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Brief paper

Model-based $\mathcal{H}_2$ adaptive filter for 3D positioning and tracking systems[☆]

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ABSTRACT

In this work, a cascade of two estimators is proposed as the solution for a joint parameter and state estimation problem associated with a target maneuvering in three-dimensional space. A model for the target that depends on its angular speed is considered and only the target position is measured. A parameter identifier is used to obtain estimates of the target angular speed, which are then fed into an adaptive filter that estimates the position, linear velocity, and linear acceleration of the target. The synthesis of the parameter identifier resorts to Lyapunov techniques and the adaptive filter is synthesized using $\mathcal{H}_2$ optimization strategies. Under persistence of excitation conditions, the error in the angular speed identification and the error in the target state estimates provided by the $\mathcal{H}_2$ adaptive filter are: (i) proved to converge exponentially fast to zero in the deterministic setup, i.e., in the absence of noise, and (ii) proved to be bounded when bounded stochastic disturbances are considered and there is an upper bound on the target linear velocity and angular speed. To assess the proposed methods, simulations showing that the aforementioned stability and convergence properties hold, even when the estimates provided by an Extended Kalman Filter (EKF) diverge, are presented.

1. Introduction

The problem of 3D target positioning and tracking has been widely studied over recent decades, mainly due to the great impact that the availability of reliable estimates for the position of a target has on the performance of many robotic applications. Some examples of such applications appear, for instance, in the contexts of security and surveillance, trajectory determination, human–computer interaction, and air traffic control, see Bar-Shalom, Rong-Li, and Kirubarajan (2001), Kolodziej and Hjelm (2006), Lepetit and Fua (2005), and Saeedi, Lawrence, and Lowe (2006).

Positioning and tracking consist in using measurements provided by one or more sensors, at fixed locations or at moving platforms, to estimate the state of a moving object, which is usually composed of its position, velocity, and sometimes acceleration. To estimate such quantities, a dynamical model for the maneuvering target is usually considered, see the comprehensive survey in Rong Li and Jilkov (2003). Typical models depend on the target angular velocity, or on its magnitude, the target angular speed. However, most of the time this quantity is not known and measurements of its value are not available. In fact, in most applications, only the target position is measured. Therefore, strategies that consider several models for the target differing, for instance, in their structure or in the value of the target angular speed have been used. In Bar-Shalom et al. (2001) and Gaspar and Oliveira (2011), tracking systems based on Interacting Multiple Models and on a Multiple Model Adaptive Estimator can be found, respectively. In this work, an alternative approach based on parameter identification strategies and on adaptive filtering is proposed.

The problem of estimating the position, velocity, and acceleration of a target maneuvering in 3D space using only measurements of its position is tackled by resorting to a cascade of a parameter identifier and an $\mathcal{H}_2$ adaptive filter. The first estimates the target angular speed and the second combines these estimates with measurements of the target position to estimate the target state.

The problem at hand could have been addressed using other strategies, such as robust linear filtering, for instance. However, the model considered for the target, like other state-space models used in target tracking, is unstable. In this case, the system has three eigenvalues at the origin of the complex plane and three pairs of complex conjugate eigenvalues in the imaginary axis. The work

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in Xie, de Souza, and Soh (1994) addresses the problem of robust filtering design for uncertain linear systems with unstable modes and subject to norm-bounded parameter uncertainty in both the state and the output matrices. However, it is assumed that the uncertainty does not affect the unstable modes of the system, which is not the case. Moreover, it is easy to show that any linear filter designed for the system considered in this work using a wrong model for the target dynamics will be biased. Other approaches inspired, for instance, in Lyapunov theory or backstepping, see Krstic, Kanellakopoulos, and Kokotovic (1995), have also failed, since both strategies require the observation of the target velocity and acceleration, which are not available for measurement.

The main contributions of this work are:

(1) a new $\mathcal{H}_2$ adaptive filter that estimates the position, linear velocity, and linear acceleration of a target using only position observations;
(2) a new parameter identifier that estimates the (assumed constant) target angular speed — the structure of this identifier is different from the usual approaches, since there is only one unknown parameter, but there are several measurements depending on its value;
(3) a guarantee that, under persistence of excitation conditions, the errors in the angular speed identification and in the state estimates provided by the adaptive filter converge exponentially fast to zero in the deterministic setup, i.e., in the absence of noise, and are bounded when bounded stochastic disturbances are considered and there is an upper bound on the target linear velocity and angular speed.

The proposed framework is also appropriate for systems with sensors that do not measure the target 3D position in Euclidean coordinates, such as RADAR or SONAR, see Bar-Shalom et al. (2001) and Delgado and Barreiro (2003), as long as their measurements can be transformed into 3D Euclidean position measurements.

This brief paper is organized as follows. The problem addressed in this work is formulated in Section 2, and the design and analysis of the identification procedure that estimates the target angular speed are provided in Section 3. In Section 4, an $\mathcal{H}_2$ adaptive filter for the state of the target is derived, and its stability and performance are discussed. Simulations illustrating the performance of the proposed estimators, in comparison with an Extended Kalman Filter (EKF), are presented in Section 5. Finally, in Section 6, concluding remarks are provided.

Nomenclature

In this brief paper, $|x|$ denotes the absolute value of the scalar x, $\|\mathbf{x}\|$ the Euclidean norm of the vector $\mathbf{x}$, and $\|\mathbf{X}\|$ the induced 2-norm of the matrix $\mathbf{X}$. If the vector $\mathbf{x}$ is a function of time in $\mathbb{R}^n$, $\mathbf{x} \in \mathcal{L}_2$ and $\mathbf{x} \in \mathcal{L}_\infty$ mean, respectively, that $\|\mathbf{x}\|_2 = (\int_0^\infty \|\mathbf{x}(t)\|^2 \mathrm{d}t)^{1/2}$ and $\|\mathbf{x}\|_\infty = \sup_{t\geq 0}\|\mathbf{x}(t)\|$ are finite. The notation $\mathbf{X}_{ij}$ is used to represent the entry of $\mathbf{X}$ in the i-th line and j-th column. The vector $\mathbf{e}_i$, $i = \{1, 2, 3\}$, denotes the i-th vector of the canonical basis of $\mathbb{R}^3$; tr[$\mathbf{X}$] stands for the trace of a square matrix $\mathbf{X}$, and diag$[a_1, \ldots, a_n]$ corresponds to a diagonal matrix whose diagonal entries, starting in the upper left corner, are $a_1, \ldots, a_n$ (when these entries are matrices, the resulting matrix is block diagonal). The identity and zero matrices are denoted respectively by $\mathbf{I}_k$ and $\mathbf{0}_{m\times n}$, where k corresponds to the number of rows and columns of the identity matrix, and m and n correspond, respectively, to the number of rows and columns of the matrix of zeros. Finally, $\otimes$ denotes the Kronecker product, $\delta(t)$ the Dirac delta function, and min$[a, b]$ the minimum of the elements a and b.

2. Problem formulation

The problem addressed in this brief paper is that of tracking and locating a target maneuvering in three-dimensional space using observations of its position. The target position, linear velocity, and

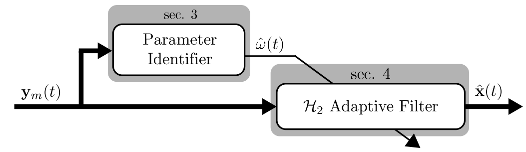


Fig. 1. Parameter identifier and adaptive filter connection.

linear acceleration in the inertial (Cartesian) frame are denoted by $\mathbf{p} = [x\ y\ z]^T$, $\mathbf{v} = [\dot{x}\ \dot{y}\ \dot{z}]^T$, and $\mathbf{a} = [\ddot{x}\ \ddot{y}\ \ddot{z}]^T$, respectively, where the dot represents the time derivative. Using this notation, the state $\mathbf{x} = [x\ \dot{x}\ \ddot{x}\ y\ \dot{y}\ \ddot{y}\ z\ \dot{z}\ \ddot{z}]^T \in \mathbb{R}^9$ of the target is considered to evolve according to the *3D Planar Constant-Turn Model*

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\omega)\mathbf{x}(t) + \mathbf{B}\mathbf{d}(t), \tag{1}$$

as presented in Rong Li and Jilkov (2003), where

$$\mathbf{F}(\omega) = \mathrm{diag}\left[\,\bar{\mathbf{F}}(\omega), \bar{\mathbf{F}}(\omega), \bar{\mathbf{F}}(\omega)\,\right], \qquad \bar{\mathbf{F}}(\omega) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & 0 \end{bmatrix},$$

$$\mathbf{B} = \mathbf{I}_3 \otimes \mathbf{b}, \qquad \mathbf{b} = \mathbf{e}_3,$$

and $\omega \geq 0$ is the (assumed constant, unknown, and bounded) angular speed (norm of the target angular velocity vector). The process noise is denoted by $\mathbf{d}(t) \in \mathbb{R}^3$ and time is represented by t. The eigenvalues of $\bar{\mathbf{F}}(\omega)$ are 0 and $\pm\omega j$, where $j = \sqrt{-1}$ is the imaginary unit. Thus, the nominal trajectories considered by this model are straight lines, parabolic trajectories, and ellipses.

