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Decentralized linear state observers for vehicle formations with time-varying topologies

Daniel Viegas, Pedro Batista, Paulo Oliveira, and Carlos Silvestre

***Abstract*— This paper addresses the problem of decentralized state estimation in formations of vehicles with time-varying topologies. The proposed solution relies on the implementation of a local state observer on-board each vehicle, based only on local sensing capabilities and limited communication with neighboring vehicles, to estimate its state. The effects of changes in the formation topology over time are studied resorting to switched systems theory, and sufficient conditions for exponential stability of the global estimation error dynamics are presented for two different switching laws. The results are particularized for the case of a formation of Autonomous Underwater Vehicles (AUVs), and simulation results are presented to assess the performance of the proposed solution in the presence of measurement noise.**

I. Introduction

Due to successive advances in technology, important improvements in both the miniaturization and affordability of autonomous vehicles have been made over the years, and the availability of small, relatively cheap robots has naturally sparked the interest of the community towards their use in formations. In fact, there are many applications in which the use of multiple autonomous vehicles in a cooperative manner opens interesting opportunities. To name only a couple of examples, formations of Autonomous Underwater Vehicles (AUVs) can be used to perform oceanographic sampling and minesweeping missions, see e.g. [6] and [8], while the design of automated highway systems poses problems closely related to formations, such as traffic flow control and collision avoidance [4]. In this light, it comes as no surprise that the topics of control and estimation in multi-agent formations have seen many compelling approaches and contributions, see e.g. [7] and [16].

As the number of vehicles in a formation increase, so does the dimension and complexity of the problem, and the implementation of centralized solutions might yield cripplingly high costs in both computational complexity and communication loads in the formation. One way to cope with this is to distribute the computations between all the agents in the formation, preferably relying on locally available information, that is, solve the problems in a decentralized or distributed manner, see e.g. [2] and [13].

The problem addressed in this paper is the design of a decentralized state observer for a formation of autonomous vehicles with time-varying topology. In the scenario envisioned in this work, one or more vehicles have access to absolute measurements, that is, measurements of their own state, while the rest must rely on measurements relative to other vehicles in the vicinity and limited communication with those agents in order to estimate their state. The problem is formulated as a classical state observer design problem, with an added sparsity constraint on the output injection gains to reflect the limited amount of information available to each agent. As sensing and communication in formations of vehicles can be unreliable in most practical cases, it is assumed that the measurements available to the agents can change over time, resulting in a time-varying formation topology. To address this issue, a switched systems approach is employed. The error dynamics of the decentralized state observer are formulated as a switched linear system with time-dependent switching to reflect successive alterations in the structure of the formation, and sufficient conditions for their exponential stability are derived for two different switching laws. This framework is then particularized to the practical case of a formation of AUVs. The behavior and performance of this solution in the presence of measurement noise is then assessed in simulation. The problem of decentralized state estimation in formations with fixed topologies was already addressed in previous work by the authors in [14]. Nevertheless, it is summarized in Section II, as it introduces the framework necessary to derive the results presented in the remainder of the article.

The rest of the paper is organized as follows. Section II describes the problem at hand and introduces the dynamics of the proposed decentralized state observer, while Section III studies its stability when the topology of the formation changes over time. Section IV applies the framework developed in the preceding sections to the practical case of position and velocity estimation in formations of AUVs, with simulation results detailed in Section V. Finally, Section VI summarizes the main conclusions of the paper.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. Whenever relevant, the dimensions of an $n \times n$ identity matrix are indicated as $\mathbf{I}_n$. The Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \otimes \mathbf{B}$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ represents the largest integer not larger than x.

II. Decentralized State Estimation with fixed Formation Topology

Consider a formation composed by N autonomous vehicles moving in a scenario, where each vehicle is identified by a distinct positive integer $i \in \{1, 2, \ldots, N\}$, and has access

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D. Viegas, P. Batista, P. Oliveira, and C. Silvestre are with the Institute for Systems and Robotics, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal. C. Silvestre is also with the Department of Electrical and Computer Engineering, Faculty of Science and Technology, University of Macau, Taipa, Macau. {dviegas,pbatista,pjcro,cjs}@isr.ist.utl.pt

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to a certain set of measurements and communication with neighboring vehicles, which will be specified further ahead in this section. The problem considered in this paper is the design of a decentralized state observer that allows each vehicle to estimate its state. The approach described here consists in the implementation of a local state observer on-board each vehicle. To achieve a truly decentralized structure, those local state observers must be designed such that, during operation, each vehicle only requires locally available measurements and limited communication to estimate its state. In most practical settings, it is hard or even impossible to guarantee that the measurements available to each vehicle remain the same during the whole time of operation, as they may lose or gain measurements due to a diverse array of factors, such as loss of connectivity between agents, obstructions in line-of-sight, etc. In this section, the design of a decentralized state observer is summarized for a fixed formation topology. The stability of the decentralized state observer when the structure of the formation varies over time is then studied in Section III.

