Attitude and Earth Velocity Estimation - Part I: Globally Exponentially Stable Observer

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Abstract— **The main contribution of this paper is the development of a novel attitude observer based on a triaxial high-grade rate gyro and single body-fixed vector measurements of a constant inertial vector, in contrast with typical solutions that require two of these vectors. The proposed structure consists in a cascade observer, where the angular velocity of the Earth around its own axis is extracted in the first block and the attitude is estimated in the second. The stability of the system as a whole is analyzed and the error dynamics are shown to be globally exponentially stable (GES). Simulation results are presented that illustrate the achievable performance of this solution. In a companion paper an alternative observer is proposed on the special orthogonal group and both solutions are compared.**

I. Introduction

The problem of attitude estimation has received much attention for decades now. On one hand, the knowledge of the attitude of a robotic platform is usually an essential requirement for its successful operation. On the other hand, the problem is, per se, a very interesting one from a theoretical point of view. The solution of the Wahba's problem [1] gives an algebraic estimate of the rotation matrix from body-fixed to inertial coordinates based on the vector observations provided by the sensor suite installed in the robotic platform. However, there is no filtering process involved and as such information from the angular velocity is not incorporated into the estimate of the attitude of the platform. In the survey [2] many different filtering solutions are discussed. The extended Kalman filter (EKF) and variants have been widely exploited, see for instance [3], [4], and [5]. Divergence due to the linearization of the system dynamics [2] has paved the way for the pursuit of alternative designs, in particular nonlinear observers, see e.g. [6], [7], [8], and references therein. In [9] a deterministic attitude estimator is presented based on uncertainty ellipsoids. Previous work by the authors includes [10] and [11], where two different solutions for attitude estimation are presented that avoid common problems of attitude estimation such as singularities, unwinding phenomena, or topological limitations for achieving global asymptotic stabilization, see [12] for a thorough discussion of these issues.

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Most attitude determination solutions assume the availability of, at least, two body-fixed measurements of corresponding constant vectors in inertial coordinates. Recent exceptions to this approach can be found in [13], [14], and [15], where time-varying reference vectors are considered. In this paper, a single direct measurement, in body-fixed coordinates, of a constant inertial vector is assumed available. However, an estimate of a second vector is dynamically obtained from the measurements of high-grade rate gyros, which allow to estimate the angular velocity of the Earth around its own axis.

The main contribution of this paper is the design of a novel attitude estimation solution, with globally exponentially stable (GES) error dynamics, that is based on high-grade rate gyros and single body-fixed measurements of a constant inertial vector. The solution proposed in the paper for the problem of attitude estimation is built considering essentially two strategies: i) a cascade structure is envisioned such that in the first block an estimate of the angular velocity of the Earth, expressed in body-fixed coordinates, is obtained which, together with the second measurement, allows for the estimation of the rotation matrix in the second block; and ii) the topological characteristics of some variables are ignored, embedding those in linear spaces, such that topological obstacles to obtain global stability are avoided. However, additional constructs are provided such that the topological characteristics are recovered without affecting the stability of the algorithm.

The idea of extracting the angular velocity of the Earth around its own axis from measurements of high-grade rate gyros is not novel. Indeed, for the initial alignment of an inertial navigation system (INS), gyro compassing is usually performed. However, the initial alignment requires specific maneuvers, including positions where the platform is static. Moreover, from time to time, the INS needs to be re-calibrated, otherwise errors accumulate over time and become prohibitive. The novelty introduced in this paper is that the attitude of the platform is estimated without any particular maneuver requirements. Furthermore, the estimate of the angular velocity of the Earth around its own axis is continuously updated, therefore eliminating the accumulation of errors over time.

In terms of attitude estimation solutions, there exist presently many alternatives in the literature, that range from EKFs to nonlinear observers. In the case of nonlinear observers, a popular trend is to directly consider the topological properties of the rotation matrix and therefore restrict, by construction, the estimates of the rotation matrix to the special orthogonal group $SO(3)$. Examples of this type of approach can be found in [7] and [8], which have the advantage that, by construction, all the estimates correspond to elements of $SO(3)$. The main drawbacks are the existence of topological obstructions to achieve global stability by continuous feedback and the relatively small convergence speed near unstable and saddle points, as convincingly argued in [16]. A more recent trend, which can be found, e.g., in [6], [10], [11], and [15] consists in embedding the estimates in linear spaces, thus avoiding the topological obstructions to achieve global stability. The advantage is that the error dynamics feature global asymptotic (or exponential) stability, while the disadvantage is that the direct estimates of the observers do not necessarily belong to the special orthogonal group $SO(3)$. However, additional estimates can be built, that belong to $SO(3)$, without loss of global convergence, and several approaches are available, see e.g. [6] and [15]. In this paper the latter approach is considered, although another attitude observer could be used in the second block of the cascade system, provided that the stability analysis held when the attitude observer is fed by variables corrupted by disturbances converging globally exponentially fast to zero. An alternative solution is presented in the companion paper [17], where an observer directly built on SO(3) is proposed, and both solutions are discussed and compared.

