# Set-Valued Observers for Attitude and Rate Gyro Bias Estimation

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Abstract— This paper addresses the problem of attitude and rate gyro bias estimation using Set-Valued Observers (SVOs). We propose a singularity-free solution that exploits rate gyro measurements and vector observations corrupted by bounded noise providing bounds on the attitude of the rigid body and on the bias of the rate gyros. The sensor readings are fused directly by the SVO, without intermediate attitude computation. By taking advantage of the powerful new multi-core processors it is possible to increase the estimator convergence rate without compromising the execution time. The feasibility of the technique is demonstrated in simulation.

# I. INTRODUCTION

For most modern underwater, aerial, and space vehicles, accurate attitude determination is a key requirement. The navigation systems of these platforms frequently rely on information provided by sensors such as rate gyros, accelerometers, magnetometers, star-trackers, and sun sensors [3]. The development of new low-cost sensors, in particular the micro-electro-mechanical systems (MEMS), have greatly contributed to reducing the costs of middle grade performance navigation systems. However, these sensors, in particular the rate gyros, suffer form non-negligible noise and bias, that, if not taken into account, might degrade significantly the attitude estimates [10]. In this paper, we propose a solution based on Set-Valued Observers (SVOs) that simultaneously solves the attitude and rate gyro bias estimation problem.

With two non-parallel vector observations in the bodyfixed reference frame, it is possible to compute lowbandwidth estimates of the attitude of a rigid body with respect to the inertial reference frame - see [28], [13], [15] and references therein. To increase the accuracy of the estimates, the vector observations are often fused with the high-bandwidth rate gyro measurements. The rate gyro measurements are affected, not only, by noise, but also by bias that deteriorates the attitude estimates. The bias is, in general, slowly time-varying, which reduces the effectiveness of offline calibration and corroborates the necessity of online parameter estimation. The Extended Kalman Filter (EFK) has been extensively used to blend the information of rate gyros and vector observations. The interested reader is further referred to [12], [16], [26]. A survey on these estimators, among others, such as particle filters and the unscented Kalman filter, can be found in [6]. More recently,

nonlinear observers have attracted the attention of the scientific community – see for instance [25], [17], [11], [27].

When a probabilistic description of the measurement noise is not available and only magnitude bounds are known, the previous estimators are not reliable. In such circumstances, an estimator that computes the set of possible states given the sensor information is more suitable. The work in [22] discusses the state estimation problem for systems with bounded inputs, while in [4] and [14] a similar problem, but using a set-membership description for model uncertainty, is addressed. New advances in the framework of these estimators, known as SVOs [1], are presented in [24], [19], [20]. The work in [21] proposes an attitude estimator where the sensor measurements and the filter state are bounded by uncertainty ellipsoids. However, this estimator relies on the linearization of the system to propagate the uncertainty ellipsoids, which, in the presence of fast dynamics, may lead to inaccurate estimates.

In a previous work by the authors [5], it is proposed an SVO-based solution for attitude estimation assuming that the sensor measurements have additive uncertainties characterized by polytopes. If the uncertainties are solely present on the vector observation, the uncertainty on the estimates is guaranteed to be the smallest possible.

In this paper, we propose an estimator based on SVOs that simultaneously calculates the attitude of a rigid body and the rate gyro bias, by directly exploiting angular velocity measurements and vector observations. We propose a solution that considers measurements with uncertainties, defined by polytopes, that guarantees that the true state of the system is inside the estimated set, as long as the assumptions on the bounds on the measurements are satisfied. Unlike typical approaches in the literature, no linearization is required.

In summary, the main contributions of this paper are as follows:

- The development of a singularity-free SVO for attitude estimation, which takes into account rate gyro bias.
- The development of an SVO for the online estimation of the bias of the rate gyros.
- The assessment, in simulation, of the proposed technique, highlighting the advantages and shortcomings in comparison with the alternatives in the literature.

The remainder of this article is organized as follows. In Section II, the attitude estimation problem is introduced and the available sensor information is described. The main contribution of this work is presented in Section III, where the SVOs for attitude and rate gyro bias estimation is developed. Section IV illustrates the properties of the proposed solution in simulation. Finally, some concluding remarks are presented in Section V.

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# NOMENCLATURE

To enhance the readability of this paper, we introduce the following notation. The set of special orthogonal matrices is denoted by SO(3) and the associated algebra is denoted by  $\mathfrak{so}(3)$ . The skew-symmetric operator in  $\mathbb{R}^3$  is denoted by  $[.]_{\times} : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ , which satisfies  $[\mathbf{v}]_{\times} \mathbf{w} = \mathbf{v} \times \mathbf{w}$ ,  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ . The Kronecker product of matrices is denoted by  $\mathbf{A} \otimes \mathbf{B}$  and satisfies

$$\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix}a_{11}\mathbf{B}&\cdots&a_{1n}\mathbf{B}\\\vdots&\ddots&\vdots\\a_{m1}\mathbf{B}&\cdots&a_{mn}\mathbf{B}\end{bmatrix}.$$