A. Dynamics of the vehicles and local observers

Suppose that the state $\mathbf{x}_i(t) \in \mathbb{R}^{n_L}$ of vehicle i follows

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_L\mathbf{x}_i(t) + \mathbf{B}_L\mathbf{u}_i(t),$$

where $\mathbf{u}_i(t) \in \mathbb{R}^{m_L}$ is the input of the system, and $\mathbf{A}_L \in \mathbb{R}^{n_L \times n_L}$ and $\mathbf{B}_L \in \mathbb{R}^{n_L \times m_L}$ are given constant matrices.

Regarding the available measurements, suppose that one or more vehicles have access to measurements of their own state, denoted as "absolute" measurements for convenience, yielding the Linear Time-Invariant (LTI) system

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{A}_L\mathbf{x}_i(t) + \mathbf{B}_L\mathbf{u}_i(t) \\ \mathbf{y}_i(t) = \mathbf{C}_L\mathbf{x}_i(t) \end{cases}, \tag{1}$$

where $\mathbf{y}_i(t) \in \mathbb{R}^{o_L}$ is the output of the system, and $\mathbf{C}_L \in \mathbb{R}^{o_L \times n_L}$. It is assumed that the pair $(\mathbf{A}_L, \mathbf{C}_L)$ is observable.

For the other vehicles, suppose that each one has access to measurements of its state relative to N_i other vehicles in the vicinity, yielding the dynamic system

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{A}_L\mathbf{x}_i(t) + \mathbf{B}_L\mathbf{u}_i(t) \\ \mathbf{y}_i(t) = \mathbf{C}_i\Delta\mathbf{x}_i(t) \end{cases}, \tag{2}$$

in which $\mathbf{y}_i(t) \in \mathbb{R}^{o_L \times N_i}$, $\mathbf{C}_i = \mathbf{I}_{N_i} \otimes \mathbf{C}_L$, and

$$\Delta\mathbf{x}_i(t) := \begin{bmatrix} \mathbf{x}_i(t) - \mathbf{x}_{\theta_{i,1}}(t) \\ \vdots \\ \mathbf{x}_i(t) - \mathbf{x}_{\theta_{i,N_i}}(t) \end{bmatrix} \in \mathbb{R}^{n_L N_i},\ \theta_{i,j} \in \Theta_i,$$

where

$$\Theta_i := \{\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,N_i} \,|\, \theta_{i,j} \in \{1, \ldots, N\},\ j = 1, \ldots, N_i\}$$

is the set of other vehicles corresponding to the relative measurements available to vehicle i. Furthermore, assume that vehicle i can exchange data with those vehicles through communication. In particular, assume that those vehicles send updated state estimates to vehicle i.

For the vehicles which have access to absolute measurements, since the pair $(\mathbf{A}_L, \mathbf{C}_L)$ is observable, it is straightforward to design a local state observer with globally asymptotically stable error dynamics for the LTI system (1), see [1]. For vehicle i, its dynamics follow

$$\begin{cases} \dot{\hat{\mathbf{x}}}_i(t) = \mathbf{A}_L\hat{\mathbf{x}}_i(t) + \mathbf{B}_L\mathbf{u}_i(t) + \mathbf{L}_i(\mathbf{y}_i(t) - \hat{\mathbf{y}}_i(t)) \\ \hat{\mathbf{y}}_i(t) = \mathbf{C}_L\hat{\mathbf{x}}_i(t) \end{cases}, \tag{3}$$

where $\hat{\mathbf{x}}_i(t) \in \mathbb{R}^{n_L}$ is the state estimate, and $\mathbf{L}_i \in \mathbb{R}^{n_L \times o_L}$ is a constant matrix of observer gains, to be computed.

Regarding the vehicles which have access to relative measurements, the design process is slightly different. First, note that, since vehicle i receives state estimates from the other vehicles corresponding to its relative measurements, it can build an estimate of $\Delta\mathbf{x}_i(t)$,

$$\Delta\hat{\mathbf{x}}_i(t) := \begin{bmatrix} \hat{\mathbf{x}}_i(t) - \hat{\mathbf{x}}_{\theta_{i,1}}(t) \\ \vdots \\ \hat{\mathbf{x}}_i(t) - \hat{\mathbf{x}}_{\theta_{i,N_i}}(t) \end{bmatrix} \in \mathbb{R}^{n_L N_i},\ \theta_{i,j} \in \Theta_i.$$

Thus, the following local observer structure for the system (2) follows naturally:

$$\begin{cases} \dot{\hat{\mathbf{x}}}_i(t) = \mathbf{A}_L\hat{\mathbf{x}}_i(t) + \mathbf{B}_L\mathbf{u}_i(t) + \mathbf{L}_i(\mathbf{y}_i(t) - \hat{\mathbf{y}}_i(t)) \\ \hat{\mathbf{y}}_i(t) = \mathbf{C}_i\Delta\hat{\mathbf{x}}_i(t) \end{cases}, \tag{4}$$

with $\mathbf{L}_i \in \mathbb{R}^{n_L \times o_L N_i}$.