The measurements $\mathbf{y}_m(t) \in \mathbb{R}^3$ of the position of the target with respect to the inertial reference frame are a linear function of the target state, and can be written as

$$\mathbf{y}_m(t) = \mathbf{p}(t) + \mathbf{D}\mathbf{n}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{n}(t), \tag{2}$$

where $\mathbf{n}(t) \in \mathbb{R}^3$ denotes the measurement noise, $\mathbf{C} = \mathbf{I}_3 \otimes \mathbf{e}_1^T$, and $\mathbf{D} = \mathbf{I}_3$. Both the process and the observation noises are assumed to be bounded stochastic disturbances, i.e., $\beta_d = \|\mathbf{d}\|_\infty$ and $\beta_n = \|\mathbf{n}\|_\infty$ are finite.

The problem addressed in this brief paper is stated next.

Problem statement. Consider a target maneuvering in 3D space according to the model in (1), with constant, unknown, and bounded angular speed. Moreover, assume that measurements of the target position, as described in (2), are available. In this case, design two estimators, one for the target state and the other for its angular speed, such that the errors in both cases (i) converge exponentially fast to zero when no process and observation noises are present, and (ii) are bounded when bounded noise is considered and there is an upper bound on the target linear velocity.

To solve this problem, a cascade of a parameter identifier and an adaptive filter is proposed, see Fig. 1. In the figure, $\hat{\omega}(t)$ denotes the estimates of the target angular speed ω and $\hat{\mathbf{x}}(t)$ the estimates of the target state $\mathbf{x}(t)$.

3. Angular speed identification

In this section, the design and analysis in continuous-time of a parameter identifier that estimates the angular speed of a target moving according to the model in (1) are provided. This identifier resorts only to position measurements obtained as in (2), and builds on strategies commonly used in adaptive control, see Ioannou and Fidan (2006) and Sastry and Bodson (1989).

The design of the parameter identifier uses the convolution differentiation properties in Proposition 1.

Proposition 1 (*Convolution Differentiation Rules*)**.** *If the convolution of the functions $f(t)$ and $g(t)$, over the range $[0, t]$, is given by $f(t)$*

$* g(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$, *then*

$$\dot{f}(t) * g(t) = f(t) * \dot{g}(t) + f(t)g(0) - f(0)g(t) \quad (3)$$

$$\dddot{f}(t) * g(t) = f(t) * \dddot{g}(t) + f(t)\ddot{g}(0) - f(0)\ddot{g}(t) + \dot{f}(t)\dot{g}(0) - \dot{f}(0)\dot{g}(t) + \ddot{f}(t)g(0) - \ddot{f}(0)g(t). \quad (4)$$

Proof. Using the product and chain rules of differentiation, it is easy to obtain (3) since

$$\dot{f}(t) * g(t) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\big(f(\tau)g(t-\tau)\big)\mathrm{d}\tau - \int_0^t f(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}g(t-\tau)\mathrm{d}\tau$$
$$= f(t)g(0) - f(0)g(t) - \int_0^t f(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}g(t-\tau)\mathrm{d}\tau.$$

If this relation is applied successively to $\dddot{f}(t) * g(t)$, then (4) follows immediately. □

From the model in (1), it is easy to conclude that $\dot{\mathbf{a}}(t) = \alpha\mathbf{v}(t) + \mathbf{d}(t)$, i.e., that $\dddot{\mathbf{p}}(t) = \alpha\dot{\mathbf{p}}(t) + \mathbf{d}(t)$, where $\alpha = -\omega^2$. To avoid the use of differentiators, let us start by filtering the entries of the vectors in the previous expression with a filter with impulsive response $h_f(t)$, which leads to

$$\dddot{\mathbf{p}}(t) * h_f(t) = \alpha\dot{\mathbf{p}}(t) * h_f(t) + \mathbf{d}(t) * h_f(t).$$

Note that $\mathbf{u}(t) * h_f(t)$, where $\mathbf{u}(t)$ represents a generic vector, denotes the convolution of each one of the entries of $\mathbf{u}(t)$ with the impulsive response $h_f(t)$, over the finite range $[0, t]$. According to (3) and (4), the previous expression can be rewritten in the form

$$\begin{aligned}&\mathbf{p}(t) * \big[\dddot{h}_f(t) + \delta(t)\ddot{h}_f(0)\big]\\ &= \alpha\mathbf{p}(t) * \big[\dot{h}_f(t) + \delta(t)h_f(0)\big] + \mathbf{p}(0)\ddot{h}_f(t) + \dot{\mathbf{p}}(0)\dot{h}_f(t)\\ &\quad + \big(\ddot{\mathbf{p}}(0) - \alpha\mathbf{p}(0)\big)h_f(t) - \dot{\mathbf{p}}(t)\dot{h}_f(0)\\ &\quad - \ddot{\mathbf{p}}(t)h_f(0) + \mathbf{d}(t) * h_f(t),\end{aligned} \quad (5)$$

given that $\mathbf{p}(t) = \mathbf{p}(t) * \delta(t)$, since $\delta(t-\tau) = 0, \forall_{\tau > t}$.

For simplicity, consider the notation

$$\begin{aligned}&h_1(t) = h_f(t), && H_1(s) = H_f(s)\\ &h_2(t) = \dot{h}_f(t) + \delta(t)h_f(0), && H_2(s) = sH_f(s)\\ &h_3(t) = \dddot{h}_f(t) + \delta(t)\ddot{h}_f(0), && H_3(s) = s^3H_f(s) - s^2h_f(0) - s\dot{h}_f(0)\end{aligned}$$

where s is the Laplace operator, $H_f(s)$ the Laplace transform of $h_f(t)$, and $H_i(s)$ the Laplace transform of the impulsive response $h_i(t), i = 1, 2, 3$.

In (5), the target velocity and acceleration are not measured, thus the terms $\dot{\mathbf{p}}(t)\dot{h}_f(0)$ and $\ddot{\mathbf{p}}(t)h_f(0)$ are unknown. These quantities can be removed from the equation by designing the impulsive response $h_f(t)$ in such a way that $h_f(t)$ and $\dot{h}_f(t)$ verify $h_f(0) = \dot{h}_f(0) = 0$. Moreover, for $h_1(t)$, $h_2(t)$, and $h_3(t)$ to be impulsive responses of filters that are realizable by linear time-invariant state-space systems, $H_1(s)$, $H_2(s)$, and $H_3(s)$ must be proper rational functions, see Rugh (1996). Since s^3 has degree 3, this can be accomplished by choosing $H_f(s)$ to be a stable filter of the form $H_f(s) = 1/\Lambda(s)$, where $\Lambda(s)$ is a third-order monic Hurwitz polynomial, e.g., $\Lambda(s) = (s+\lambda)^3, \lambda > 0$. By calculating the inverse Laplace transform of $H_f(s)$, the functions $h_f(t) = t^2e^{-\lambda t}u(t)/2$ and $\dot{h}_f(t) = t(1-\lambda t/2)e^{-\lambda t}u(t)$, where $u(t)$ denotes the continuous-time unit step function, see Oppenheim, Willsky, and Hamid (1983), result. Note that the condition $h_f(0) = \dot{h}_f(0) = 0$ is verified. The transfer function $H_f(s) = 1/\Lambda(s)$ leads to

$$H_1(s) = \frac{1}{\Lambda(s)}, \quad H_2(s) = \frac{s}{\Lambda(s)}, \quad \text{and} \quad H_3(s) = \frac{s^3}{\Lambda(s)}.$$