The paper is organized as follows. The problem statement and the nominal system dynamics are introduced in Section II, while the observer design is detailed in Section III. Simulation results are presented in Section IV and Section V summarizes the main results of the paper.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented by $\mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_\mathrm{n})$. For $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ represents the inner and cross product, respectively. The special orthogonal group is denoted by $SO(3) := \left\{\mathbf{X} \in \mathbb{R}^{3\times 3} : \mathbf{X}\mathbf{X}^T = \mathbf{X}^T\mathbf{X} = \mathbf{I} \wedge \det(\mathbf{X}) = 1\right\}$. For convenience, define also the transpose operator $(.)^T$, and notice that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

II. Problem statement

Consider a robotic platform where a set of three, high-grade, orthogonally mounted rate gyros are mounted, in addition to another sensor, possibly inertial, that measures, in the reference frame of the platform, a vector that is constant in some inertial frame. Further consider that the rate gyros also measure the angular velocity of the Earth, expressed in the frame of the sensors. Loosely speaking, the problem addressed in the paper is that of determining the attitude of the platform with these sensor measurements.

To properly set the problem framework, let $\{I\}$ denote a local inertial coordinate reference frame, e.g. the North-East-Down (NED) coordinate frame with origin fixed to some point of the Earth, and denote by $\{B\}$ the so-called body-fixed frame, attached to the platform. Notice that due to the rotation and curvature of the Earth, the NED coordinate frame is not truly inertial but for local navigation it can be considered so. It is assumed, without loss of generality, that this is also the frame of the sensors. Let $\mathbf{R}(t) \in SO(3)$ denote the rotation matrix from $\{B\}$ to $\{I\}$, which satisfies

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left[\boldsymbol{\omega}(t)\right], \tag{1}$$

where $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is the angular velocity of $\{B\}$ with respect to $\{I\}$, expressed in $\{B\}$, and $\mathbf{S}\left[\boldsymbol{\omega}(t)\right] \in \mathbb{R}^{3\times 3}$ is the skew-symmetric matrix that encodes the cross product, i.e., $\mathbf{S}\left(\mathbf{x}\right)\mathbf{y} = \mathbf{x} \times \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

The measurements of the high-grade set of rate gyros are given by

$$\boldsymbol{\omega}_m(t) = \boldsymbol{\omega}(t) + \boldsymbol{\omega}_E(t), \tag{2}$$

where $\boldsymbol{\omega}_E(t) \in \mathbb{R}^3$ is the angular velocity of the Earth around its own axis, expressed in $\{B\}$. Denote by ${}^I\boldsymbol{\omega}_E \in \mathbb{R}^3$ the angular velocity of the Earth around its own axis expressed in $\{I\}$. Then,

$${}^I\boldsymbol{\omega}_E = \mathbf{R}(t)\boldsymbol{\omega}_E(t) \tag{3}$$

for all time.

Let $\mathbf{m}(t) \in \mathbb{R}^3$ denote the measurements of the second sensor, which measures, in body-fixed coordinates, a vector that, when expressed in inertial coordinates, is assumed known and constant. Let ${}^I\mathbf{m} \in \mathbb{R}^3$ denote the inertial vector corresponding to $\mathbf{m}(t)$. Then, in a similar fashion to the angular velocity of the Earth around its own axis, one has that

$${}^I\mathbf{m} = \mathbf{R}(t)\mathbf{m}(t) \tag{4}$$

for all time.

The following assumptions are considered throughout the paper.

Assumption 1: The inertial vector ${}^I\mathbf{m}$ is not parallel to the angular velocity of the Earth ${}^I\boldsymbol{\omega}_E$, i.e., ${}^I\boldsymbol{\omega}_E \times {}^I\mathbf{m} \neq \mathbf{0}$.

Assumption 2: The rate gyro measurements are bounded for all time.