B. Global estimation error dynamics

For analysis purposes, all the local state observers can be taken as a whole, resulting in a decentralized state observer for the formation, as the local state observer of each vehicle relies only on locally available measurements and limited communication to estimate its state. To study the stability properties of such a decentralized state observer, it is necessary to consider the error dynamics of the whole formation. To do so, it is convenient to introduce some concepts of graph theory, see e.g. [15], as vehicle formations such as the one considered in this paper can be compactly described by a directed graph. A directed graph, or digraph, $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ is composed by a set $\mathcal{V}$ of vertices together with a set of directed edges $\mathcal{E}$, which are ordered pairs of vertices. Such an edge can be expressed as $e = (a, b)$, meaning that edge e is incident on vertices a and b, directed towards b. Now, consider the vehicle formation described in the previous section. This kind of formation can be associated with a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each vertex represents a distinct vehicle, and an edge (a, b) means that vehicle b has access to a measurement relative to vehicle a, and also to its state estimate. To represent the absolute measurements available to some of the vehicles, define a special set of edges of the form $(0, i)$, connected to only one vertex, which represents the absolute state measurement available to vehicle i. Fig. 1 depicts a few examples of such formation graphs. For a graph $\mathcal{G}$ with n_v vertices and n_e edges, the entries of its incidence matrix $\mathbf{S}_\mathcal{G} \in \mathbb{R}^{n_v \times n_e}$ follow

$$[\mathbf{S}_\mathcal{G}]_{jk} = \begin{cases} 1, & \text{edge } k \text{ incident on } j \text{, directed towards it,} \\ -1, & \text{edge } k \text{ incident on } j \text{, directed away from it,} \\ 0, & \text{edge } k \text{ not incident on } j. \end{cases}$$

The global dynamics of the formation can be represented by the LTI system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_g\mathbf{x}(t) + \mathbf{B}_g\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}_g\mathbf{x}(t) \end{cases}, \tag{5}$$

where $\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}_1^T(t) & \ldots & \mathbf{x}_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{n_L N}$ is the state of the whole formation, $\mathbf{y}(t) := \begin{bmatrix} \mathbf{y}_1^T(t) & \ldots & \mathbf{y}_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{o_L O}$ the output of the system, O being the total number of measurements in the whole formation, and $\mathbf{u}(t) := \begin{bmatrix} \mathbf{u}_1^T(t) & \ldots & \mathbf{u}_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{m_L N}$ is the input of the system.

The matrices $\mathbf{A}_g$, $\mathbf{B}_g$, and $\mathbf{C}_g$ are built from the dynamics of the individual agents, following

$$\begin{cases} \mathbf{A}_g = \mathbf{I}_N \otimes \mathbf{A}_L \\ \mathbf{B}_g = \mathbf{I}_N \otimes \mathbf{B}_L \\ \mathbf{C}_g = \mathbf{S}_{\mathcal{G}}^T \otimes \mathbf{C}_L \end{cases}.$$

The local state observers can also be grouped in a similar way, yielding

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) := \mathbf{A}_g \hat{\mathbf{x}}(t) + \mathbf{B}_g \mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \hat{\mathbf{y}}(t)) \\ \hat{\mathbf{y}}(t) := \mathbf{C}_g \hat{\mathbf{x}}(t) \end{cases}, \quad (6)$$

where $\hat{\mathbf{x}}(t) := \begin{bmatrix} \hat{\mathbf{x}}_1^T(t) & \hat{\mathbf{x}}_2^T(t) & \dots & \hat{\mathbf{x}}_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{n_L N}$ is the global state estimate of the decentralized state observer, and $\mathbf{L} \in \mathbb{R}^{n_L N \times o_L O}$ is the matrix of observer gains. To account for the fact that each local observer only has access to some measurements, $\mathbf{L}$ must follow a special structure, or sparsity constraint. More specifically, define an augmented incidence matrix, $\mathbf{S}'_{\mathcal{G}} = \mathbf{S}_{\mathcal{G}} \otimes \mathbf{1}_{n_L, o_L} \in \mathbb{R}^{n_L N \times o_L O}$, where $\mathbf{1}_{n,m}$ is a $n \times m$ matrix whose entries are all equal to 1. Then, the individual entries of $\mathbf{L}$ follow

$$\begin{cases} [\mathbf{S}'_{\mathcal{G}}]_{jk} = 1 \Rightarrow \mathbf{L}_{jk} \text{ can be set to an arbitrary value} \\ [\mathbf{S}'_{\mathcal{G}}]_{jk} \neq 1 \Rightarrow \mathbf{L}_{jk} = 0. \end{cases} \quad (7)$$