The  $3 \times 3$  matrices whose elements are zeros except the element ij, and whose elements are ones, are denoted by  $\mathbf{E}_{i,j}$  and  $\mathbf{1}_{3\times 3}$ , respectively. The operator  $\operatorname{Vec}(\mathbf{M})$  stacks the columns of the  $m \times n$  matrix  $\mathbf{M}$  into a long  $(9 \times 1) \times 1$  vector and the inverse operation of  $\operatorname{Vec}(.)$  is denoted as  $\operatorname{Mat}(.)$ . The matrix norm  $||.||_{\max}$  is defined as the maxmum of the absolute value of all matrix elements, i.e.,  $\|\mathbf{A}\|_{\max} := \max\{|a_{ij}|\}$ . Consider a polytope defined by  $\{\mathbf{x} \in \mathbb{R}^{n_x} : \mathbf{Ax} \leq \mathbf{b}\}$ . Then, define the Fourier-Motzkin projection [9] as

$$(\bar{\mathbf{A}}, \bar{\mathbf{b}}) := \mathrm{FM}(\mathbf{A}, \mathbf{b}, n),$$

where  $n = n_x - \bar{n}_x > 0$ , and  $\bar{\mathbf{A}}$  and  $\bar{\mathbf{b}}$  satisfy, for all  $\bar{\mathbf{x}} \in \mathbb{R}^{\bar{n}_x}$ ,

$$\bar{\mathbf{A}}\bar{\mathbf{x}} \leq \bar{\mathbf{b}} \Leftrightarrow \exists_{\mathbf{x} \in \mathbb{R}^n} : \mathbf{A} \begin{bmatrix} \bar{\mathbf{x}} \\ \mathbf{x} \end{bmatrix} \leq \mathbf{b}.$$

Finally, define  $\operatorname{Set}(\mathbf{M}, \mathbf{m}) := \{ \mathbf{x} \in \mathbb{R}^m : \mathbf{M}\mathbf{x} \leq \mathbf{m} \}$ , and  $\operatorname{MatSet}(\mathbf{M}, \mathbf{m}) := \{ \mathbf{Y} \in \mathbb{R}^{a \times b} : \mathbf{M} \operatorname{Vec}(\mathbf{Y}) \leq \mathbf{m} \}.$ 

# **II. PROBLEM FORMULATION**

In this section, we introduce the attitude estimation problem using vector observations and biased angular velocity measurements. The vector observation provides instantaneous information about attitude while the angular velocity characterizes the time-evolution of attitude.

The rotation matrix is an attractive attitude representation, since unlike others, such as the Euler Angles, is free of singularities, which is a necessary condition for the results being global. Let  $\mathcal{R}$  be the rotation matrix from the reference frame  $\{I\}$ , which is assumed to be inertial, to the body-fixed reference frame  $\{B\}$ .

The kinematics of the rotation matrix  $\mathcal{R}$  is given by

$$\mathcal{R} = [\boldsymbol{\omega}]_{\times} \mathcal{R},\tag{1}$$

where  $\boldsymbol{\omega}$  is the angular velocity of the inertial frame with respect to the rigid body, measured in the body reference frame. This continuous-time model is not suitable to be implemented in a digital system. However, for a sufficiently small sampling period, T, we can approximate the angular velocity between sampling times by a constant function, and use the Euler approximation (see [7, pp. 126]) of (1), which is given by

$$\mathcal{R}(k+1) = \exp(T\boldsymbol{\omega}(k))\mathcal{R}(k), \qquad (2)$$

where  $\exp(.) : \mathfrak{so}(3) \mapsto SO(3)$  is the exponential map on the special orthogonal group, and  $\mathfrak{so}(3)$  is the Lie algebra associated with SO(3). An advantage of this approximation is the linearity on  $\mathcal{R}$  of the time-varying discrete-time model in (2).

A triad of rate gyros fixed in reference frame  $\{B\}$  measures  $\omega_r \in \mathbb{R}^3$ , which is the angular velocity corrupted by an unknown bias term  $\mathbf{b} \in \mathbb{R}^3$  and bounded noise  $\mathbf{n} \in \mathbb{R}^3$ so that

$$\boldsymbol{\omega}_r = \boldsymbol{\omega} + \mathbf{b} + \mathbf{n}, \quad \|\mathbf{n}\|_{\max} \le \bar{n}. \tag{3}$$

On-board sensors, such as, magnetometers, star trackers, among others, provide vector observations expressed in body frame coordinates, i.e.,

$${}^{\scriptscriptstyle B}\mathbf{v}_i = \mathcal{R}^{\scriptscriptstyle I}\mathbf{v}_i,\tag{4}$$

where  $i = 1, ..., N_v$ , and  $N_v$  is the number of different vector observations and no three of which are collinear, or, in the matrix form,

$$^{B}\mathbf{V}=\mathcal{R}^{I}\mathbf{V},$$
(5)

where  ${}^{B}\mathbf{V} = [{}^{B}\mathbf{v}_{1} \dots {}^{B}\mathbf{v}_{N_{v}}]$  such  ${}^{I}\mathbf{V} = [{}^{I}\mathbf{v}_{1} \dots {}^{I}\mathbf{v}_{N_{v}}]$ . If the linear acceleration is negligible in comparison with the gravity, the measurements of tri-axial accelerometers may also be suitable to be used as vector observation.