According to the notation and reasoning above, the relation in (5) can be rewritten as a function of $\mathbf{y}_m(t)$:

$$\underbrace{\mathbf{y}_m(t) * h_3(t)}_{\boldsymbol{\psi}(t)} = \alpha\underbrace{\mathbf{y}_m(t) * h_2(t)}_{\boldsymbol{\phi}(t)} + \underbrace{\mathbf{n}(t) * \big[h_3(t) - \alpha h_2(t)\big] + \mathbf{d}(t) * h_1(t)}_{\boldsymbol{\xi}(t)} + \underbrace{\mathbf{p}_0\ddot{h}_f(t) + \mathbf{v}_0\dot{h}_f(t) + (\mathbf{a}_0 - \alpha\mathbf{p}_0)h_f(t)}_{\mathbf{q}(t)}, \quad (6)$$

where $\mathbf{p}_0$, $\mathbf{v}_0$, and $\mathbf{a}_0$ denote the initial values of $\mathbf{p}(t)$, $\mathbf{v}(t)$, and $\mathbf{a}(t)$, respectively. In this formula, $\mathbf{q}(t)$ is a term that comes from the initial conditions and $\boldsymbol{\xi}(t) \in \mathbb{R}^3$ is a signal that results from filtering the process and observation noises with filters with impulsive responses $h_1(t)$ and $h_3(t) - \alpha h_2(t)$, i.e., with filters with transfer functions $1/\Lambda(s)$ and $(s^3 - \alpha s)/\Lambda(s)$, respectively. Since the process and observation noises, as well as the quantities $\mathbf{p}_0$, $\mathbf{v}_0$, $\mathbf{a}_0$, and α, are not known, the signals $\boldsymbol{\xi}(t)$ and $\mathbf{q}(t)$ are also not known. It is straightforward to show that $\|\mathbf{q}(t)\|$ converges exponentially fast to zero, thus this term vanishes with time. Some properties will also be inferred for $\|\boldsymbol{\xi}(t)\|$ in Section 3.3, which will ensure that having the two unknown terms $\boldsymbol{\xi}(t)$ and $\mathbf{q}(t)$, in (6), is not a problem. Moreover, $\boldsymbol{\psi}(t) \in \mathbb{R}^3$ and $\boldsymbol{\phi}(t) \in \mathbb{R}^3$ are signals obtained by filtering the entries of the measurement vector $\mathbf{y}_m(t)$with filters with transfer functions $H_3(s)$ and $H_2(s)$, respectively. Considering a state-space framework, the i-th entry $\boldsymbol{\psi}_i(t)$ of $\boldsymbol{\psi}(t)$ and the i-th entry $\boldsymbol{\phi}_i(t)$ of $\boldsymbol{\phi}(t)$ are obtained by filtering the i-th entry $\mathbf{y}_{m_i}(t)$ of the measurements $\mathbf{y}_m(t)$ with the following causal linear time-invariant filters:

$$\begin{cases}\dot{\mathbf{x}}_{\psi_i}(t) = \mathbf{A}_\psi\mathbf{x}_{\psi_i}(t) + \mathbf{B}_\psi\mathbf{y}_{m_i}(t)\\ \boldsymbol{\psi}_i(t) = \mathbf{C}_\psi\mathbf{x}_{\psi_i}(t) + D_\psi\mathbf{y}_{m_i}(t),\end{cases} \quad \mathbf{x}_{\psi_i}(0) = \mathbf{0}_{3\times1}, \quad (7)$$

and

$$\begin{cases}\dot{\mathbf{x}}_{\phi_i}(t) = \mathbf{A}_\phi\mathbf{x}_{\phi_i}(t) + \mathbf{B}_\phi\mathbf{y}_{m_i}(t)\\ \boldsymbol{\phi}_i(t) = \mathbf{C}_\phi\mathbf{x}_{\phi_i}(t),\end{cases} \quad \mathbf{x}_{\phi_i}(0) = \mathbf{0}_{3\times1}, \quad (8)$$

where $\mathbf{x}_{\psi_i}(t) \in \mathbb{R}^3$ and $\mathbf{x}_{\phi_i}(t) \in \mathbb{R}^3$ denote the state vectors of each filter. For $\Lambda(s) = (s+\lambda)^3, \lambda > 0$, we have

$$\mathbf{A}_\psi = \mathbf{A}_\phi = \begin{bmatrix}-3\lambda & -3\lambda^2 & -\lambda^3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \quad \mathbf{B}_\psi = \mathbf{B}_\phi = \mathbf{e}_1,$$

$$\mathbf{C}_\psi = \begin{bmatrix}-3\lambda & -3\lambda^2 & -\lambda^3\end{bmatrix}, \quad \mathbf{C}_\phi = \mathbf{e}_2^T,$$

and $D_\psi = 1$, see Rugh (1996).

In this work, no probability distribution is considered for the unknown parameter. This is to keep the proposed methods as general as possible, without particularizing the algorithms for a given target or experiment.

3.1. Angular speed adaptive law

In order to derive an adaptive law that provides estimates for α, consider the estimate $\hat{\boldsymbol{\psi}}(t)$ of $\boldsymbol{\psi}(t)$, with expression

$$\hat{\boldsymbol{\psi}}(t) = \hat{\alpha}(t)\boldsymbol{\phi}(t), \quad (9)$$

obtained resorting to an estimate $\hat{\alpha}(t)$ of the unknown parameter α, at time t. Since the value of α is unknown, the error $\tilde{\alpha}(t) = \alpha - \hat{\alpha}(t)$ in its estimation is not available. However, the estimation error $\boldsymbol{\varepsilon}(t) = (\boldsymbol{\psi}(t) - \hat{\boldsymbol{\psi}}(t))/m_\phi^2(t)$ can be computed using the position measurements and reflects the difference between α and $\hat{\alpha}(t)$:

$$\boldsymbol{\varepsilon}(t) = \frac{\tilde{\alpha}(t)\boldsymbol{\phi}(t)}{m_\phi^2(t)} + \frac{\boldsymbol{\xi}(t)}{m_\phi^2(t)} + \frac{\mathbf{q}(t)}{m_\phi^2(t)}. \quad (10)$$

The term $m_\phi^2(t)$ is a normalization signal that guarantees that the entries of $\boldsymbol{\phi}(t)/m_\phi(t)$ are bounded, and is sometimes used in the context of parameter identification, see examples in Ioannou and Fidan (2006) and Sastry and Bodson (1989). This property is useful in the analysis of the convergence of the estimates $\hat{\alpha}(t)$ to the real parameter α, when $\boldsymbol{\phi}(t)$ is not guaranteed to be bounded. In this work, the signal $m_\phi^2(t) = 1 + \mu\, \boldsymbol{\phi}^T(t)\boldsymbol{\phi}(t)$, $\mu > 0$, is used.

Estimates $\hat{\alpha}(t)$ of the unknown parameter α can be obtained by minimizing the cost function

$$J(\hat{\alpha}(t)) = \frac{\|\boldsymbol{\varepsilon}(t)\|^2\, m_\phi^2(t)}{2} = \frac{\|\boldsymbol{\psi}(t) - \hat{\alpha}(t)\boldsymbol{\phi}(t)\|^2}{2\, m_\phi^2(t)}, \tag{11}$$

which depends quadratically on the estimation error $\boldsymbol{\varepsilon}(t)$. The minimization of this function with respect to $\hat{\alpha}(t)$ is performed using the normalized (the normalization signal $m_\phi^2(t)$ is considered) gradient method $\dot{\hat{\alpha}}(t) = -\gamma \nabla J(\hat{\alpha}(t))$, where $\gamma > 0$ is a constant usually referred to as the adaptation gain and $\nabla J(\hat{\alpha}(t))$ is the gradient of $J(\hat{\alpha}(t))$ with respect to $\hat{\alpha}(t)$. The following adaptive law results:

$$\dot{\hat{\alpha}}(t) = \gamma \boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t), \qquad \hat{\alpha}(0) = \hat{\alpha}_0, \tag{12}$$

where $\hat{\alpha}_0$ denotes the initial estimate of α.