The global error of the decentralized state observer (6), $\tilde{\mathbf{x}}(t) \in \mathbb{R}^{n_L N}$, is defined as $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$. Taking its time derivative and using (5) and (6) yields

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A}_g - \mathbf{L}\mathbf{C}_g)\tilde{\mathbf{x}}(t).$$

To stabilize the error dynamics of the decentralized state observer, one must find $\mathbf{L}$ subject to the sparsity constraint (7) such that the matrix $(\mathbf{A}_g - \mathbf{L}\mathbf{C}_g)$ is Hurwitz. This problem was addressed by the authors in [14] and will not be elaborated on here.

III. Decentralized State Estimation with time-varying Formation Topology

This section extends the decentralized state observer presented in the previous section to the case where the topology of the formation changes over time. The changes in topology considered here consist in the gain or loss of measurements and communication between agents, which can be represented by the addition or removal of edges from the formation graph. When faced with changes in the formation, the local observers must adapt to the new topology as, in general, observer gains computed for a given structure may result in unstable error dynamics when applied to a different formation topology. Thus, to extend the decentralized state observer to the time-varying formation case, three problems must be addressed:

1) When a change is detected in the formation, the vehicles have to determine the new formation structure, and then select and apply new, suitable observer gains.
2) A strategy must be chosen for the local observers to cope with the changes in the formation until the new topology is computed and suitable gains are applied.
3) The error dynamics will alternate between periods of guaranteed stability (when the local observers are all operating with suitable gains) and potential instability (after a change in the formation, while the vehicles are adapting to the new topology). The stability of the whole process over time must be ensured.

A. Selection of new observer gains

To determine the new formation topology in a decentralized fashion, the vehicle(s) that lose or gain measurements could spread a warning message through the formation, which could then trigger the synchronous execution of an algorithm such as the one detailed in Table I in each vehicle. Defining graph distance between two vertices as the number of edges in the shortest undirected path between those two vertices, the number of iterations of the algorithm in Table I is bounded as follows:

Lemma 1: Suppose that the algorithm in Table I is started synchronously in each vehicle and that the formation graph is weakly connected. Then, the number of iterations of the algorithm in any vehicle of the formation is, at most, equal to d_G+1, where d_G is the maximum graph distance between two vertices of the formation graph.

The proof is omitted due to space constraints. However, note that, due to the way the problem was initially formulated, weak connectivity of the formation's measurement graph implies strong connectivity of the communication graph.

TABLE I

Algorithm for formation graph determination, for vehicle i

1) Initialization: create a table to store the edges, E_i, and initialize it with the currently known edges, that is, the measurements available to the vehicle. Create a control vector $\mathbf{v}_i \in \mathbb{R}^N$, and set it to zero except for the i-th component, which is set to 1.
2) Send E_i and $\mathbf{v}_i$ to neighboring agents (the ones to which communication is available) and receive the same from them.
3) Compare E_i with its counterparts received through communication, and add any previously unknown edges. For each nonzero component in each received $\mathbf{v}_j$, set the corresponding component in $\mathbf{v}_i$ to 1.
4) If all components of $\mathbf{v}_i$ are equal to 1, and if no changes were made to $\mathbf{v}_i$ and E_i in this iteration, stop the algorithm. Otherwise, go to 2).

After the formation graph is determined, the vehicles must select and apply suitable observer gains. As the gains for a given formation structure can be computed beforehand [14], one way to do this would be to store a database of observer gains for a large number of possible formations on-board each vehicle. Since the local observer gains are constant matrices of relatively low dimension, nowadays it is perfectly feasible to store hundreds or even thousands of precomputed observer gains in each vehicle. However, if the new formation graph is not found in the database, the agent can look for a subgraph of it, and apply the corresponding observer gains.

B. Behavior during transition periods

Regarding the second problem, i.e. the strategy followed when gaining or losing measurements, several approaches can be envisioned. The one followed in the simulations in Section V, is for the vehicle to propagate the dynamics in open loop if measurements were lost, but to keep the old gains temporarily when it obtains new measurements. This strategy has the advantage that the transition periods when new measurements appear in the formation will still be stable, limiting the instability periods to the cases where one or more vehicles lose measurements.

