Corollary 12: If f is smooth, then the set $\mathcal{A}$ has zero measure if and only if the set $\phi_t(\mathcal{A})$ has zero measure.

Lemma 13: The local inset of an equilibrium point is measurable under Assumption 1.

Proof: Denote the neighborhood of the equilibrium point as W and the local inset as $\mathcal{Z}_W$, with a slight abuse of notation. The local inset $\mathcal{Z}_W$ is characterized by the intersection of a "stability" and a "convergence" sets, given by $\mathcal{Z}_W = \mathcal{S} \cap \mathcal{C}$ where

$$\mathcal{S} = \{x \in W : \phi_t(x) \in W \text{ for all } t \geq 0\}$$
$$\mathcal{C} = \{x \in W : \forall \epsilon, \exists T, \forall t > T,\ \|\phi_t(x) - x_u\| < \epsilon\}.$$

The set $\mathcal{S}$ can be described by $\mathcal{S} = \bigcap_{k \in \mathbb{N}_0} \mathcal{S}_k$ where

$$\mathcal{S}_k = \{x \in W : \phi_t(x) \in W \text{ for all } t \in [k\ k+1]\}.$$

By the continuous dependence of $\phi_t(x)$ on the initial conditions [4], [16], and on t, for each $x \in \mathcal{S}_k$ there exists δ sufficiently small, such that $\|x - y\| < \delta \Rightarrow \phi_t(y) \in W$ for the compact interval $t \in [k\ k+1]$. Consequently, the set $\mathcal{S}_k$ is open, thus measurable, and the set $\mathcal{S}$ is measurable.

The set $\mathcal{C}$ can be described by $\mathcal{C} = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}_0} \mathcal{C}_{n,k}$, where

$$\mathcal{C}_{n,k} = \left\{x \in W : \|\phi_t(x) - x_u\| < \frac{1}{n} \text{ for all } t \geq k\right\}.$$

The set $\mathcal{C}_{n,k}$ is measurable, by the same arguments used for the measurability of $\mathcal{S}$. Consequently, $\mathcal{C}$ is a countable union and intersection of measurable sets and is measurable, which concludes the proof. ■

Lemma 14: Consider an equilibrium point x_w and let $W \in \mathbb{R}^n$ be a neighborhood of x_w. Under Assumption 1, the local inset $\mathcal{Z}_W(x_w)$ has zero measure if and only if the global inset $\mathcal{W}(x_w)$ has zero measure.

Proof: ($\Leftarrow$) Immediate from $\mathcal{Z}_W(x_w) \subset \mathcal{W}(x_w)$. ($\Rightarrow$) Every trajectory of the global inset enters the local inset for t large enough, so the global inset can be described as $\mathcal{W}(x_w) = \bigcup_{k \in \mathbb{N}_0} \mathcal{T}_k$ where $\mathcal{T}_k = \{x \in \mathbb{R}^n : \phi_t(x) \in \mathcal{Z}_W(x_w) \text{ for all } t \geq k\}$. The set $\mathcal{T}_k$ satisfies $\phi_k(\mathcal{T}_k) \subset \mathcal{Z}_W(x_w)$ and hence $\phi_k(\mathcal{T}_k)$ has zero measure and, by Corollary 12, $\mathcal{T}_k$ has zero measure. Consequently $\mathcal{W}(x_w)$ is a countable union of zero measure sets, and hence has zero measure. ■

Appendix B
Attitude Observer Formulation

This section briefly introduces the observer used in Section III to illustrate the proposed stability analysis techniques. The attitude observer is designed to estimate the orientation of a rigid body with respect to a fixed inertial frame, by merging angular velocity measurements, with vectors observations obtained in body coordinates. The derivation of the observer is based on the work presented in [18] and similar solutions are found in [6], [9]. The rigid body kinematics are described by

$$\dot{\bar{\mathcal{R}}} = \bar{\mathcal{R}} \cdot (\bar{\omega})_\times$$

where $\bar{\mathcal{R}}$ is the rotation matrix from body frame to the inertial frame coordinates, and $\bar{\boldsymbol{\omega}}$ is the body angular velocity expressed in body coordinates. The body angular velocity is measured by a rate gyro sensor triad, and the measurement model is

$$\boldsymbol{\omega}_r = \bar{\boldsymbol{\omega}} + \mathbf{u}_\omega \tag{15}$$

where $\mathbf{u}_\omega$ is a measurement disturbance.