The first is a mild assumption and can be considered standard as most attitude estimation solutions assume the existence of at least two known non-parallel inertial vectors, which are measured in body-fixed coordinates. The difference in this paper is that $\boldsymbol{\omega}_E(t)$ is not measured. Instead, it is also explicitly estimated. The second is a technical assumption that is evidently verified for all systems in practice, as one cannot have arbitrarily large angular velocities.

The problem addressed in the paper is that of designing an observer, with globally exponentially stable error dynamics, for the rotation matrix $\mathbf{R}(t)$ from $\{B\}$ to $\{I\}$ based on the rate gyro measurements $\boldsymbol{\omega}_m(t)$, the additional measurements $\mathbf{m}(t)$ and the knowledge of both ${}^I\boldsymbol{\omega}_E$ and ${}^I\mathbf{m}$.

III. Observer design

A. Earth rotation observer

This section presents the derivation and stability analysis of the observer for the angular velocity of the Earth around its own axis. In order to do so, take the time derivative of the vector measurement $\mathbf{m}(t)$, which using (1), (2), and (4) can be written as

$$\dot{\mathbf{m}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t) - \boldsymbol{\omega}_E(t)\right]\mathbf{m}(t). \tag{5}$$

Similarly, take the time derivative $\boldsymbol{\omega}_E(t)$, which from (1)-(3) can be written as

$$\dot{\boldsymbol{\omega}}_E(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t)\right]\boldsymbol{\omega}_E(t). \tag{6}$$

From (5) one is suggested that, with measurements of $\mathbf{m}(t)$ and $\boldsymbol{\omega}_m(t)$, it might be possible to estimate $\boldsymbol{\omega}_E(t) \times \mathbf{m}(t)$. This is indeed the case, as it is shown next.

Define $\mathbf{x}_1(t) := \mathbf{m}(t)$ and $\mathbf{x}_2(t) := \boldsymbol{\omega}_E(t) \times \mathbf{m}(t)$. Then, from (5) and (6) one may write

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}_1(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t)\right]\mathbf{x}_1(t) + \mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t) - \boldsymbol{\omega}_E(t)\right]\mathbf{x}_2(t) \end{array} \right. . \tag{7}$$

The form of the time derivative of $\mathbf{x}_2(t)$ in (7) is undesirable as it still depends on $\boldsymbol{\omega}_E(t)$, when one at this time has chosen to estimate $\boldsymbol{\omega}_E(t) \times \mathbf{m}(t)$. Consider as an orthonormal basis the set of vectors

$$\left\{ \frac{\mathbf{x}_1(t)}{\|\mathbf{x}_1(t)\|}, \frac{\mathbf{x}_2(t)}{\|\mathbf{x}_2(t)\|}, \frac{\mathbf{x}_1(t) \times \mathbf{x}_2(t)}{\|\mathbf{x}_1(t) \times \mathbf{x}_2(t)\|} \right\}. \tag{8}$$

Notice that this set is always well-defined under Assumption 1. Define also

$$A_{21} := \frac{\left({}^I\boldsymbol{\omega}_E \cdot {}^I\mathbf{m}\right)^2 - \left\|{}^I\boldsymbol{\omega}_E\right\|^2 \left\|{}^I\mathbf{m}\right\|^2}{\left\|{}^I\mathbf{m}\right\|^2}$$

and

$$A_{22} := \frac{\left({}^I\boldsymbol{\omega}_E \cdot {}^I\mathbf{m}\right) \left\|{}^I\boldsymbol{\omega}_E \times {}^I\mathbf{m}\right\|^2}{\left\|{}^I\mathbf{m} \times \left({}^I\boldsymbol{\omega}_E \times {}^I\mathbf{m}\right)\right\|^2}.$$

Then, using norm, rotation, and cross-product properties, as well as the three-dimensional case of the Binet-Cauchy identity, decomposing the term $\mathbf{S}\left[\boldsymbol{\omega}_E(t)\right]\mathbf{x}_2(t)$ using the orthonormal basis (8) it is possible to rewrite the system dynamics (7) as

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}_1(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t)\right]\mathbf{x}_1(t) + \mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) = A_{21}\mathbf{x}_1(t) - \mathbf{S}\left[\boldsymbol{\omega}_m(t) - A_{22}\mathbf{x}_1(t)\right]\mathbf{x}_2(t) \end{array} \right. . \tag{9}$$