3.2. Angular speed convergence—deterministic framework

For convergence study purposes, let us start by considering a deterministic framework, i.e., consider that the process and observation noises introduced in Section 2 are not present (the influence of these noises is addressed in Section 3.3). In this case, the proposed adaptive law ensures that the estimation error $\boldsymbol{\varepsilon}(t)$ converges to zero, but does not imply that $\hat{\alpha}(t)$ converges to α. In order to guarantee this property, some conditions must be imposed on $\boldsymbol{\phi}(t)$. These conditions are derived in Theorem 4, whose proof depends on Definition 2 and Lemma 3.

Definition 2 (*Rugh, 1996*)**.** The linear state equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, $\mathbf{x}(t_0) = \mathbf{x}_0$, is called uniformly exponentially stable (UES) if there exist finite positive constants γ_u, λ_u such that for any initial time instant t_0 and any initial condition $\mathbf{x}_0$, the corresponding solution satisfies $\|\mathbf{x}(t)\| \le \gamma_u e^{-\lambda_u(t-t_0)}\|\mathbf{x}_0\|$, $t \ge t_0$.

Lemma 3 (*Zhang, Ioannou, & Chien, 1994*)**.** *Consider the system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{u}(t)$. If $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ is UES and $\|\mathbf{u}(t)\|$ is exponentially decaying, then $\|\mathbf{x}(t)\|$ converges to zero exponentially fast.*

In Ioannou and Fidan (2006), stability and convergence guarantees for identification algorithms where several parameters are considered, and $\boldsymbol{\psi}(t)$ is a scalar, can be found. In this work, these properties are generalized for cases where there is only one unknown parameter, but $\boldsymbol{\psi}(t)$ is a vector, see Theorem 4. The reasoning used to prove this generalization is completely different from that in Ioannou and Fidan (2006).

Theorem 4. *In the deterministic case, the identifier structure described previously, combined with the normalized gradient algorithm (12), guarantees that $\hat{\alpha}(t)$ converges to the nominal parameter α exponentially fast, if $\|\overline{\boldsymbol{\phi}}(t)\| = \|\frac{\boldsymbol{\phi}(t)}{m_\phi(t)}\|$ is persistently exciting.*

Proof. Let the parameter estimation error be given by $\tilde{\alpha}(t) = \alpha - \hat{\alpha}(t)$. Since α is constant, when the process and observation noises are not considered we have

$$\dot{\tilde{\alpha}}(t) = -\gamma \boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t) = -\gamma \frac{\|\boldsymbol{\phi}(t)\|^2}{m_\phi^2(t)}\tilde{\alpha}(t) - \gamma \frac{\mathbf{q}^T(t)\boldsymbol{\phi}(t)}{m_\phi^2(t)}, \tag{13}$$

with $\tilde{\alpha}(t_0) = \tilde{\alpha}_0$, where t_0 denotes the initial time instant and $\tilde{\alpha}_0$ the initial parameter estimation error. Moreover, if $\|\overline{\boldsymbol{\phi}}(t)\|$ is persistently exciting (PE), there exist $\theta_0 > 0$ and $T_0 > 0$ such that $\int_t^{t+T_0} \|\overline{\boldsymbol{\phi}}(\tau)\|^2 \mathrm{d}\tau \ge \theta_0 T_0$, $\forall t \ge 0$, see the definition of persistence of excitation in Ioannou and Fidan (2006).

To prove this theorem, let us start by proving that the homogeneous part of (13) is UES if $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE. Consider the continuously differentiable function

$$V(t, \tilde{\alpha}(t)) = \int_t^{t+T_0} \tilde{\alpha}^2(\tau)\mathrm{d}\tau, \quad \forall\, t \ge 0. \tag{14}$$

Since the solution of the homogeneous equation is given by

$$\tilde{\alpha}(\tau) = \tilde{\alpha}(t)e^{-\gamma \int_t^\tau \|\overline{\boldsymbol{\phi}}(\sigma)\|^2 \mathrm{d}\sigma}, \quad \tau \ge t, \tag{15}$$

the function in (14) can be written in the form

$$V(t, \tilde{\alpha}(t)) = \int_t^{t+T_0} \tilde{\alpha}^2(t)e^{-2\gamma \int_t^\tau \|\overline{\boldsymbol{\phi}}(\sigma)\|^2 \mathrm{d}\sigma}\mathrm{d}\tau, \quad \forall\, t \ge 0. \tag{16}$$

Moreover, $\|\overline{\boldsymbol{\phi}}(t)\|$ is bounded, i.e., $\beta = \sup_{\tau \ge 0}\|\overline{\boldsymbol{\phi}}(\tau)\|$ is a finite constant, thus $0 \le \int_t^\tau \|\overline{\boldsymbol{\phi}}(\sigma)\|^2 \mathrm{d}\sigma \le \beta^2(\tau - t)$, $\tau \ge t$. Using these inequalities and the expression in (16), it is possible to conclude that

$$\frac{1 - e^{-2\gamma\beta^2 T_0}}{2\gamma\beta^2}\tilde{\alpha}^2(t) \le V(t, \tilde{\alpha}(t)) \le T_0\tilde{\alpha}^2(t), \quad \forall\, t \ge 0.$$

From (14), the derivative of $V(t, \tilde{\alpha}(t))$ with respect to time is $\dot{V}(t, \tilde{\alpha}(t)) = \tilde{\alpha}^2(t + T_0) - \tilde{\alpha}^2(t)$. If $\|\overline{\boldsymbol{\phi}}(t)\|$ is assumed to be PE and (15) is used with $\tau = t + T_0$, it is straightforward to show that there exist $\theta_0 > 0$ and $T_0 > 0$ such that

$$\dot{V}(t, \tilde{\alpha}(t)) \le -\left(1 - e^{-2\theta_0 T_0}\right)\tilde{\alpha}^2(t), \quad \forall\, t \ge 0.$$

If such $\theta_0 > 0$ and $T_0 > 0$ are considered, then there exist positive constants $k_1 = (1 - e^{-2\gamma\beta^2 T_0})/(2\gamma\beta^2)$, $k_2 = T_0$, and $k_3 = 1 - e^{-2\theta_0 T_0}$, such that $k_1\tilde{\alpha}^2(t) \le V(t, \tilde{\alpha}(t)) \le k_2\tilde{\alpha}^2(t)$ and $\dot{V}(t, \tilde{\alpha}(t)) \le -k_3\tilde{\alpha}^2(t)$, for all $t \ge 0$. Therefore, if $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE, the homogeneous equation associated with the time-varying system in (13) is UES, see Theorem 4.10 in Khalil (2002).

Since $\|\mathbf{q}(t)\|$ is exponentially decaying, $\|\boldsymbol{\phi}(t)/m_\phi(t)\|$ is bounded, and $m_\phi(t) \ge 1$, the norm of the term $-\gamma \mathbf{q}^T(t)\boldsymbol{\phi}(t)/m_\phi^2(t)$, in (13), converges exponentially fast to zero. Therefore, according to Lemma 3, $|\tilde{\alpha}(t)|$ also converges to zero exponentially fast. □

When $\mathbf{v}(0)$ and $\mathbf{a}(0)$ are not both null, the signal $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE, which is easily understood by analyzing the trajectories associated with the model in (1). Therefore, according to Theorem 4, in a deterministic framework $\hat{\alpha}(t)$ is guaranteed to converge to α exponentially fast unless $\mathbf{v}(0) = \mathbf{a}(0) = \mathbf{0}$, i.e., unless the target does not move, which was expected since trying to identify the target angular speed ω does not make sense in such a situation.