We assume the sensor measurements to be corrupted by noise characterized by compact polytopes. Thus, a generic measurement  $\mathbf{q} \in \mathbb{R}^n$  belongs to the compact convex polytope defined by an  $m \times n$  real matrix  $\mathbf{M} \in \mathbb{R}^{m \times n}$  and by a vector  $\mathbf{m} \in \mathbb{R}^m$ , i.e.,  $\mathbf{q} \in \text{Set}(\mathbf{M}, \mathbf{m})$ . The measurement is thus provided as a set, rather than as a point.

Definition 1: Given  ${}^{I}\mathbf{v}_{i}$ ,  $i = 1, ..., N_{v}$ ,  $\mathcal{R}$  is compatible with a set of observations, S, if there exists  ${}^{B}\mathbf{v} \in S$  such that (4) is satisfied.

In the next lemma, we show how the output of the system relates with the state, i.e., the time-varying rotation matrix.

Lemma 1: ([5, Lemma 1]) Assume that the vector observations  ${}^{B}\mathbf{v}_{i}(k)$ ,  $i = 1, ..., N_{v}$ , at each time k satisfy  ${}^{B}\mathbf{v}_{i}(k) \in \operatorname{Set}(\mathbf{M}_{v_{i}}(k), \mathbf{m}_{v_{i}}(k))$ . Then, there exist a matrix  $\mathbf{M}_{m}$  and a vector  $\mathbf{m}_{m}$  such that

$$\mathbf{M}_m(k) \operatorname{Vec}(\mathcal{R}(k)) \leq \mathbf{m}_m(k),$$

if and only if  $\mathcal{R}(k)$  is compatible with the set of observations, Set $(\mathbf{M}_{\mathbf{v}}(k), \mathbf{m}_{\mathbf{v}}k)$ , where

$$\mathbf{M}_{\mathbf{v}}(k) = \begin{bmatrix} \mathbf{M}_{v_1}(k) & 0 \\ & \ddots & \\ 0 & & \mathbf{M}_{v_{N_v}}(k) \end{bmatrix}, \quad \mathbf{m}_{\mathbf{v}}(k) = \begin{bmatrix} \mathbf{m}_{v_1}(k) \\ \vdots \\ \mathbf{m}_{v_{N_v}}(k) \end{bmatrix}.$$

*Proof:* The vector observations  ${}^{B}\mathbf{v}_{i}(k)$ ,  $i = 1, ..., N_{v}$ , at each time k, satisfy  ${}^{B}\mathbf{v}_{i}(k) \in \operatorname{Set}(\mathbf{M}_{v_{i}}(k), \mathbf{m}_{v_{i}}(k))$ , i.e.,  $\operatorname{Vec}({}^{B}\mathbf{V}(k))$  satisfy  $\mathbf{M}_{\mathbf{v}}(k)\operatorname{Vec}({}^{B}\mathbf{V}(k)) \leq \mathbf{m}_{\mathbf{v}}(k)$ . On the other hand, it follows from (5) that  $\operatorname{Vec}({}^{B}\mathbf{V}(k)) = \mathbf{Q}(k)\operatorname{Vec}(\mathcal{R}(k))$ , where  $\mathbf{Q}(k)$  is given by

$$\mathbf{Q}(k) = \begin{bmatrix} {}^{I}v_{11}\mathbf{I} & {}^{I}v_{21}\mathbf{I} & {}^{I}v_{31}\mathbf{I} \\ \vdots & \vdots & \vdots \\ {}^{I}v_{1N_{v}}\mathbf{I} & {}^{I}v_{2N_{v}}\mathbf{I} & {}^{I}v_{3N_{v}}\mathbf{I} \end{bmatrix}$$

and  ${}^{I}v_{ij}$  is the element in the line *i* and column *j* of matrix  ${}^{I}\mathbf{V}(k)$ . Hence, we have that

$$\underbrace{\mathbf{M}_{\mathbf{v}}(k)\mathbf{Q}(k)}_{\mathbf{M}(k)}\operatorname{Vec}(\mathcal{R}(k)) \leq \underbrace{\mathbf{m}_{\mathbf{v}}(k)}_{\mathbf{m}(k)}.$$

The objective of the present work is to estimate the smallest set that contains the attitude of a rigid body using the available sensor suite, i.e., to obtain the set-valued attitude estimate with the smallest possible uncertainty set.

# III. ATTITUDE AND RATE GYRO BIAS ESTIMATION USING SVOS

The aim of this section is to design an SVO for attitude and gyro bias estimation. Throughout the remainder of this paper we assume that the angular velocity  $\omega(.)$  is constant between sampling times.