Consider the observer for (9) given by

$$\left\{ \begin{array}{rl} \dot{\hat{\mathbf{x}}}_1(t) = & -\mathbf{S}\left[\boldsymbol{\omega}_m(t)\right]\hat{\mathbf{x}}_1(t) + \hat{\mathbf{x}}_2(t) \\ & +\alpha_1\left[\mathbf{x}_1(t) - \hat{\mathbf{x}}_1(t)\right] \\ \dot{\hat{\mathbf{x}}}_2(t) = & A_{21}\mathbf{x}_1(t) - \mathbf{S}\left[\boldsymbol{\omega}_m(t) - A_{22}\mathbf{x}_1(t)\right]\hat{\mathbf{x}}_2(t) \\ & +\alpha_2\left[\mathbf{x}_1(t) - \hat{\mathbf{x}}_1(t)\right] \end{array} \right. , \tag{10}$$

where $\hat{\mathbf{x}}_1(t) \in \mathbb{R}^3$ and $\hat{\mathbf{x}}_2(t) \in \mathbb{R}^3$ correspond to the estimates of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, respectively, and $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$ are observer gains. Let $\tilde{\mathbf{x}}_1(t) := \mathbf{x}_1(t) - \hat{\mathbf{x}}_1(t)$ and $\tilde{\mathbf{x}}_2(t) := \mathbf{x}_2(t) - \hat{\mathbf{x}}_2(t)$ denote the estimation errors. Then, from (9) and (10) one may write the error dynamics

$$\left\{ \begin{array}{rl} \dot{\tilde{\mathbf{x}}}_1(t) = & -\left(\alpha_1\mathbf{I} + \mathbf{S}\left[\boldsymbol{\omega}_m(t)\right]\right)\tilde{\mathbf{x}}_1(t) + \tilde{\mathbf{x}}_2(t) \\ \dot{\tilde{\mathbf{x}}}_2(t) = & -\alpha_2\tilde{\mathbf{x}}_1(t) - \mathbf{S}\left[\boldsymbol{\omega}_m(t) - A_{22}\mathbf{x}_1(t)\right]\tilde{\mathbf{x}}_2(t) \end{array} \right. . \tag{11}$$

The following theorem addresses the stability and convergence properties of (11).

Theorem 1: Consider the state observer (10) and suppose that the observer gains α_1 and α_2 are positive. Further suppose that Assumptions 1 and 2 hold. Then, the origin of the error dynamics (11) is a globally exponentially stable equilibrium point.

Proof: Due to space limitations, only a sketch of the proof is presented. Consider the compact error definition

$$\mathbf{z}_1(t) := \left[\begin{array}{c} \tilde{\mathbf{x}}_1(t) \\ \tilde{\mathbf{x}}_2(t) \end{array} \right] \in \mathbb{R}^6$$

and notice that the error dynamics (11) can be written as the linear time-varying system $\dot{\mathbf{z}}_1(t) = \mathbf{A}_1(t)\mathbf{z}_1(t)$, with

$$\mathbf{A}_1(t) := \left[\begin{array}{cc} -\left(\alpha_1\mathbf{I} + \mathbf{S}\left[\boldsymbol{\omega}_m(t)\right]\right) & \mathbf{I} \\ -\alpha_2\mathbf{I} & -\mathbf{S}\left[\boldsymbol{\omega}_m(t) - A_{22}\mathbf{x}_1(t)\right] \end{array} \right].$$

Define the Lyapunov-like function

$$V\left(\mathbf{z}_1(t)\right) := \mathbf{z}_1^T(t)\mathbf{P}\mathbf{z}_1(t),$$

with

$$\mathbf{P} = \left[\begin{array}{cc} \frac{1}{2}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2\alpha_2}\mathbf{I} \end{array} \right] \in \mathbb{R}^{6\times 6}.$$

The proof now follows in a similar fashion to that of [18, Example 8.11]. ■

The observer (10) allows to estimate the component of the angular velocity of the Earth that is orthogonal to the vector measurement $\mathbf{m}(t)$. This is sufficient to obtain an algebraic estimate of the rotation matrix $\mathbf{R}(t)$, as at this point one has access to two vectors in body-fixed coordinates, $\mathbf{m}(t)$ and $\boldsymbol{\omega}_E(t) \times \mathbf{m}(t)$ whose counterparts in inertial coordinates are also available, given by ${}^I\mathbf{m}$ and ${}^I\boldsymbol{\omega}_E \times {}^I\mathbf{m}$, respectively. However, that is not sufficient to obtain a filtered estimate of the rotation matrix as the complete angular velocity of the Earth is required, not just the component orthogonal to $\mathbf{m}(t)$. Indeed, notice that from (1) and (2) one has

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left[\boldsymbol{\omega}_m(t) - \boldsymbol{\omega}_E(t)\right]. \tag{12}$$

Nevertheless, one already has all the elements that allow to reconstruct the angular velocity of the Earth, as established in the following proposition.