The vector observations are a function of the rigid body's attitude. The vectors coordinates are known and time-invariant in inertial frame, e.g., Earth's magnetic and gravitational fields, and measured in body coordinates by on-board sensors such as magnetometers and pendulums, among others. The vector measurement is expressed by

$$\mathbf{h}_{r,i} = {}^B\bar{\mathbf{h}}_i$$

where ${}^B\bar{\mathbf{h}}_i = \bar{\mathcal{R}}'{}^I\bar{\mathbf{h}}_i$, the leading superscripts B and I denote that the vector is expressed respectively in body and inertial coordinates, $i = 1 \ldots n$ is the vector index, and n is the number of vector measuring sensors.

The vector measurements $\mathbf{h}_{r,i}$ are introduced in the observer by means of a conveniently defined linear coordinate transformation, which is briefly described, for further details see [18]. The transformed vectors expressed in inertial and body frames are respectively given by

$$ {}^I\bar{\mathbf{x}}_j := \sum_{i=1}^{n} a_{ij}\,{}^I\bar{\mathbf{h}}_i \Leftrightarrow {}^I\bar{\mathbf{X}} := {}^I\bar{\mathbf{H}}\mathbf{A} \tag{16a}$$

$$ {}^B\bar{\mathbf{x}}_j := \sum_{i=1}^{n} a_{ij}\bar{\mathbf{h}}_{r,i} \Leftrightarrow {}^B\bar{\mathbf{X}} := \mathbf{H}_r\mathbf{A} \tag{16b}$$

where matrix $\mathbf{A} := [a_{ij}] \in \mathrm{M}(n)$ is invertible by construction, and ${}^I\bar{\mathbf{X}} := [{}^I\bar{\mathbf{x}}_1 \ldots {}^I\bar{\mathbf{x}}_n]$, ${}^B\bar{\mathbf{X}} := [{}^B\bar{\mathbf{x}}_1 \ldots {}^B\bar{\mathbf{x}}_n]$, ${}^I\bar{\mathbf{H}} := [{}^I\bar{\mathbf{h}}_1 \ldots {}^I\bar{\mathbf{h}}_n]$, $\mathbf{H}_r := [\mathbf{h}_{r,1} \ldots \mathbf{h}_{r,n}]$, ${}^I\bar{\mathbf{X}}, {}^B\bar{\mathbf{X}}, {}^I\bar{\mathbf{H}}, \mathbf{H}_r \in \mathrm{M}(3,n)$. In this work, the transformation $\mathbf{A}$ is defined such that ${}^I\bar{\mathbf{X}}{}^I\bar{\mathbf{X}}' = \mathbf{I}$, to shape uniformly the directionality introduced by the vector readings. Also, it is assumed that there are at least two noncollinear vectors ${}^I\mathbf{h}_i$, so that all rotational degrees of freedom are observable, see [18] and references therein for a discussion on the present observer characteristics.

The proposed observer estimates the attitude of the rigid body by computing the kinematics

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}} \cdot (\hat{\boldsymbol{\omega}})_\times \tag{17}$$

where $\hat{\mathcal{R}}$ is the estimated attitude and $\hat{\boldsymbol{\omega}}$ is the feedback term constructed to compensate for the attitude estimation error.

The attitude observer estimates the rotation matrix by exploiting the non-ideal angular velocity measurements (15) and the vector observations (16b) in the feedback term $\hat{\boldsymbol{\omega}}$, for more details on the adopted observer see [18]. In the case where the term $\mathbf{u}_\omega$ is considered an unmodeled and bounded disturbance, the feedback law $\hat{\boldsymbol{\omega}}$ is defined as

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}} \cdot (\hat{\boldsymbol{\omega}})_\times,\ \hat{\boldsymbol{\omega}} = \hat{\mathcal{R}}'{}^I\bar{\mathbf{X}}{}^B\bar{\mathbf{X}}'\boldsymbol{\omega}_r - k_\omega \sum_{i=1}^{n}\left(\hat{\mathcal{R}}'{}^I\bar{\mathbf{x}}_i\right) \times {}^B\bar{\mathbf{x}}_i \tag{18}$$

where $k_\omega \in \mathbb{R}^+$ is the feedback gain. When the sensor disturbance is modeled as an unknown time-invariant bias, i.e., $\dot{\mathbf{u}}_\omega = 0$, the observer can be augmented to dynamically compensate $\mathbf{u}_\omega$. In this case, the feedback law $\hat{\boldsymbol{\omega}}$ and the bias estimate are defined as