# A. Attitude estimation

At each time k, the proposed estimator provides a set containing the current state of the system in (2). The rate gyro bias is modeled as an unknown constant vector, b, and the set of possible bias values is given by the polytope B. Our goal is to reduce the volume of the sets containing the rotation matrix and the rate gyro bias as time goes by. To that end, we introduce the vectors  $\mathbf{b}_0$  and  $\mathbf{b}_{\delta}$ , that satisfy

$$\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_\delta$$

where  $\mathbf{b}_0$  is the center of *B*.

The system dynamics in (2) can be rewritten in the form

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k),\tag{6}$$

where  $\mathbf{x}(k) = \operatorname{Vec}(\mathcal{R}(k))$  and  $\mathbf{A}(k) = \mathbf{I}_{3\times 3} \otimes$  $\exp(T[\boldsymbol{\omega}(k)]_{\times})$ . Due to the rate gyro bias and the measurement noise, the angular velocity  $\omega(k)$ , and consequently A(k), cannot be obtained from the measurements. However, there exists  $\Delta_i(k) \in \mathbb{R}$ , such that matrix  $\mathbf{A}(k)$  satisfies the following sum

$$\mathbf{A}(k) = \mathbf{A}_0(k) + \sum_{i=1}^{9} \mathbf{A}_i(k) \Delta_i(k), \quad |\Delta_i(k)| \le 1$$
 (7)

where  $\mathbf{A}_0(k) = \mathbf{I}_{3\times 3} \otimes \exp(T[\boldsymbol{\omega}_r - \mathbf{b}_0]_{\times}), \ \mathbf{A}_i(k) = \mathbf{I}_{3\times 3} \otimes$  $\gamma \mathbf{E}_{m,n}, m, n = 1, 2, 3, i = 3(n-1) + m$ , and exploiting the result from Appendix A

$$\gamma = \frac{1}{2} \left( \left( \exp(2T(\bar{\omega}_r + \bar{b_0} + \bar{b_\delta} + \bar{n})) \right) - \exp(2T(\bar{\omega}_r + \bar{b}_0)) \right)$$

where  $\bar{\omega}_r = \|\boldsymbol{\omega}_r\|_{\max}$ ,  $\bar{b}_0 = \|\mathbf{b}_0\|_{\max}$ ,  $\bar{b}_{\delta} = \|\mathbf{b}_{\delta}\|_{\max}$ . From (6) and (7), we obtain

$$\mathbf{x}(k+1) = \mathbf{A}_0(k)\mathbf{x}(k) + \sum_{i=1}^{9} \mathbf{A}_i(k)\Delta_i(k)\mathbf{x}(k).$$
 (8)

Assume that the state  $\mathbf{x}(k)$  is not known but that it is contained inside the compact and convex polytope defined by the known matrix  $\mathbf{M}(k)$  and vector  $\mathbf{m}(k)$ , i.e.,

$$\mathbf{x}(k) \in \operatorname{Set}(\mathbf{M}(k), \mathbf{m}(k)).$$

Due to the presence of noise in the angular velocity measurements, which is reflected in the uncertainty in  $\Delta_i$ ,  $i = 1, \ldots, 9$ , the set of feasible states at time k+1,  $\mathbf{x}(k+1)$ , is not convex, in general.

Nevertheless, we will see next that, by considering specific realizations of (8) and using SVOs to obtain the polytope that contains the state for each particular realization, we can derive a set containing the state  $\mathbf{x}(k+1)$ .

As such, consider now a realization of (8) where  $\Delta_i(k) =$  $\Delta_i^*, |\Delta_i^*| \leq 1, i = 1, \dots, 9$  and denote by  $\mathbf{A}_{\Delta^*}$  the corresponding uncertainty map, i.e.,  $\mathbf{A}_{\Delta^*} = \mathbf{A}_1^* \Delta_1^* + \cdots + \mathbf{A}_9^* \Delta_9^*$ . For each  $A_{\Delta^*}$ , the technique described in [24] can be used to design an SVO which computes a set-valued estimate of the state of the system. Indeed, if the matrix  $A_0(k) + A_{\Delta^*}$ is non-singular<sup>1</sup>, we can write the following inequality as a constraint for the state  $\mathbf{x}(k+1)$ 

$$\underbrace{\begin{bmatrix} \mathbf{M}(k)(\mathbf{A}_0(k) + \mathbf{A}_{\Delta^*})^{-1} \\ \mathbf{M}_m(k+1) \end{bmatrix}}_{\mathbf{M}_*(k+1)} \mathbf{x}(k+1) \leq \underbrace{\begin{bmatrix} \mathbf{m}(k) \\ \mathbf{m}_m(k+1) \end{bmatrix}}_{\mathbf{m}_*(k+1)}.$$
 (9)

In other words, for  $\Delta_i(k) = \Delta_i^*$ ,  $i = 1, \dots, 9$ ,  $\mathbf{x}(k+1) \in$  $\begin{array}{l} \mathrm{Set}\,(\mathbf{M}_*(k+1),\mathbf{m}_*(k+1)).\\ \mathrm{Let}\,\,v_i,\,i=1,\ldots,2^9 \mbox{ denote a vertex of the hyper-cube} \end{array}$ 

$$H := \{\delta \in \mathbb{R}^9 : |\delta| \le 1\},\tag{10}$$

where  $v_i = v_j \Leftrightarrow i = j$ . Then, we denote by  $\hat{X}_{v_i}(k+1)$ the set of points  $\mathbf{x}(k+1)$  that satisfy (9) where  $\mathbf{A}_{\Delta^*} = \mathbf{A}_{v_i}$ and with  $\mathbf{x}(k) \in \text{Set}(\mathbf{M}(k), \mathbf{m}(k))$ . Notice that  $X_{v_i}(k+1)$ can be obtained by resorting to (9).