Proposition 1: Consider the estimates $\hat{\mathbf{x}}_1(t)$ and $\hat{\mathbf{x}}_2(t)$ given by the state observer (10) and define an estimate of the angular velocity of the Earth as

$$\begin{aligned} \hat{\boldsymbol{\omega}}_E(t) = & \frac{{}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E}{\|{}^I\mathbf{m}\|^2}\hat{\mathbf{x}}_1(t) \\ & + \frac{\|{}^I\mathbf{m}\|^2\|{}^I\boldsymbol{\omega}_E\|^2 - \left({}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E\right)^2}{\|{}^I\mathbf{m}\times\left({}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m}\right)\|^2}\hat{\mathbf{x}}_1(t) \times \hat{\mathbf{x}}_2(t). \end{aligned} \tag{13}$$

Denote by $\tilde{\boldsymbol{\omega}}_E(t) := \boldsymbol{\omega}_E(t) - \hat{\boldsymbol{\omega}}_E(t)$ the estimation error of the angular velocity of the Earth. Then, under the conditions of Theorem 1, $\tilde{\boldsymbol{\omega}}_E(t)$ converges exponentially fast to zero for all initial conditions $\hat{\mathbf{x}}_1(t_0)$ and $\hat{\mathbf{x}}_2(t_0)$.

Proof: Using the orthonormal basis (8) and several norm, rotation, and cross product properties, the error of the estimate of the angular velocity of the Earth can be written as

$$\begin{aligned}\tilde{\boldsymbol{\omega}}_E(t) = &\frac{{}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E}{\|{}^I\mathbf{m}\|^2}\tilde{\mathbf{x}}_1(t)\\ &+\frac{\|{}^I\mathbf{m}\|^2\|{}^I\boldsymbol{\omega}_E\|^2-\left[{}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E\right]^2}{\|{}^I\mathbf{m}\times({}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m})\|^2}\tilde{\mathbf{x}}_1(t)\times\mathbf{x}_2(t)\\ &+\frac{\|{}^I\mathbf{m}\|^2\|{}^I\boldsymbol{\omega}_E\|^2-\left[{}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E\right]^2}{\|{}^I\mathbf{m}\times({}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m})\|^2}\mathbf{x}_1(t)\times\tilde{\mathbf{x}}_2(t)\\ &-\frac{\|{}^I\mathbf{m}\|^2\|{}^I\boldsymbol{\omega}_E\|^2-\left[{}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E\right]^2}{\|{}^I\mathbf{m}\times({}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m})\|^2}\tilde{\mathbf{x}}_1(t)\times\tilde{\mathbf{x}}_2(t).\end{aligned}\tag{14}$$

Using simple norm inequalities, one can now write, from (14), that

$$\begin{aligned}\|\tilde{\boldsymbol{\omega}}_E(t)\| \leq &\frac{\left|{}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E\right|}{\|{}^I\mathbf{m}\|^2}\|\tilde{\mathbf{x}}_1(t)\|\\ &+\frac{\|{}^I\mathbf{m}\|^2\|{}^I\boldsymbol{\omega}_E\|^2-\left[{}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E\right]^2}{\|{}^I\mathbf{m}\times({}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m})\|^2}\|\mathbf{x}_2(t)\|\,\|\tilde{\mathbf{x}}_1(t)\|\\ &+\frac{\|{}^I\mathbf{m}\|^2\|{}^I\boldsymbol{\omega}_E\|^2-\left[{}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E\right]^2}{\|{}^I\mathbf{m}\times({}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m})\|^2}\|\mathbf{x}_1(t)\|\,\|\tilde{\mathbf{x}}_2(t)\|\\ &+\frac{\|{}^I\mathbf{m}\|^2\|{}^I\boldsymbol{\omega}_E\|^2-\left[{}^I\mathbf{m}\cdot{}^I\boldsymbol{\omega}_E\right]^2}{\|{}^I\mathbf{m}\times({}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m})\|^2}\|\tilde{\mathbf{x}}_1(t)\|\,\|\tilde{\mathbf{x}}_2(t)\|.\end{aligned}\tag{15}$$