3.3. Angular speed convergence—stochastic framework

When a stochastic framework is considered, i.e., when the process and observation noises, $\mathbf{d}(t)$ and $\mathbf{n}(t)$ respectively, introduced in Section 2 are taken into account, the error $\tilde{\alpha}(t)$ associated with the estimation of the target angular speed cannot be expected to converge exactly to zero. However, it is possible to prove that this error converges to the vicinity of zero if some conditions are imposed on $\mathbf{d}(t)$, $\mathbf{n}(t)$, and $\|\overline{\boldsymbol{\phi}}(t)\|$. These conditions are stated in Theorem 5.

Theorem 5. *If the process and observation noises, $\mathbf{d}(t)$ and $\mathbf{n}(t)$ respectively, are bounded and $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE, then the normalized* *gradient algorithm (12) guarantees that there exist finite positive constants γ_1, λ_1, and $\beta_{\tilde{\alpha}}$ such that*

$$|\tilde{\alpha}(t)| \leq \gamma_1 e^{-\lambda_1(t-t_0)} + \beta_{\tilde{\alpha}}, \quad \forall\, t \geq t_0. \tag{17}$$

Proof. In the stochastic setup, we have

$$\dot{\tilde{\alpha}}(t) = -\gamma \frac{\|\boldsymbol{\phi}(t)\|^2}{m_\phi^2(t)}\tilde{\alpha}(t) - \gamma \frac{\mathbf{q}^T(t)\boldsymbol{\phi}(t)}{m_\phi^2(t)} - \gamma \frac{\boldsymbol{\xi}^T(t)\boldsymbol{\phi}(t)}{m_\phi^2(t)}, \tag{18}$$

with $\tilde{\alpha}(t_0) = \tilde{\alpha}_0$, where t_0 denotes the initial time instant and $\tilde{\alpha}_0$ the initial parameter estimation error. The term $\boldsymbol{\xi}(t)$ results from filtering the process and observation noises with causal linear time-invariant filters that are UES, which implies that they are also uniformly bounded-input, bounded-output stable, see Rugh (1996). Therefore, if the noises are bounded, i.e., if $\beta_d = \|\mathbf{d}\|_\infty$ and $\beta_n = \|\mathbf{n}\|_\infty$ are finite, then there exist finite positive constants η_d and η_n such that the forced responses of the filters guarantee that $\|\boldsymbol{\xi}(t)\| \leq \eta_d\beta_d + \eta_n\beta_n$, for any t_0.

As argued in the proof of Theorem 4, the homogeneous part of (18) is UES if $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE. Thus, in this case there exist finite positive constants $\gamma_{\tilde{\alpha}}$ and $\lambda_{\tilde{\alpha}}$ such that the state transition matrix associated with this equation verifies $\|\Phi_{\tilde{\alpha}}(t,\tau)\| \leq \gamma_{\tilde{\alpha}} e^{-\lambda_{\tilde{\alpha}}(t-\tau)}$ for all t, τ such that $t \geq \tau$, see Rugh (1996). Since $\|\mathbf{q}(t)\|$ is exponentially decaying, there exist $\gamma_q > 0$ and $\lambda_q > 0$ such that $\|\mathbf{q}(t)\| \leq \gamma_q e^{-\lambda_q(t-t_0)}\|\mathbf{q}(t_0)\|$. Moreover, $\|\overline{\boldsymbol{\phi}}(\tau)\| \leq \beta$ and $m_\phi(\tau) \geq 1$, therefore the solution of (18) verifies

$$\begin{aligned}|\tilde{\alpha}(t)| \leq{}& \gamma_{\tilde{\alpha}} e^{-\lambda_{\tilde{\alpha}}(t-t_0)}|\tilde{\alpha}(t_0)| + \gamma\gamma_{\tilde{\alpha}}\beta \\ &\times \int_{t_0}^{t} e^{-\lambda_{\tilde{\alpha}}(t-\tau)}\left(\gamma_q e^{-\lambda_q(\tau-t_0)}\|\mathbf{q}(t_0)\| + \eta_d\beta_d + \eta_n\beta_n\right)\mathrm{d}\tau.\end{aligned}$$

By computing the integral, it is easy to show that the previous expression can be written as a sum of several terms that converge exponentially fast to zero and a term that is an upper bound for the parameter estimation error after the initial transient. Thus, there exist finite positive constants γ_1, λ_1, and $\beta_{\tilde{\alpha}}$ such that (17) holds. □

According to Theorem 5, when the process and observation noises are bounded, $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE, and the initial transient vanishes, the norm of the error in the estimation of the unknown parameter verifies $|\tilde{\alpha}(t)| \leq \beta_{\tilde{\alpha}}$, which guarantees that the angular speed estimates converge to the vicinity of the target angular speed.

3.4. Gradient projection method

The parameter $\alpha = -\omega^2$ to be estimated cannot be positive. Therefore, instead of minimizing (11) for all $\hat{\alpha}(t) \in \mathbb{R}$, we want to constrain the estimation to be within the convex subset $\mathcal{S} \triangleq \{\hat{\alpha}(t) \in \mathbb{R} : \alpha_1 \leq \hat{\alpha}(t) \leq \alpha_2\}$ of $\mathbb{R}$, where $\alpha_1 \leq \alpha_2 \leq 0$. This is done using the gradient projection method (GPM), see Ioannou and Fidan (2006). According to this method, instead of (12), the new adaptive law

$$\dot{\hat{\alpha}}(t) = \begin{cases} \gamma\boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t), & \text{if } \alpha_1 < \hat{\alpha}(t) < \alpha_2 \\ & \text{or if } \hat{\alpha}(t) = \alpha_1 \text{ and } \boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t) \geq 0, \\ & \text{or if } \hat{\alpha}(t) = \alpha_2 \text{ and } \boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t) \leq 0, \\ 0, & \text{otherwise,}\end{cases} \tag{19}$$

is used. This law retains the properties derived in the absence of projection, while guaranteeing that $\hat{\alpha}(t) \in [\alpha_1, \alpha_2]$, for all t, as long as $\hat{\alpha}_0 \in \mathcal{S}$ and $\alpha \in \mathcal{S}$. The proof of this statement is omitted here due to space constraints.

The angular speed estimation strategy described in this section is summarized below and illustrated in Fig. 2.

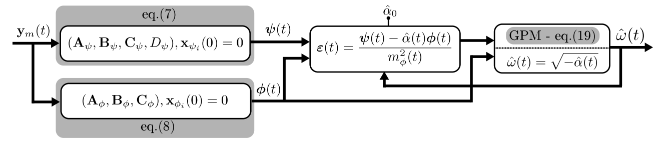


Fig. 2. Angular speed estimation strategy.

Algorithm 1. Estimation of the target angular speed:

(1) obtain $\boldsymbol{\psi}(t)$ and $\boldsymbol{\phi}(t)$ by filtering the measurements $\mathbf{y}_m(t)$ with the filters presented in (7) and (8);
(2) compute the estimation error $\boldsymbol{\varepsilon}(t) = (\boldsymbol{\psi}(t) - \hat{\alpha}(t)\boldsymbol{\phi}(t))/m_\phi^2(t)$, $\hat{\alpha}(0) = \hat{\alpha}_0$;
(3) compute $\hat{\alpha}(t)$ using the gradient projection method presented in (19);
(4) obtain the target angular speed estimates $\hat{\omega}(t) = \sqrt{-\hat{\alpha}(t)}$.

4. $\mathcal{H}_2$ adaptive filter

In this section, a continuous-time $\mathcal{H}_2$ adaptive filter that estimates the state of a target moving according to (1), resorting only to measurements of the target position and estimates of its angular speed, is proposed. The stability and performance of the filter are studied.