C. Error dynamics as a switched system

If the decentralized state observer operates according to what was discussed in the preceding subsections, its error dynamics can be represented in the following manner: starting at time instant t_0, the formation graph is $\mathcal{G}_1 = (V_1, E_1)$, and the error dynamics follow $\dot{\tilde{\mathbf{x}}}(t) = \mathbf{\Lambda}_1\tilde{\mathbf{x}}(t)$, with $\mathbf{\Lambda}_1$ Hurwitz. Now, suppose that at $t = t_1$ there is a change in the formation topology and assume that, after some time interval no longer than a known constant τ_u, the vehicles are able to determine the new formation graph and synchronously apply suitable observer gains. During that time, the error dynamics of the decentralized state observer will follow $\dot{\tilde{\mathbf{x}}}(t) = \mathbf{\Lambda}_2\tilde{\mathbf{x}}(t)$, where $\mathbf{\Lambda}_2$ depends on the strategy adopted by the vehicles when losing or gaining measurements, and is possibly unstable. Then, at $t = t_2$, the new observer gains are applied, and the decentralized state observer has stable error dynamics $\dot{\tilde{\mathbf{x}}}(t) = \mathbf{\Lambda}_3\tilde{\mathbf{x}}(t)$. Now, suppose that these changes in formation topology continue to happen over time, and the error dynamics alternate sequentially between periods of stability and instability. This scenario can be represented by the linear switched system [10][11]

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}_{\sigma(t)}\boldsymbol{x}(t), \tag{8}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state, $\sigma(t) : [t_0, \infty[\rightarrow \mathbb{N}_P = \{1, 2, \ldots, P\}$ is a piecewise constant switching signal, and $\mathbf{A}_\sigma$ takes values in a family of $n \times n$ matrices, $\boldsymbol{\mathcal{A}} := \{\mathbf{A}_p : p \in \mathbb{N}_P\}$. With no loss of generality, assume that $\mathbf{A}_p$ is stable for $1 \leq p \leq q$, and unstable for $q < p \leq P$. Then, there exist scalars $a_1 > 0, a_2 > 0, \ldots, a_P > 0$ and $\lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_P > 0$ such that

$$\begin{cases} \|e^{\mathbf{A}_p t}\| \leq e^{a_p - \lambda_p t}, & 1 \leq p \leq q \\ \|e^{\mathbf{A}_p t}\| \leq e^{a_p + \lambda_p t}, & q < p \leq P \end{cases}. \tag{9}$$

Denote the switching times by $t_1, t_2, t_3, \ldots$ Then, to reflect the dynamics of the problem at hand, the switching signal satisfies the following assumption:

Assumption 1: For $t_{2j-2} \leq t < t_{2j-1},\ j \in \mathbb{N}$, the switching signal follows $1 \leq \sigma(t) \leq q$. For $t_{2j-1} \leq t < t_{2j},\ j \in \mathbb{N}$, the switching signal follows $q < \sigma(t) \leq P$. Furthermore, there exists $\tau_u > 0$ such that $(t_{2j} - t_{2j-1}) \leq \tau_u$ for all $j \in \mathbb{N}$.

This assumption encodes the sequential switching between stable and unstable estimation error dynamics into the switching signal, and sets an upper bound τ_u on the duration of the instability periods.

D. Stability of the switched system

Concerning the duration of the stable periods, two different cases are considered. In the first one, the minimum activation time of the stable subsystems is bounded below by a constant:

Assumption 2: For all $j \in \mathbb{N}$, $(t_{2j-1} - t_{2j-2}) \geq \tau_s$, for some $\tau_s > 0$.

This assumption simply states that the stable periods must last for at least τ_s. The second case, inspired by the results in [17], adapts the concept of average dwell time introduced in [9] to the scenario considered in this paper:

Assumption 3: Let $N_\sigma^u(t_0, t)$ denote the number of switchings to unstable subsystems in the interval $]t_0, t]$. Then, the switching signal $\sigma(t)$ follows

$$N_\sigma^u(t_0, t) \leq N_0 + \frac{t - t_0}{\tau_a},\ \forall t \geq t_0,$$

for some constants N_0, $\tau_a > 0$.

This assumption states that, on average, the time interval between two consecutive switchings to unstable configurations will be no less than τ_a, and the chatter bound N_0 is included to account for an eventual limited number of faster switchings. If Assumption 1 is also verified, it follows that the duration of each successive "stable/unstable" pair will be, on average, no less than τ_a.

The following result presents a sufficient condition for the stability of (8) for the first case.

Theorem 1: Consider the linear switched system (8), assume that the switching signal $\sigma(t)$ verifies Assumptions 1 and 2, and let

$$\alpha^* := \sup_{\substack{1 \leq k \leq q \\ q < l \leq P}} \{a_k - \lambda_k\tau_s + a_l + \lambda_l\tau_u\}.$$

If $\alpha^* < 0$, then the state $\boldsymbol{x}(t)$ of (8) follows

$$\|\boldsymbol{x}(t)\| \leq e^{a - \lambda(t - t_0)}\|\boldsymbol{x}(t_0)\|,\ \forall t \geq t_0,$$

for some $a > 0$ and any $0 < \lambda \leq \lambda^*$, where $\lambda^* := -\frac{\alpha^*}{\tau_s + \tau_u}$.