Recall that both $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ have constant norm. Moreover, under the conditions of Theorem 1, both $\tilde{\mathbf{x}}_1(t)$ and $\tilde{\mathbf{x}}_2(t)$ converge globally exponentially fast to zero. Therefore, as the upper bound in (15) consists in a sum of decaying exponentials, it follows that there is a single decaying exponential that bounds the sum from above, therefore concluding the proof. ■

B. Attitude observer

In the previous section an observer was proposed that allows one to obtain filtered estimates of two vectors in body-fixed coordinates, $\hat{\mathbf{m}}(t)$ and $\hat{\mathbf{x}}_2(t)$, whose counterparts in inertial coordinates, i.e., ${}^I\mathbf{m}$ and ${}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m}$, respectively, are also known. In addition, a filtered estimate of the angular velocity of the Earth was also obtained. In this section, the latter is used to drive the dynamics of a filtered estimate of the rotation matrix, while the two vectors in body-fixed coordinates, along with their counterparts in inertial coordinates, are used in the error injection term to drive the estimation error to zero.

At this point, any attitude observer that uses directly vector measurements could be employed, provided that the design is robust to exponentially decaying perturbations. Indeed, the attitude observer is fed with estimates whose error converges to zero exponentially fast, not with the actual quantities. One such design is provided in [10] and it is employed in this section with the appropriate adaptations. The design is here streamlined and further details can be found in [10].

Consider a column representation of the rotation matrix $\mathbf{R}(t)$ given by

$$\mathbf{z}_2(t) = \begin{bmatrix}\mathbf{r}_1(t)\\ \mathbf{r}_2(t)\\ \mathbf{r}_3(t)\end{bmatrix} \in \mathbb{R}^9,$$

where

$$\mathbf{R}(t) = \begin{bmatrix}\mathbf{r}_1^T(t)\\ \mathbf{r}_2^T(t)\\ \mathbf{r}_3^T(t)\end{bmatrix},\ \mathbf{r}_i(t)\in\mathbb{R}^3,\ i=1,\ldots,3.$$

Then, from (12), it follows that

$$\dot{\mathbf{z}}_2(t) = -\mathbf{S}_3\left[\boldsymbol{\omega}_m(t)-\boldsymbol{\omega}_E(t)\right]\mathbf{z}_2(t),$$

where

$$\mathbf{S}_3(\mathbf{x}) := \operatorname{diag}\left(\mathbf{S}(\mathbf{x}),\mathbf{S}(\mathbf{x}),\mathbf{S}(\mathbf{x})\right)\in\mathbb{R}^{9\times 9},\ \mathbf{x}\in\mathbb{R}^3.$$

Let ${}^I\mathbf{m} = \begin{bmatrix}{}^Ix_{11} & {}^Ix_{12} & {}^Ix_{13}\end{bmatrix}^T$, ${}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m} = \begin{bmatrix}{}^Ix_{21} & {}^Ix_{22} & {}^Ix_{23}\end{bmatrix}^T$, and ${}^I\mathbf{m}\times\left({}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m}\right) = \begin{bmatrix}{}^Ix_{31} & {}^Ix_{32} & {}^Ix_{33}\end{bmatrix}^T$. Notice that

$${}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m} = \mathbf{R}(t)\mathbf{x}_2(t)\tag{16}$$

and

$${}^I\mathbf{m}\times\left({}^I\boldsymbol{\omega}_E\times{}^I\mathbf{m}\right) = \mathbf{R}(t)\left[\mathbf{x}_1(t)\times\mathbf{x}_2(t)\right].\tag{17}$$

From (4), (16), and (17) it is possible to write

$$\mathbf{v}(t) = \mathbf{C}_2\mathbf{z}_2(t),$$

where

$$\mathbf{v}(t) = \begin{bmatrix}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\\ \mathbf{x}_1(t)\times\mathbf{x}_2(t)\end{bmatrix}\in\mathbb{R}^9$$

and

$$\mathbf{C}_2 = \begin{bmatrix}{}^Ix_{11}\mathbf{I} & {}^Ix_{12}\mathbf{I} & {}^Ix_{13}\mathbf{I}\\ {}^Ix_{21}\mathbf{I} & {}^Ix_{22}\mathbf{I} & {}^Ix_{23}\mathbf{I}\\ {}^Ix_{31}\mathbf{I} & {}^Ix_{32}\mathbf{I} & {}^Ix_{33}\mathbf{I}\end{bmatrix}.$$

Notice that, under Assumption 1, matrix $\mathbf{C}_2$ has full rank.