If, instead of the target angular speed ω, $\alpha = -\omega^2$ is used, the model in (1) for the target can be written as an affine parameter dependent system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\alpha)\mathbf{x}(t) + \mathbf{B}\mathbf{d}(t), \tag{20}$$

where $\mathbf{A}(\alpha) = \operatorname{diag}\left[\overline{\mathbf{A}}(\alpha), \overline{\mathbf{A}}(\alpha), \overline{\mathbf{A}}(\alpha)\right] \in \mathbb{R}^{9\times 9}$ and

$$\overline{\mathbf{A}}(\alpha) = \underbrace{\begin{bmatrix}\mathbf{0}_{3\times 1} & \mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}}_{\overline{\mathbf{A}}_0} + \alpha \underbrace{\begin{bmatrix}\mathbf{0}_{3\times 1} & \mathbf{e}_3 & \mathbf{0}_{3\times 1}\end{bmatrix}}_{\overline{\mathbf{A}}_1}.$$

Moreover, consider that the target angular speed is bounded, i.e., that there exist $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$ such that $\alpha \in [\alpha_1, \alpha_2]$. If estimates $\hat{\alpha}(t)$, obtained according to (19), and measurements $\mathbf{y}_m(t)$, as defined in (2), of the target position are used, the following adaptive filter for the state $\mathbf{x}(t)$, with structure motivated by a linear filter, results:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(\hat{\alpha}(t))\hat{\mathbf{x}}(t) + \mathbf{L}(\mathbf{y}_m(t) - \hat{\mathbf{y}}(t)), \qquad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0, \tag{21}$$

where $\hat{\mathbf{x}}(t)$ is an estimate of $\mathbf{x}(t)$, $\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$, and $\hat{\mathbf{x}}_0$ denotes the initial conditions of the filter. The vector $\mathbf{L} \in \mathbb{R}^{9\times 3}$ is the gain of the filter.

The dynamics of the state estimation error $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ associated with the filter can be written in the form

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}(t) ={}& (\mathbf{A}(\alpha) - \mathbf{L}\mathbf{C} - \tilde{\alpha}(t)\mathbf{A}_1)\tilde{\mathbf{x}}(t) \\ &+ \tilde{\alpha}(t)\mathbf{A}_1\mathbf{x}(t) + \mathbf{B}\mathbf{d}(t) - \mathbf{L}\mathbf{D}\mathbf{n}(t),\end{aligned} \tag{22}$$

where $\mathbf{A}_1 = \operatorname{diag}\left[\overline{\mathbf{A}}_1, \overline{\mathbf{A}}_1, \overline{\mathbf{A}}_1\right] \in \mathbb{R}^{9\times 9}$.

4.1. Filter stability—deterministic framework

In order to study the stability of the proposed filter, let us start by considering a deterministic framework, i.e., consider that the process and observation noises introduced in Section 2 are not present. In this case, conditions on the gain $\mathbf{L}$ that ensure that the error of the filter in (21) converges exponentially fast to zero can be imposed. These conditions are provided in Theorem 6. The influence of the noise on the stability of the filter is addressed in the next section.

Theorem 6. *When a deterministic framework is considered and $\|\overline{\boldsymbol{\phi}}(t)\|$ is persistently exciting, the error of the filter in (21), with $\hat{\alpha}(t)$ computed resorting to (19) and gain $\mathbf{L}$ chosen to guarantee that both*

$\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_1) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ and $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_2) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ are UES for given values of $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$, converges to zero exponentially fast.

Proof. In the deterministic case, the dynamics of the error of the filter in (21), presented in (22), has the form

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A}(\hat{\alpha}(t)) - \mathbf{LC})\tilde{\mathbf{x}}(t) + \tilde{\alpha}(t)\mathbf{A}_1\mathbf{x}(t).$$

If $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE, there exist finite positive constants c_p, λ_p such that $\|\tilde{\alpha}(t)\mathbf{A}_1\mathbf{x}(t)\| \leq c_p e^{-\lambda_p(t-t_0)}, \forall t \geq t_0$, given that $\|\tilde{\alpha}(t)\mathbf{A}_1\mathbf{x}(t)\| \leq \|\tilde{\alpha}(t)\|.\|\mathbf{A}_1\|.\|\mathbf{x}(t)\|$ and that $\|\mathbf{x}(t)\|$ is either bounded or dominated by a polynomial of degree two, due to the trajectories considered by the target model in (1). Since, according to Theorem 4 and Section 3.4, $\tilde{\alpha}(t)$ converges to zero exponentially fast when $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE and no noise is considered, and $\|\mathbf{A}_1\|$ is a finite positive constant, the exponential dominates the other terms in the expression. Therefore, according to Lemma 3, if $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE and $\mathbf{L}$ is chosen in such a way that $\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A}(\hat{\alpha}(t)) - \mathbf{LC})\tilde{\mathbf{x}}(t)$ is UES, then the error of the filter is guaranteed to converge to zero exponentially fast.

Consider that there exist a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{9\times 9}$ and a matrix $\mathbf{W} \in \mathbb{R}^{9\times 3}$ such that the linear matrix inequalities (LMIs)

$$\begin{cases} \mathbf{A}^T(\alpha_1)\mathbf{P} + \mathbf{PA}(\alpha_1) - \mathbf{C}^T\mathbf{W}^T - \mathbf{WC} < -\mathbf{Q} \\ \mathbf{A}^T(\alpha_2)\mathbf{P} + \mathbf{PA}(\alpha_2) - \mathbf{C}^T\mathbf{W}^T - \mathbf{WC} < -\mathbf{Q} \end{cases} \tag{23}$$

are verified for a given symmetric positive semidefinite matrix $\mathbf{Q} \in \mathbb{R}^{9\times 9}$, i.e., consider that there exists a gain $\mathbf{L} = \mathbf{P}^{-1}\mathbf{W}$ such that $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_1) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ and $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_2) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ are UES. In this case, according to Theorem 3.2 in Amato (2006), $\mathbf{L} = \mathbf{P}^{-1}\mathbf{W}$ makes $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ quadratically stable for $\alpha \in [\alpha_1, \alpha_2]$, see details about quadratic stability in Amato (2006). Moreover, from Theorems 2.6 and 3.1 in Amato (2006), if $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ is quadratically stable for $\alpha \in [\alpha_1, \alpha_2]$, then $\dot{\tilde{\mathbf{x}}}(t) = \left[\mathbf{A}(\hat{\alpha}(t)) - \mathbf{LC}\right]\tilde{\mathbf{x}}(t)$ is uniformly asymptotically stable, since $\hat{\alpha}(t) \in [\alpha_1, \alpha_2]$. For linear time-varying systems, uniform asymptotic stability is equivalent to uniform exponential stability, see Rugh (1996); therefore, under the stated assumptions, $\dot{\tilde{\mathbf{x}}}(t) = \left[\mathbf{A}(\hat{\alpha}(t)) - \mathbf{LC}\right]\tilde{\mathbf{x}}(t)$ is UES, which, according to Lemma 3, guarantees that the error of the filter converges to zero exponentially fast. □

In Theorem 6, conditions to be imposed on $\mathbf{L}$ that ensure the convergence of the filter error exponentially fast to zero, in the deterministic case, were presented. It is possible to show that there always exists a gain $\mathbf{L}$ verifying these conditions if $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE. The proof of this statement is omitted here due to space constraints.

4.2. Filter stability—stochastic framework

When both the process and the observation noise are considered, it is possible to prove that the filter estimation error converges to the vicinity of zero and that, after the initial transient, its maximum norm has an upper bound if some conditions are imposed on $\mathbf{d}(t)$, $\mathbf{n}(t)$, $\|\overline{\boldsymbol{\phi}}(t)\|$, and on the target maximum linear velocity, see Theorem 7.