Proof: Let $v \in \mathbb{N}_P$, $w \in \mathbb{N}_P$, $t_v \in \mathbb{R}$, and $t_w \in \mathbb{R}$, and assume that $1 \leq v \leq q$, $q < w \leq P$, $t_v \geq \tau_s$, and $t_w \leq \tau_u$. Then, it follows from the definition of λ^* that, for any $0 < \lambda \leq \lambda^*$,

$$e^{a_w + \lambda_w t_w + a_v - \lambda_v t_v} \leq e^{-\lambda(t_w + t_v)}. \tag{10}$$

Let p_l denote the value of $\sigma(t)$ between t_l and t_{l+1}. Then, for any $j > 0$ and any t such that $t_j \leq t \leq t_{j+1}$, the state of the system (8) follows

$$\boldsymbol{x}(t) = e^{\mathbf{A}_{p_j}(t - t_j)}e^{\mathbf{A}_{p_{j-1}}(t_j - t_{j-1})}\ldots e^{\mathbf{A}_{p_0}(t_1 - t_0)}\boldsymbol{x}(t_0),$$

and its norm verifies the inequality

$$\|\boldsymbol{x}(t)\| \leq \|e^{\mathbf{A}_{p_j}(t - t_j)}\| \prod_{l=1}^{j}\left(\|e^{\mathbf{A}_{p_{l-1}}(t_l - t_{l-1})}\|\right)\|\boldsymbol{x}(t_0)\|. \tag{11}$$

Now, suppose that at time t the system (8) is in an unstable configuration, and let $j^* = \lfloor j/2 \rfloor$. Then, using (9) and (10) in (11) yields

$$\begin{aligned}\|\boldsymbol{x}(t)\| &\leq e^{-\lambda(t - t_{j-1})}\prod_{l=1}^{j^*}\left(e^{-\lambda(t_{2l} - t_{2l-2})}\right)\|\boldsymbol{x}(t_0)\| \\ &\leq e^{-\lambda(t - t_0)}\|\boldsymbol{x}(t_0)\|.\end{aligned}$$

On the other hand, if at time t the system (8) is in a stable configuration, using the same reasoning it follows that

$$\|\boldsymbol{x}(t)\| \leq e^{a_j - \lambda(t - t_0)}\|\boldsymbol{x}(t_0)\|,$$

thus concluding the proof. ■

The following result presents a sufficient condition for the stability of (8) for the second case.

Theorem 2: Consider the switched linear system (8), assume that the switching signal $\sigma(t)$ verifies Assumptions 1 and 3, and let

$$\alpha_a^* := a_s + a_u + \lambda_s\tau_u + \lambda_u\tau_u - \lambda_s\tau_a,$$

where

$$a_s = \sup_{1 \leq k \leq q}\{a_k\},\ a_u = \sup_{q < k \leq P}\{a_k\},$$
$$\lambda_s = \inf_{1 \leq k \leq q}\{\lambda_k\},\ \text{and}\ \lambda_u = \sup_{q < k \leq P}\{\lambda_k\}.$$

Then, if $\alpha_a^* < 0$, the state $\boldsymbol{x}(t)$ of (8) follows

$$\|\boldsymbol{x}(t)\| \leq e^{a-\lambda(t-t_0)}\|\boldsymbol{x}(t_0)\|, \ \forall t \geq t_0,$$

for some $a > 0$ and any $0 < \lambda \leq \lambda_a^*$, where $\lambda_a^* := -\frac{\alpha_a^*}{\tau_a}$.

The proof, which follows a similar method to the proof for Theorem 1, is omitted due to lack of space.

IV. Decentralized linear position and velocity estimation in a formation of AUVs

This section details an application of the results introduced in the previous sections to a practical case: decentralized position and velocity estimation in a formation of AUVs. Consider a formation composed by N AUVs, and suppose that each has sensors mounted on-board which give access to either measurements of its own position in an inertial reference coordinate frame $\{I\}$, or measurements of its position relative to one or more AUVs in the vicinity. Furthermore, each of those vehicles transmits an estimate of its own inertial position to AUV i. In underwater applications, the relative measurements can be provided by an Ultra-short Baseline (USBL) positioning system in an inverted configuration [12]. The inertial measurements can be provided, e.g., by a Long Baseline (LBL), or by an USBL positioning system.