Consider the attitude observer given by

$$\begin{aligned}\dot{\hat{\mathbf{z}}}_2(t) \;=\; &-\mathbf{S}_3\left[\boldsymbol{\omega}_m(t)-\hat{\omega_E}(t)\right]\hat{\mathbf{z}}_2(t)\\ &+\mathbf{C}_2^T\mathbf{Q}^{-1}\left[\hat{\mathbf{v}}(t)-\mathbf{C}_2\hat{\mathbf{z}}_2(t)\right],\end{aligned}\tag{18}$$

where $\mathbf{Q}=\mathbf{Q}^T\in\mathbb{R}^{9\times 9}$ is a positive definite matrix and

$$\hat{\mathbf{v}}(t) = \begin{bmatrix}\hat{\mathbf{x}}_1(t)\\ \hat{\mathbf{x}}_2(t)\\ \hat{\mathbf{x}}_1(t)\times\hat{\mathbf{x}}_2(t)\end{bmatrix}.$$

Define the error variable $\tilde{\mathbf{z}}_2(t) = \mathbf{z}_2(t)-\hat{\mathbf{z}}_2(t)$. Then, the observer error dynamics are given by

$$\dot{\tilde{\mathbf{z}}}_2(t) = \left[\mathbf{A}_2(t)-\mathbf{S}_3\left[\tilde{\boldsymbol{\omega}}_E(t)\right]\right]\tilde{\mathbf{z}}_2(t)+\mathbf{u}_2(t),\tag{19}$$

where

$$\mathbf{A}_2(t) = -\mathbf{C}_2^T\mathbf{Q}^{-1}\mathbf{C}_2-\mathbf{S}_3\left[\boldsymbol{\omega}(t)\right]$$

and

$$\mathbf{u}_2(t) = \mathbf{S}_3\left(\tilde{\boldsymbol{\omega}}_E(t)\right)\mathbf{z}_2(t)+\mathbf{C}_2^T\mathbf{Q}^{-1}\tilde{\mathbf{v}}(t),$$

with $\tilde{\mathbf{v}}(t) := \mathbf{v}(t)-\hat{\mathbf{v}}(t)$.

The following theorem is the main result of this section.

Theorem 2: Consider the attitude observer (18), where the estimates of the vector observations, $\hat{\mathbf{v}}(t)$, are obtained using the state observer (10) and the estimates of the angular velocity of the Earth, $\hat{\omega_E}(t)$, are given by (13). Then, under the conditions of Theorem 1, and assuming that $\mathbf{Q}$ is a positive definite matrix, it follows that the origin of the error dynamics (19) is a globally exponentially stable equilibrium point.

Proof: The proof is similar to that of [10, Theorem 3]. It is ommitted due to space limitations. ■

C. Further discussion

The estimates of the rotation matrix provided by the attitude observer (18) do not necessarily belong to $SO(3)$, although they converge asymptotically to elements of $SO(3)$. As previously discussed, this is a choice of design that allows one to achieve global stability. Nevertheless, if one insists in having explicit estimates on $SO(3)$, there exist several methods in the literature to obtain such constructs from the estimates provided by (18). As this has been discussed previously, the reader is referred to [10] and [15] for such alternatives.

IV. Simulation results

Simulation results are presented in this section to demonstrate the performance achieved with the proposed solution. These are only preliminary results and extensive Monte Carlo simulations will be carried out in the future, prior to experimental validation.

The local inertial frame was considered as the NED frame, centered at a latitude of $\varphi = 38.7138°$, a longitude of $\psi = 9.1394°$, and at sea level. The norm of the angular velocity of the Earth was set to $\left\|{}^{I}\boldsymbol{\omega}_E\right\| = 7.2921150 \times 10^{-5}$ rad/s, which corresponds approximately to 15 degrees per hour. Thus, in the NED frame, one has ${}^{I}\boldsymbol{\omega}_E = \left\|{}^{I}\boldsymbol{\omega}_E\right\| \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \end{bmatrix}^T$. As for the vector measurement $\mathbf{m}(t)$, it is assumed that magnetic field measurements are available. However, any other inertial vector could have been considered. In this case, ${}^{I}\mathbf{m}$ was set according to the 11^{th} generation of International Geomagnetic Reference Field for the latitude, longitude, and altitude previously described. Notice that, with this choice, Assumption 1 is satisfied.