Theorem 7. *Consider the filter in (21), with $\hat{\alpha}(t)$ computed resorting to (19) and gain $\mathbf{L}$ chosen to guarantee that both $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_1) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ and $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_2) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ are UES for given values of $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$. Moreover, assume that $\|\overline{\boldsymbol{\phi}}(t)\|$ is persistently exciting and that the process noise $\mathbf{d}(t)$, the observation noise $\mathbf{n}(t)$, and the target linear velocity $\mathbf{v}(t)$ are bounded. In this case, there exists a finite positive constant $\beta_{\tilde{\mathbf{x}}}$ such that, after an initial transient, the filter estimation error verifies*

$$\|\tilde{\mathbf{x}}(t)\| \leq \beta_{\tilde{\mathbf{x}}}, \quad \forall\, t \geq t_0. \tag{24}$$

Proof. Consider that the process and observation noises are bounded, i.e., that $\beta_d = \|\mathbf{d}\|_\infty$ and $\beta_n = \|\mathbf{n}\|_\infty$ are finite, and that $\|\overline{\boldsymbol{\phi}}(t)\|$ is PE. If the target angular speed estimates are obtained using the gradient adaptive law in (19), which guarantees that $|\tilde{\alpha}(t)| \leq |\alpha_2 - \alpha_1|$, for all $t \geq t_0$, then, according to Theorem 5, there exist finite positive constants γ_1, λ_1, and $\beta_{\tilde{\alpha}}$ such that

$$|\tilde{\alpha}(t)| \leq \min\left[|\alpha_2 - \alpha_1|, \gamma_1 e^{-\lambda_1(t-t_0)} + \beta_{\tilde{\alpha}}\right], \quad \forall\, t \geq t_0. \tag{25}$$

Moreover, if the gain $\mathbf{L}$ is chosen so that both $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_1) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ and $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_2) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ are UES for any given values of $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$, then, as argued in the proof of Theorem 6, $\dot{\tilde{\mathbf{x}}}(t) = \left[\mathbf{A}(\hat{\alpha}(t)) - \mathbf{LC}\right]\tilde{\mathbf{x}}(t)$ is also UES. Therefore, there exist finite positive constants $\gamma_{\tilde{x}}$ and $\lambda_{\tilde{x}}$ such that the state transition matrix associated with this linear state equation verifies $\|\Phi_{\tilde{x}}(t,\tau)\| \leq \gamma_{\tilde{x}} e^{-\lambda_{\tilde{x}}(t-\tau)}$ for all t, τ such that $t \geq \tau$.

When noise is considered, the target state estimation error evolves according to (22). Due to the structure of $\mathbf{A}_1$, this expression can be written as

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A}(\hat{\alpha}(t)) - \mathbf{LC})\tilde{\mathbf{x}}(t) + \mathbf{B}\left(\mathbf{d}(t) + \tilde{\alpha}(t)\mathbf{v}(t)\right) - \mathbf{LDn}(t).$$

If the target linear velocity $\mathbf{v}(t)$ is bounded, i.e., if $\beta_v = \|\mathbf{v}\|_\infty$ is finite, then, using the variation of constants method, see Rugh (1996), yields

$$\begin{aligned}\|\tilde{\mathbf{x}}(t)\| \leq\; & \|\Phi_{\tilde{x}}(t,t_0)\|.\|\tilde{\mathbf{x}}(t_0)\| + \beta_v\|\mathbf{B}\| \\ & \times \int_{t_0}^{t} \|\Phi_{\tilde{x}}(t,\tau)\| \min\left[|\alpha_2 - \alpha_1|, \gamma_1 e^{-\lambda_1(\tau-t_0)} + \beta_{\tilde{\alpha}}\right] \mathrm{d}\tau \\ & + \left(\|\mathbf{B}\|\beta_d + \|\mathbf{L}\|.\|\mathbf{D}\|\beta_n\right)\int_{t_0}^{t} \|\Phi_{\tilde{x}}(t,\tau)\|\mathrm{d}\tau\end{aligned}$$

for all $t \geq t_0$. By computing the integrals it is straightforward to show that, under the stated assumptions, there exists a finite positive constant $\beta_{\tilde{\mathbf{x}}}$, given by

$$\beta_{\tilde{\mathbf{x}}} = \frac{\gamma_{\tilde{x}}}{\lambda_{\tilde{x}}}\left(\|\mathbf{B}\|\beta_v \min\left[|\alpha_2 - \alpha_1|, \beta_{\tilde{\alpha}}\right] + \|\mathbf{B}\|\beta_d + \|\mathbf{L}\|.\|\mathbf{D}\|\beta_n\right), \tag{26}$$

such that, after an initial transient, the filter estimation error verifies (24). □

In Theorem 7, some assumptions necessary to ensure the existence of a finite positive constant $\beta_{\tilde{\mathbf{x}}}$ such that, after an initial transient, (24) is verified, were presented. It is possible to prove that, under these assumptions, a gain $\mathbf{L}$ guaranteeing the existence of such a constant can always be found. The proof of this statement is omitted here due to space constraints.

4.3. Design of the gain of the $\mathcal{H}_2$ filter

In this section, a LMI-based strategy for the design of the gain of the $\mathcal{H}_2$ adaptive filter is proposed, see Boyd, El Ghaoui, Feron, and Balakrishnan (1994) and Oliveira (2002) for details about the design of $\mathcal{H}_2$ filters using LMIs.

When noise is considered, the dynamics of the error $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ associated with the estimates provided by the filter in (21) can be written in the form

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A}(\alpha) - \mathbf{LC})\,\tilde{\mathbf{x}}(t) + \mathbf{B}[\mathbf{d}(t) + \tilde{\alpha}(t)\hat{\mathbf{v}}(t)] - \mathbf{LDn}(t),$$

where $\hat{\mathbf{v}}(t)$ corresponds to the target linear velocity estimates, i.e., $\hat{\mathbf{v}}(t) = [\hat{\mathbf{x}}_2(t)\;\; \hat{\mathbf{x}}_5(t)\;\; \hat{\mathbf{x}}_8(t)]^T$, where $\hat{\mathbf{x}}_k(t)$ denotes the k-th entry of $\hat{\mathbf{x}}(t)$. The value of $\tilde{\alpha}(t)\hat{\mathbf{v}}(t) \in \mathbb{R}^3$ is unknown, as it depends on the error $\tilde{\alpha}(t)$ in the estimation of α, whose impact on the filter performance we want to minimize. This term affects the estimation error $\tilde{\mathbf{x}}(t)$ in the same way $\mathbf{d}(t)$ does (through $\mathbf{B}$), i.e., they both corrupt directly the error associated with the target acceleration estimates. Thus, for design purposes, a single disturbance vector $\boldsymbol{\delta}(t) = \mathbf{d}(t) + \tilde{\alpha}(t)\hat{\mathbf{v}}(t)$, comprising the contribution of both terms, is considered. By concatenating this disturbance with the noise that corrupts the measurements of the target position into a single vector, the generalized disturbance vector $\mathbf{w}(t) = [\boldsymbol{\delta}^T(t) \quad \mathbf{n}^T(t)]^T \in \mathbb{R}^6$ results. Rewriting the dynamics of the error as a function of this disturbance yields

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A}(\alpha) - \mathbf{LC})\,\tilde{\mathbf{x}}(t) + \left(\mathbf{B}_w - \mathbf{LD}_{yw}\right)\mathbf{w}(t), \tag{27}$$

where $\mathbf{D}_{yw} = [\mathbf{0}_{3\times 3} \quad \mathbf{D}]$ and $\mathbf{B}_w = [\mathbf{B} \quad \mathbf{0}_{9\times 3}]$.

For performance purposes, only the target position estimation error $\mathbf{e}(t) = \mathbf{C}\tilde{\mathbf{x}}(t) \in \mathbb{R}^3$ is considered. The gain $\mathbf{L}$ of the filter is found by minimizing the maximum of the $\mathcal{H}_2$ norm of the systems obtained from $\mathbf{w}(t)$ to $\mathbf{e}(t)$ when α, in (27), is replaced by α_1 and α_2; see Oliveira (2002) for details about the design of $\mathcal{H}_2$ filters using LMIs. The $\mathcal{H}_2$ norm obtained with this strategy is an upper bound for the $\mathcal{H}_2$ norm of the real system since $\alpha \in [\alpha_1, \alpha_2]$, see Becker and Packard (1994).

It is straightforward to conclude that the gain $\mathbf{L}$ found using this method guarantees that both $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_1) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ and $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_2) - \mathbf{LC}]\,\tilde{\mathbf{x}}(t)$ are UES for given values of $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$. Therefore, this gain ensures that the error of the filter in (21), with $\hat{\alpha}(t)$ obtained as in (19), verifies (24) if the assumptions stated in Theorem 7 hold.

5. Simulation results

In this section, continuous-time simulation results illustrating the performance of the proposed parameter identification procedure and adaptive filter are presented.

For comparison purposes, results obtained with an Extended Kalman Filter, see Gelb (2001), are also provided. This filter was designed for the nonlinear system that results from augmenting the state $\mathbf{x}(t) \in \mathbb{R}^9$, of (1), with the target angular speed ω. The new state variable was modeled as a Wiener process, see Rong Li and Jilkov (2003). The model considered for the measurements was the one introduced in (2).