Further define

$$\hat{X}(k+1) := \operatorname{co}\left\{\hat{X}_{v_1}(k+1), \dots, \hat{X}_{v_{2^9}}(k+1)\right\}, \quad (11)$$

where  $co\{p_1, \ldots, p_m\}$  is the smallest convex set containing the points  $p_1, \ldots, p_m$ , also known as convex hull of  $p_1, \ldots, p_m$ . The set of all possible states at time k+1 is, in general, non-convex even if X(k) is convex, thus we are going to use  $\hat{X}(k+1)$  to over bound it. Since  $\hat{X}(k+1)$  is the convex hull of a finite number of polytopes, it can be written in the form  $Set(\mathbf{M}(k+1), \mathbf{m}(k+1))$ .

The following theorem describes the proposed attitude SVO, which considers the rate gyro bias and noise. This estimator can be interpreted as generalization of the observer proposed in [5].

Theorem 1: Assume that there is a triad of rate gyros that measure the rigid body angular velocity corrupted by noise and constant bias, that measurements  ${}^{B}\mathbf{v}_{i}$ ,  $i = 1, \ldots, N_{v}$ under the conditions of Lemma 1 are available, and that there exist a matrix  $\mathbf{M}(k)$  and a vector  $\mathbf{m}(k)$ , such that

$$\mathcal{R}(k) \in \operatorname{MatSet}(\mathbf{M}(k), \mathbf{m}(k)) \cap \operatorname{SO}(3).$$

Then, the set  $MatSet(\mathbf{M}(k+1), \mathbf{m}(k+1)) \cap SO(3)$  as defined previously contains all the matrices  $\mathcal{R}(k+1)$  that satisfy (2) and that are compatible with the observations at time k+1.

*Proof:* By assumption, the state of the system at time k belongs to the set

$$X(k) := \left\{ \mathbf{x} \in \mathbb{R}^9 : \mathbf{M}(k) \mathbf{x} \le \mathbf{m}(k), \operatorname{Mat}(\mathbf{x}) \in \operatorname{SO}(3) \right\}.$$

Also, define X(k+1) as the set of all possible states of the system at time k + 1.

Equation (9) defines the set of states at time k + 1 that satisfy (7) and are compatible with the measurements. By evaluating (9) for  $\Delta = [\Delta_1, \dots, \Delta_9]$  in the vertices of H,

<sup>1</sup>See Remark 1 for the case where the matrix  $A_0(k) + A_{\Delta^*}$  is singular.

defined in (10), one obtains  $\hat{X}_{v_1}(k+1), ..., \hat{X}_{v_{2^9}}(k+1)$ . It was shown in [20] that, since  $\Delta(k)$  can be obtained by convex combinations of the vertices of  $\mathcal{H}$ , the state of the system,  $\mathbf{x}(k+1)$ , is inside the set generated by the convex hull in (11). Thus,  $\hat{X}(k+1)$ , is an overbound of X(k+1) and an overbound to the space of possible solutions of  $\mathcal{R}(k+1)$ , and is given by

$$\mathcal{R}(k+1) \in \operatorname{MatSet}(\mathbf{M}(k+1), \mathbf{m}(k+1)) \cap \operatorname{SO}(3).$$

*Remark 1:* If,  $A(k) + A_{\Delta^*}$  is singular or ill-conditioned, one can write the inequality

$$\underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{A}_0(k) - \mathbf{A}_{\Delta^*}(k) \\ -\mathbf{I} & \mathbf{A}_0(k) + \mathbf{A}_{\Delta^*}(k) \\ \mathbf{M}_m(k+1) & 0 \\ 0 & \mathbf{M}(k) \end{bmatrix}}_{\tilde{\mathbf{M}}} \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{x}(k) \end{bmatrix} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ \mathbf{m}_m(k+1) \\ \mathbf{m}(k) \end{bmatrix}}_{\tilde{\mathbf{m}}},$$

and then use the Fourier-Motzkin projection [9] to compute  $\mathbf{M}_*(k+1)$  and  $\mathbf{m}_*(k+1)$  such that  $\mathbf{M}_*(k+1)\mathbf{x}(k+1) \leq \mathbf{m}_*(k+1)$ , i.e.,

$$(\mathbf{M}_*(k+1), \mathbf{m}_*(k+1)) = \mathrm{FM}(\mathbf{M}, \tilde{\mathbf{m}}, 3).$$

If there is no uncertainties in the rate gyro measurements, we know the exact model dynamics and the resulting set that contains the solution is not conservative. The next corollary highlights this result.