Let $\{B_i\}$ denote a coordinate frame attached to AUV i, denominated in the sequel as the body-fixed coordinate frame associated with the i-th AUV. The linear motion of AUV i can be modeled by the dynamic system

$$\begin{cases} \dot{\mathbf{p}}_i(t) = \mathbf{R}_i(t)\mathbf{v}_i(t) \\ \dot{\mathbf{v}}_i(t) = -\mathbf{S}(\boldsymbol{\omega}_i(t))\mathbf{v}_i(t) + \mathbf{g}_i(t) + \mathbf{a}_i(t) \\ \dot{\mathbf{g}}_i(t) = -\mathbf{S}(\boldsymbol{\omega}_i(t))\mathbf{g}_i(t) \\ \mathbf{y}_i(t) = \mathbf{p}_i(t) \end{cases} .$$

where $\mathbf{p}_i(t) \in \mathbb{R}^3$ is the inertial position of the vehicle, $\mathbf{v}_i(t) \in \mathbb{R}^3$ denotes its velocity relative to $\{I\}$, expressed in body-fixed coordinates of the i-th AUV, $\mathbf{R}_i(t) \in SO(3)$ is the rotation matrix from $\{B_i\}$ to $\{I\}$, $\boldsymbol{\omega}_i(t) \in \mathbb{R}^3$ is the angular velocity of $\{B_i\}$, expressed in body-fixed coordinates of the i-th AUV, and $\mathbf{S}(\boldsymbol{\omega})$ is the skew-symmetric matrix such that $\mathbf{S}(\boldsymbol{\omega})\boldsymbol{x}$ is the cross product $\boldsymbol{\omega} \times \boldsymbol{x}$. It is assumed that an Attitude and Heading Reference System (AHRS) installed on-board each AUV provides measurements of both $\mathbf{R}_i(t)$ and $\boldsymbol{\omega}_i(t)$. It is also assumed that the AUV has access to linear acceleration measurements, denoted by $\mathbf{a}_i(t) \in \mathbb{R}^3$, and $\mathbf{g}_i(t) \in \mathbb{R}^3$ is the acceleration of gravity, expressed in body-fixed coordinates of the i-th AUV. Even though the acceleration of gravity is usually well-known, it is treated as an unknown variable as small errors in the estimation of the attitude of the vehicle may lead to significant errors in the acceleration compensation. Using in each vehicle the Lyapunov state transformation introduced in [3],

$$\begin{bmatrix} \mathbf{x}_i^1(t) \\ \mathbf{x}_i^2(t) \\ \mathbf{x}_i^3(t) \end{bmatrix} := \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_i(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_i(t) \end{bmatrix} \begin{bmatrix} \mathbf{p}_i(t) \\ \mathbf{v}_i(t) \\ \mathbf{g}_i(t) \end{bmatrix}, \qquad (12)$$

which preserves stability and observability properties [5], and making $\mathbf{u}_i(t) := \mathbf{R}_i(t)\mathbf{a}_i(t)$, the system dynamics can be written as the LTI system (1), with $n_L = 9$, $m_L = 3$, $o_L = 3$,

$$\mathbf{A}_L = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n_L \times n_L}, \ \mathbf{B}_L = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n_L \times m_L},$$

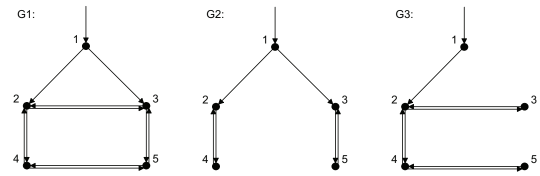

Fig. 1. Formation graphs considered in the simulations

and $\mathbf{C}_L = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{o_L \times n_L}$.

Regarding the second case, i.e., when the AUV has access to relative position measurements and receives position estimates from the corresponding vehicles, a similar representation can be achieved. The relative position measurements available to AUV i are denoted by

$$\Delta\mathbf{p}_i(t) := \begin{bmatrix} \mathbf{p}_i(t) - \mathbf{p}_{\theta_{i,1}}(t) \\ \vdots \\ \mathbf{p}_i(t) - \mathbf{p}_{\theta_{i,N_i}}(t) \end{bmatrix} \in \mathbb{R}^{3N_i}, \ \theta_{i,j} \in \Theta_i, \qquad (13)$$

where Θ_i is defined as in Section II. The position estimates received through communication are denoted by $\hat{\mathbf{p}}_{\theta_{i,j}}(t) \in \mathbb{R}^3$. Taking the relative position measurements (13) as the output and applying (12) yields the system (2), where $\mathbf{A}_L$ and $\mathbf{B}_L$ are defined as in the previous case, and $\mathbf{C}_i = \mathbf{I}_{N_i} \otimes \mathbf{C}_L \in \mathbb{R}^{o_L N_i \times n_L N_i}$. Following this, each AUV can implement a local state estimator with dynamics (3) or (4), depending on the available measurements. On the subject of computing suitable output injection gains for each local observer such that the global error dynamics are stable and achieve a certain performance under measurement noise, please refer to previous work by the authors in [14].