In this section, measurements of the target position in spherical coordinates obtained with a single PTZ (acronym for pan, tilt, and zoom) camera are used. These measurements can be obtained using the strategies proposed in Gaspar and Oliveira (2011), for instance, and are transformed to Cartesian coordinates using a nonlinear transformation, see examples in Bar-Shalom et al. (2001), which leads to the model in (2) for the position measurements.

In the simulations presented in this section, the intrinsic parameters are the ones from a 215 PTZ camera from AXIS, and the target angular speed is considered to belong to the interval [0, 0.5] rad/s. The parameters $\mu = \gamma = 10^{-10}$ and the Hurwitz polynomial $\Lambda(s) = (s + \lambda)^3$, $\lambda = 0.2$, were used in the design of the parameter identifier. In the design of the $\mathcal{H}_2$ filter, the intensities of the process and observation noises were tuned by replacing $\mathbf{b} = \mathbf{e}_3$ and $\mathbf{D} = \mathbf{I}_3$ by $\mathbf{b} = 10\,\mathbf{e}_3$ and $\mathbf{D} = 100\,\mathbf{I}_3$. This strategy led to a gain $\mathbf{L}$ with non-null entries of the form $\mathbf{L}_{1,1} = \mathbf{L}_{4,2} = \mathbf{L}_{7,3} = 1.33$, $\mathbf{L}_{2,1} = \mathbf{L}_{5,2} = \mathbf{L}_{8,3} = 0.77$, and $\mathbf{L}_{3,1} = \mathbf{L}_{6,2} = \mathbf{L}_{9,3} = 0.13$.

The measurements of the center of the target in the images and the measurements of its distance with respect to the camera are corrupted by uniformly distributed noise, with values in the intervals $[-10, 10]$ pixel and $[-1, 1]$ m, respectively. For the design of the EKF, the process noise that affects the target acceleration and the measurement noise that corrupts the measurements of the target position are considered to have power spectral density matrices $10^2\mathbf{I}_3$ mm^{2}Hz^5 and $100^2\mathbf{I}_3$ mm^{2}Hz^{-1}, respectively. The power spectral density considered for the noise that affects the target angular speed is 10^{-6} rad^2 Hz^3.

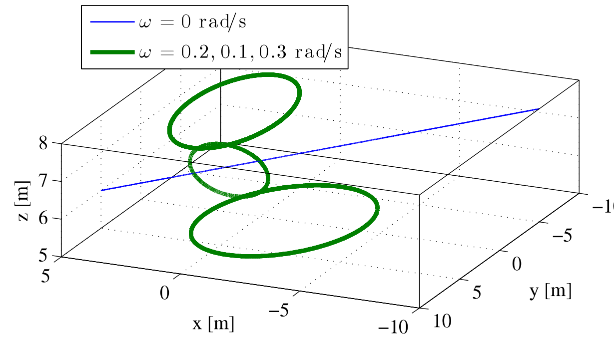


Fig. 3. Trajectories described by the target.

In the following, two experiments are reported. The first illustrates the performance of the proposed estimators when the target moves along a straight line ($\omega = 0$ rad/s), and the second illustrates their performance when the target angular speed varies over time. The trajectories described by the target in the two situations are shown in Fig. 3.

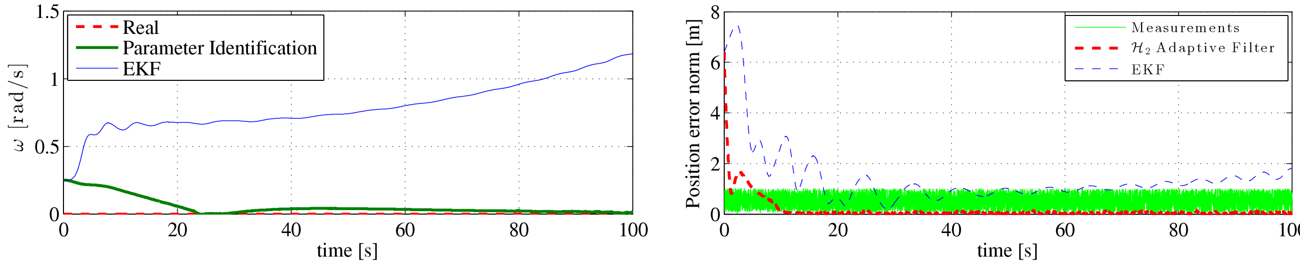
In Fig. 4(a), the target angular speed estimates provided by the identification procedure proposed in Section 3 and by the EKF, for the first experiment, are depicted. As can be seen, the estimates provided by the parameter identifier converge to the vicinity of the target real angular speed $\omega = 0$ rad/s, whereas the EKF diverges.

The results obtained with the $\mathcal{H}_2$ adaptive filter in the first experiment are depicted in Fig. 4(b). These results are compared with the estimates provided by the EKF and with the measurements of the target position computed resorting to the aforementioned nonlinear transformation. As expected from the performance of the EKF in the estimation of the target angular speed, its estimates for the target position diverge. Even though the EKF diverges, the error in the position estimates provided by the $\mathcal{H}_2$ adaptive filter, and the error in the estimation of the target angular speed, converge to the vicinity of zero. These results are in accordance with the stability guarantees derived in Theorems 5 and 7. Moreover, the steady-state performance of the adaptive filter is significantly better than that obtained with the measurements of the target position.

The results obtained in the second experiment, which considers a trajectory for the target with three different angular speed values, are presented in Fig. 5. As can be seen, the angular speed identification strategy proposed in Section 3 is robust to variations in the parameter to be estimated, since the angular speed estimates converge to the real angular speed even after abrupt changes in its value. The degradation in the performance of the position estimates obtained with the $\mathcal{H}_2$ adaptive filter, around time instants 100 and 200 s, is due to the transients observed in the estimates provided by the parameter identifier when the target changes its angular speed.

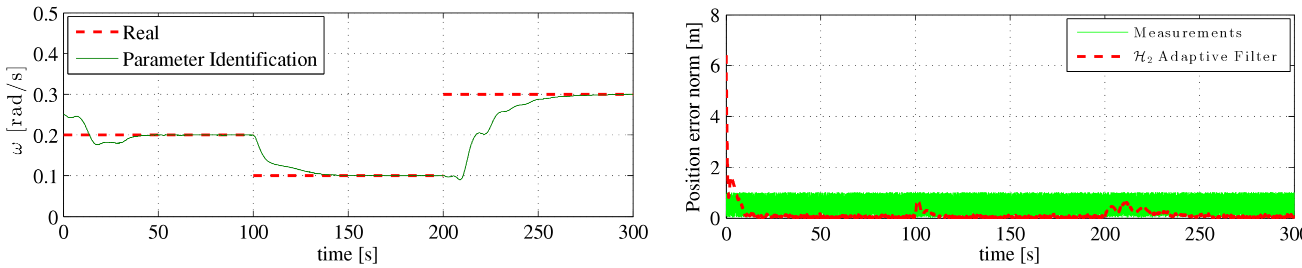
6. Conclusions

In this work, the problem of estimating the position, linear velocity, and linear acceleration of a target maneuvering in 3D space was addressed. A model for the target that depends on its angular speed was considered and only measurements of the target position were used. This problem was tackled resorting to a cascade of a parameter identifier, which estimates the angular speed of the target, and an $\mathcal{H}_2$ adaptive filter, which combines the angular speed estimates with measurements of the target position to estimate the target state. Under persistence of excitation conditions and for experiments where the process noise, the observation noise, the target linear velocity, and the target angular



(a) Angular speed identification. (b) Position estimation.

Fig. 4. Performance analysis for $\omega = 0$ rad/s.



(a) Angular speed identification. (b) Position estimation.

Fig. 5. Performance analysis for a target with changing angular speed.

speed are bounded, the errors associated with the proposed estimators were proved to converge to the vicinity of zero. Simulations showing that the convergence and stability guarantees derived in this brief paper hold, even when the estimates provided by an Extended Kalman Filter diverge, were presented.

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