*Corollary 1:* Assume that there is a triad of rate gyros that measure the rigid body angular without uncertainty, i.e.,

$$\bar{n} = 0, \qquad \bar{b}_{\delta} = 0,$$

that measurements  ${}^{B}\mathbf{v}_{i}$ ,  $i = 1, ..., N_{v}$  under the conditions of Lemma 1 are available, and that there exist a matrix  $\mathbf{M}(k)$ and a vector  $\mathbf{m}(k)$ , such that

$$\mathcal{R}(k) \in \operatorname{MatSet}(\mathbf{M}(k), \mathbf{m}(k)) \cap \operatorname{SO}(3).$$

Then, the estimator, at each time k, provides the smallest set compatible with (2) and the vector observations.

Proof: Under the conditions of the corollary,

$$\mathbf{x}(k+1) = \mathbf{A}_0(k)\mathbf{x}(k),\tag{12}$$

where  $\mathbf{A}_0(k)$  is fully known. Let us now define the set  $X(k+1) = \{ \tilde{\mathbf{x}} \in \mathbb{R}^9 : \mathbf{M}(k+1) \tilde{\mathbf{x}} \le \mathbf{m}(k+1), \operatorname{Mat}(\tilde{\mathbf{x}}) \in \operatorname{SO}(3) \}$ , where

$$\mathbf{M}(k+1) = \begin{bmatrix} \mathbf{M}(k)\mathbf{A}_0^{-1}(k) \\ \mathbf{M}_m(k+1) \end{bmatrix}, \ \mathbf{m}(k+1) = \begin{bmatrix} \mathbf{m}(k) \\ \mathbf{m}_m(k+1) \end{bmatrix}$$

By Theorem 1, the set X(k+1) contain all the solutions of (2) at k+1, which are compatible with the vector observation.

To show that we obtain the smallest set, we also need to demonstrate that all the states  $\mathbf{x}(k+1) \in X(k+1)$  are compatible with (2) and with

$$\mathbf{x}(k) \in \{ \tilde{\mathbf{x}} \in \mathbb{R}^9 : \mathbf{M}(k) \tilde{\mathbf{x}} \le \mathbf{m}(k), \operatorname{Mat}(\tilde{\mathbf{x}}) \in \operatorname{SO}(3) \}.$$

The state at k + 1 satisfies

$$\mathbf{M}(k)\mathbf{A}_0^{-1}(k)\mathbf{x}(k+1) \le \mathbf{m}(k),$$

and by using (12), which, under the conditions of the corollary, is equivalent to (2), in the previous equation, we recover

$$\mathbf{M}(k)\mathbf{x}(k) \le \mathbf{m}(k)$$

Finally, Proposition 1 ensures that if we consider  $\mathbf{x}(k)$ ,  $\mathbf{x}(k+1) \in \mathbb{R}^9$  instead of  $Mat(\mathbf{x}(k))$ ,  $Mat(\mathbf{x}(k+1)) \in SO(3)$  we add no conservatism to the solution.

# B. Bias estimation

We now propose an SVO for rate gyro bias estimation. The measurements provided by the rate gyros, regardless of its technology, are affected by noise and bias. The amplitude of the bias is, for most rate gyros, not negligible and might significantly deteriorate the attitude estimates [10]. The key idea in estimating the gyro bias consists in exploiting the sensor measurements to compute sets that, at each time, contain the rate gyro bias, and intersect them as time goes by, in order to reduce the associated uncertainty.

Recall that a polytope containing the states of the attitude at time k is given by  $\hat{X}(k)$ , let Y(k+1) be the set containing the rotation matrices compatible with the measurements, i.e.,

$$Y(k+1) = \operatorname{Set}(\mathbf{M}_m(k+1), \mathbf{m}_m(k+1)),$$

and define  $B_m(k)$  as the convex set containing the rate gyro bias based on the sensor measurements at time k and computed as follows.

The set  $B_m(k)$  can be obtained by inverting the attitude kinematics (2) and computing a set that contains the angular velocity at time k

$$\boldsymbol{\omega}(k) = \log(\mathcal{R}(k+1)\mathcal{R}(k)^T),$$

where  $\log(.)$  : SO(3)  $\rightarrow \mathfrak{so}(3)$  is the inverse of the exponential map in the special orthogonal group [2], which can be computed by inverting the *Rodrigues' Formula* [23, p. 65], i.e.,

$$\log(\mathbf{R}) = \begin{cases} 0 & \text{if } \theta = 0\\ \frac{\theta}{2\sin(\theta)} (\mathbf{R} - \mathbf{R}^{T}) & \text{if } \theta \neq 0 \text{ and } \theta \in (0, \pi) \end{cases},$$
(13)