It is well known that sensing and communication in underwater applications can be unreliable, and as such during a mission some AUVs may lose or gain access to some measurements and/or communication over time, which will translate in changes in the formation graph. Besides that, the geometric configuration of the formation might change dynamically yielding a new set of measurements, that is, a different formation topology. Assuming that, when one of these events occur, the AUVs are able to determine the new formation graph and synchronously apply new, suitable observer gains after a period of time no longer than τ_u, application of the results derived in Section III allow to guarantee stability of the global error dynamics when the formation graph changes over time.

V. Simulation Results

This section details the results of simulations that were carried out to assess the performance of the proposed solution in the presence of noise in the measurements. A formation of 5 AUVs was considered, and its topology switches between the three different formation graphs depicted in Fig. 1. Observer gains such that the resulting error dynamics are globally asymptotically stable were computed for each formation graph. As for the strategy that the AUVs apply to cope with the gain or loss of measurements during the transitions between stable configurations, when an AUV loses access to one or more measurements, it sets its observer gains to zero temporarily. When gaining new measurements, the local observer will maintain the previous observer gains

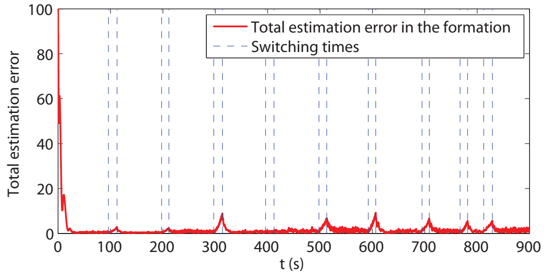

Fig. 2. Total estimation error in the formation, with noise in the measurements

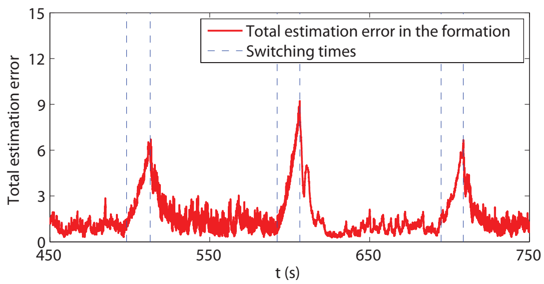

Fig. 3. Detailed view of the total estimation error in the formation, with noise in the measurements

until the end of the transition time. The duration of the stable configurations were computed by sampling a normal distribution with an expected value of $t_s = 75\ s$ and a standard deviation of $\sigma_s = 15\ s$, while the duration of the unstable transition periods follow a normal distribution with $t_u = 15\ s$ and $\sigma_u = 1\ s$. The local observers were implemented in the body-fixed coordinate space of each AUV, by reversing the Lyapunov state transformation (12). For more details on this process, see [14]. Regarding the noise in the measurements, the linear acceleration and position measurements were corrupted by additive, uncorrelated, zero-mean white Gaussian noise, with standard deviations of 0.01 (m/s^2) and 0.1 m, respectively. In addition to the noise in the position and acceleration measurements, noise was also simulated in the attitude and angular velocity measurements required for the implementation of the local state observers in the original coordinated space, as provided by an AHRS. The angular velocity measurements were corrupted by zero-mean uncorrelated white Gaussian noise, with standard deviation of 0.05 $(°/s)$. The attitude is usually parametrized by roll, pitch, and yaw Euler angles, and as such noise in the attitude measurements was simulated by adding zero-mean, uncorrelated white Gaussian perturbations to the roll, pitch, and yaw, with standard deviation of $0.03°$ for the roll and pitch, and $0.3°$ for the yaw.

The results are depicted in Figs. 2 and 3. Figure 2 depicts the evolution of the total estimation error of the decentralized state observer over time, that is, the sum of the modulus of all estimation error variables in the formation, and Fig. 3 shows a detailed view of the total estimation error after the large transient caused by the mismatch in initial conditions has settled. As it can be seen, during the unstable periods the total estimation error grows in a significant manner, while during the stable period it converges to the vicinity of zero (it does not converge to zero only due to noise in the measurements). The results reflect what was discussed in the previous section and, more generally, the behavior of linear systems subject to time-dependent switching: stability is maintained as long as the system dwells on stable configurations for a sufficient amount of time.

VI. Conclusions

This paper addressed the problem of decentralized state estimation in formations of vehicles with time-varying topologies. The proposed solution relies on the implementation of a local state observer on-board each vehicle, based only on local sensing capabilities and limited communication with neighboring vehicles, to estimate its state. The effects of changes in the formation topology over time were studied resorting to switched systems theory, and sufficient conditions for global exponential stability of the global estimation error dynamics were presented for two different switching laws. The results were particularized for the case of a formation of AUVs, and simulation results were presented that illustrate the performance of the proposed solution in the presence of measurement noise.

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