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}} \cdot (\hat{\boldsymbol{\omega}})_\times,\ \hat{\boldsymbol{\omega}} = \hat{\mathcal{R}}'{}^I\bar{\mathbf{X}}{}^B\bar{\mathbf{X}}'(\boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega) - k_\omega\mathbf{s}_\omega \tag{19a}$$

$$\dot{\hat{\mathbf{b}}}_\omega = k_{b_\omega}\mathbf{s}_\omega,\ \mathbf{s}_\omega = \sum_{i=1}^{n}\left(\hat{\mathcal{R}}'{}^I\bar{\mathbf{x}}_i\right) \times {}^B\bar{\mathbf{x}}_i \tag{19b}$$

where $k_\omega, k_{b_\omega} \in \mathbb{R}^+$ are feedback gains, and $\hat{\mathbf{b}}_\omega$ is the rate gyro bias estimate. The ISS properties of the attitude observer (18) are analyzed in Section III-A, and the almost global stability of the observer (19) is studied in Section III-B.

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José Fernandes Vasconcelos received the Licenciatura and the Ph.D. degrees in electrical engineering from the Instituto Superior Técnico (IST), Lisbon, Portugal, in 2003 and 2010, respectively.

Since 2009, he has been a Project Engineer with the Advanced Projects Department, Deimos Engenharia, Lisbon, Portugal. His research interests include nonlinear observers, inertial navigation systems, sensor calibration, and robust control with application to autonomous vehicles.

Anders Rantzer (F'01) was born in 1963. He received the Ph.D. degree in optimization and systems theory from the Royal Institute of Technology (KTH), Stockholm, Sweden.

After postdoctoral positions at KTH and at IMA, University of Minnesota, he joined the Department of Automatic Control, Lund University in 1993. He was appointed Professor of Automatic Control in Lund 1999. The academic year of 2004/05 he was Visiting Associate Faculty Member at California Institute of Technology. His research interests are in modeling, analysis and synthesis of control systems, with particular attention to uncertainty, optimization and distributed control.

Dr. Rantzer has been serving as Associate Editor of IEEE TRANSACTIONS ON AUTOMATIC CONTROL and several other journals. He is a winner of the SIAM Student Paper Competition, the IFAC Congress Young Author Price and the IET Premium Award for the best article in IEE Proceedings—Control Theory & Applications during 2006. He is a Fellow of IEEE and a member of the Royal Swedish Academy of Engineering Sciences.

Carlos Silvestre (M'07) received the Licenciatura degree in electrical engineering and the M.Sc. degree in electrical engineering and the Ph.D. degree in control science from the Instituto Superior Tecnico (IST), Lisbon, Portugal, in 1987, 1991 and 2000, respectively.

Since 2000, he has been with the Department of Electrical Engineering, IST, where he is currently an Associate Professor of Control and Robotics. Over the past years, he has conducted research in the fields of guidance, navigation, and control of air and underwater robots. His research interests include linear and nonlinear control theory, coordinated control of multiple vehicles, gain scheduled control, integrated design of guidance and control systems, inertial navigation systems, and mission control and real time architectures for complex autonomous systems with applications to unmanned air and underwater vehicles.

Paulo Jorge Oliveira (S'91-M'95) received the Licenciatura, M.Sc., and Ph.D. degrees in electrical and computer engineering in 1987, 1991, and 2002, respectively, from Instituto Superior Técnico (IST), Lisbon, Portugal.

He is an Associate Professor in the Department of Mechanical Engineering of IST and a Senior Researcher in the Institute for Systems and Robotics—Associated Laboratory. His research interests are in the area of autonomous robotic vehicles with focus on the fields of estimation, sensor fusion, navigation, positioning, and industrial automation. He is author or coauthor of more than 20 journal papers and 100 conference communications. He has participated in more than 20 European and Portuguese research projects, over the last 25 years.