where  $\mathbf{R} \in SO(3)$  and  $\theta = \arccos\left((\operatorname{tr}(\mathbf{R}) - 1)/2\right) \in [0 \ \pi]$ . If  $\theta = \pi$ , the result is not unique. However, as long as the trace of  $\mathbf{R}$  is different from minus one, this formula can be used to obtain upper and lower bounds on  $\log(\mathbf{R})$ . To this end, we over bound the sets  $\hat{X}(k)$  and Y(k+1) by a hypercube. We recall that  $\operatorname{Vec}(\mathcal{R}(k)) \in \hat{X}(k)$  and  $\operatorname{Vec}(\mathcal{R}(k + 1)) \in Y(k+1)$ . Equation (13) is then solved for  $\mathbf{R} = \mathcal{R}(k+1)\mathcal{R}(k)^T$  by resorting to interval analysis [8], which results in upper and lower bounds for  $\boldsymbol{\omega}(k)$ . As the quotient  $\frac{\theta}{\sin(\theta)}$  is a monotonically increasing function in the domain  $\theta \in [0 \ \pi]$ , the corresponding bounds can be easily computed by evaluating its value at the extremes of the interval of  $\theta$ .

The rate gyro bias is related to the true and the measured angular velocity through (3), which can be rearranged in order to isolate the bias, i.e.,

$$\mathbf{b} = \boldsymbol{\omega}_r - \boldsymbol{\omega} - \mathbf{n}, \quad \|\mathbf{n}\|_{\max} \leq \bar{n}.$$

The set  $B_m(k)$  is then computed by the sum of the vectorial angular velocity measurement with the polytopes containing the true angular velocity (computed using (13)) and the sensor noise. This sum can be accomplished using the Fourier-Motzkin projection [9]. Finally, since the bias is constant, this set is intersected with the one obtained in the previous iteration  $B(k) = B(k-1) \cap B_m(k)$ . From B(k), we extract  $\mathbf{b}_0$  and  $\bar{b}_{\delta}$ , required in the attitude SVO.



Fig. 1. Division of the uncertainty set into several sub-sets and selection of the sub-set containing the true bias.

# C. Attitude and bias estimation using more than one model

The rate gyro bias can also be seen as a parameter of (2). This fact can be exploited to increase the rate of convergence of the bias SVO and consequently of the attitude SVO. We propose a divide-and-conquer strategy. Instead of one pair of attitude SVO and bias SVO, we can have multiple pairs running in parallel. For each pair, it is assumed that the bias belongs to a different set, with the constraint that the reunion of all sets must fully cover the uncertainty space of the bias. However, only one pair contains the true value of the bias. Hence, it may happen that, as time goes by, all other pairs degenerate on empty sets for the estimates. At this point, we divide again the uncertainty space of the bias. With this approach, each pair of estimators has smaller model uncertainty and, thus, the convergence of the uncertainty set of the bias is likely to be faster.

This model invalidation strategy is typically known in the literature as *model falsification* (see [18]). In particular, a novel SVO-based approach to model falsification has been recently proposed, that uses the aforementioned line-of-thought. The interested reader is further referred to [18].

The selection of the number of estimators running in parallel depends on a compromise between the available computational power and the desired convergence rate. One great advantage of this technique is the possibility of parallelization of the computational burden by exploiting the recent advances in the multi-core processors and multiprocessors systems. This strategy, illustrated in Fig. 1, greatly increases the computational speed.

Some practical considerations should be noted.

*Remark 2:* The vector observations often have norm constraints. Although we cannot impose such restrictions directly in the SVOs, some overbounds can be derived and included in the formulation to decrease the uncertainty in the measurements

*Remark 3:* The use of this estimation scheme in very aggressive maneuvering vehicles may lead to some practical problems, since the conservatism added to the exponential map increases with the angular velocity.

*Remark 4:* The noise in sensors, like magnetometers, is typically modeled by Gaussian distributions. In such circumstances, the  $3\sigma$  empirical bound is a suitable choice, at least for some practical situations.

# **IV. SIMULATION RESULTS**

In this section, we present simulation results that illustrate the performance of the proposed solution. The simulated



Fig. 2. True attitude trajectory.



Fig. 3. Error of the upper and lower bounds of the Euler angles (a), and error of the mean of the Euler angles (b).



Fig. 4. Upper and lower bounds of the rate gyro bias (a), and error of the mean of the rate gyro bias bounds (b).

trajectory is characterized by an angular velocity with the following oscillatory profile  $\omega_x = 4.01 \sin(2\pi 0.05 kT) \text{ deg/s}$ ,  $-2.86\sin(2\pi 0.04kT)$ \_  $\deg/s$ ,  $\omega_{z}$  $\omega_y$ 3.44 sin( $2\pi 0.02kT$ ) deg/s,  $\boldsymbol{\omega} = [\omega_x \ \omega_y \ \omega_z]^T$ . The sampling time, T, and the maximum rate gyros noise =  $\bar{n}$ , are set to 0.1 s, and 0.115 deg/s,  $\|\mathbf{n}\|_{\max}$ respectively. The initial uncertainty on the rate gyro bias is  $\pm 5.73$  deg/s in each channel, while the true rate gyro bias is  $\mathbf{b} = [0.03 - 0.01 \ 0.02]^T \ \text{deg/s}$ . The directions for the vector observations in the inertial frame are given by  ${}^{I}\mathbf{v}_{1} = [1 \ 4 \ 0], {}^{I}\mathbf{v}_{2} = [3 \ 0 \ 0]^{T}, {}^{I}\mathbf{v}_{3} = [0 \ 0 \ 6]^{T}$ , and each channel of the measurements is corrupted by noise bounded by ±0.01.

Figure 2 depicts the true attitude trajectory using the Euler angles representation. In Fig. 3(a), it is shown the error of the upper and lower bounds. In Fig. 3(b), the estimation error is illustrated, showing it is always below 0.2 deg. The upper and lower bounds on the rate gyro bias, as well as its true value are depicted in Fig. 4(a). This figure shows that the initial uncertainty in the rate gyro bias is reduced from 11.46 deg/s in each component, to 1.25 deg/s, 1.35 deg/s, 2.08 deg/s in the x, y, z components, respectively. Finally, Fig. 4(b) depicts the time-evolution of the error of the central point of the bias polytope,  $b_0$ , to the true bias.

#### V. CONCLUSIONS

This paper proposed a solution for the problem of attitude and rate gyro bias estimation, based on vector observations and angular velocity, corrupted by bounded noise. We developed an SVO which considers uncertainties defined by polytopes and which guarantees that the true state of the system is inside the estimated set. The obtained solution is conservative due to the uncertainty in the measurements provided by the rate gyros. However, no linearization is required and the attitude of the rigid body is parametrized by a rotation matrix yielding estimates which are free of singularities. We showed that the new multi-core processors can be exploited to increase the estimator convergence rate without compromising the execution time. Simulation results attested the applicability of the proposed technique.

# APPENDIX

A. Bound on the exponential map of the sum of two skewsymmetric matrices

In this section, we derive a bound on the exponential map of the sum of two skew-symmetric matrices when only bounds are known for one of them.

Let  $\mathbf{k}_1 \in \mathbb{R}^3$  and  $\mathbf{k}_2 \in \mathbb{R}^3$  be two generic vectors, and define the skew-symmetric matrices  $\mathbf{K}_1 = [\mathbf{k}_1]_{\times}$ ,  $\mathbf{K}_2 = [\mathbf{k}_2]_{\times}$ . From the definition of matrix multiplication, we have that  $[\mathbf{C}]_{ij} = \sum_{k=1}^{p} [\mathbf{K}_1]_{ik} [\mathbf{K}_2]_{kj}$ , where  $\mathbf{C} = \mathbf{K}_1 \mathbf{K}_2 \in \mathbb{R}^{3 \times 3}$ , and  $[\mathbf{X}]_{ij}$  denotes the element of line *i* and column *j* of matrix  $\mathbf{X} \in \mathbb{R}^{m \times n}$ . Using the fact that, for skew-symmetric matrices, at least one of the elements of each row and each column is zero, we obtain the following inequalities

$$\|\mathbf{K}_{1}^{k}\|_{\max} \le \frac{(2\bar{k}_{1})^{k}}{2}, \qquad \|\mathbf{K}_{1}^{k}\mathbf{K}_{2}^{l}\|_{\max} \le \frac{(2\bar{k}_{1})^{k}(2\bar{k}_{2})^{l}}{2}$$

where  $\bar{k}_1 = \|\mathbf{k}_1\|_{\max}$  and  $\bar{k}_2 = \|\mathbf{k}_2\|_{\max}$ . From these inequalities, we derive an upper bound for each element of the power of the sum of two matrices  $[(\mathbf{K}_1 + \mathbf{K}_2)^k]_{ij} \leq [\mathbf{K}_1^k + \frac{1}{2}\mathbf{1}_{3\times 3}((\bar{k}_1 + \bar{k}_2)^k - \bar{k}_1^k)]_{ij}$ . Consequently, the exponential map of the sum of matrices  $\mathbf{K}_1$  and  $\mathbf{K}_2$  satisfies

$$[\exp(\mathbf{K}_{1} + \mathbf{K}_{2})]_{ij} = \sum_{k=0}^{\infty} \frac{\left[(\mathbf{K}_{1} + \mathbf{K}_{2})^{k}\right]_{ij}}{k!}$$
  
$$\leq [\exp(\mathbf{K}_{1})]_{ij} + \frac{1}{2}(\exp(2\bar{k}_{1} + 2\bar{k}_{2}) - \exp(2\bar{k}_{1})).$$

# B. Property of the multiplication of elements of the Special Orthogonal Group

The following proposition is used in Corollary 1 in order to consider the particular case when the measurements of the rate gyros are exact.

Proposition 1: For any matrix  $\mathbf{A} \in SO(3)$ ,  $\mathbf{AB} \in SO(3)$  if and only if  $\mathbf{B} \in SO(3)$ .

The proof follows directly from the properties of SO(3), however it is omitted due to space constraints.

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