The initial attitude of the platform was set to $\mathbf{R}(0) = \mathbf{I}$ and the evolution of the angular velocity is given by

$$\boldsymbol{\omega}(t) = \begin{bmatrix} 5\frac{\pi}{180}\sin\left(\frac{2\pi}{60}t\right) \\ \frac{\pi}{180}\sin\left(\frac{2\pi}{180}t\right) \\ -2\frac{\pi}{180}\sin\left(\frac{2\pi}{300}t\right) \end{bmatrix} \text{(rad/s)}.$$

In the simulations, the measurements of the magnetometer were assumed to be corrupted by zero-mean white Gaussian noise, with standard deviation of 150 nT, which corresponds to the worst case specification of the triaxial magnetometer of the nanoIMU NA02-0150F50. The rate gyro measurements noise is characterized by an angle random walk of 4°/hr/$\sqrt{\text{Hz}}$, which corresponds to the KVH DSP-3000 fiber optic gyro. A sampling frequency of 100 Hz was considered and the fourth-order Runge-Kutta method was employed in the simulations.

In order to ensure both fast convergence speed and good steady-state performance of the first observer, a set of piecewise constant gains was chosen. Notice that this does not impact the stability of the error dynamics as a finite set of transitions is considered. Indeed, for each particular choice of observer gains, the error dynamics are stable, and once the final gain is set, the results previously derived apply. These gains are described in Table I. As for the second observer,

TABLE I
OBSERVER GAINS

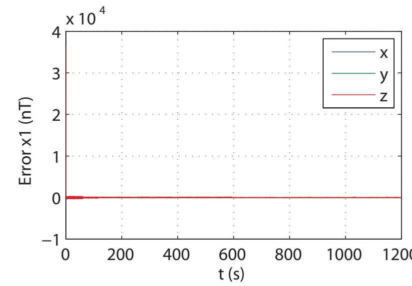
Time interval (s)	α_1	α_2
$[0, 60[$	100	10
$[60, 120[$	10	1
$[120, 240[$	5	10^{-1}
$[240, 300[$	5	5×10^{-2}
$[300, 600[$	5	25×10^{-3}
$[600, 720[$	2.5	10×10^{-3}
$[720, +\infty[$	2.5	5×10^{-3}

its gain was set to $\mathbf{Q} = 10^5 \mathbf{C}_2 \mathbf{Q}_D \mathbf{C}_2^T$, with

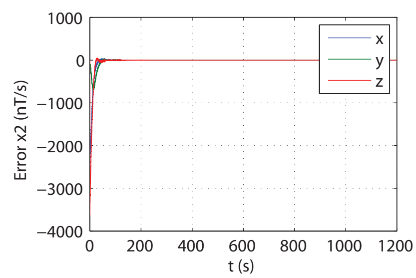
$$\mathbf{Q}_D = \text{diag}\left(\frac{20}{\|{}^{\text{I}}\mathbf{m}\|}\mathbf{I}, \frac{2 \times 10^{-2}}{\|{}^{\text{I}}\boldsymbol{\omega}_{\text{E}} \times {}^{\text{I}}\mathbf{m}\|}\mathbf{I}, \frac{10^3}{\|{}^{\text{I}}\mathbf{m} \times ({}^{\text{I}}\boldsymbol{\omega}_{\text{E}} \times {}^{\text{I}}\mathbf{m})\|}\mathbf{I}\right).$$

As in most nonlinear observers, these gains were chosen empirically, although the relative gains of the second observer are related to the error noise of its observations. The initial estimates of the first observer were set to zero, while the initial attitude estimate was set to $\hat{\mathbf{R}}(0) = \text{diag}(-1, -1, 1)$.

The initial convergence of the errors $\tilde{\mathbf{x}}_1(t)$ and $\tilde{\mathbf{x}}_2(t)$ is depicted in Fig. 1. While the convergence of the observer is fast, different gains are required, as detailed in Table I, in order to ensure an adequate steady-state level of error, as it will be detailed shortly. The initial convergence of the error $\tilde{\boldsymbol{\omega}}_E(t)$ is depicted in Fig. 2. Finally, the initial convergence of the attitude error, expressed as $\tilde{\mathbf{z}}_2(t)$, is shown in Fig. 3. These plots show that the error converges to a neighborhood of zero. In the absence of noise, the errors converge to zero. This is not shown in the paper only due to space limitations.



(a) Error $\tilde{\mathbf{x}}_1(t)$



(b) Error $\tilde{\mathbf{x}}_2(t)$

Fig. 1. Initial convergence of the errors of the first observer

In order to evaluate the performance of the attitude observer, the steady-state standard deviation of the errors is depicted in Table II. These values should be compared

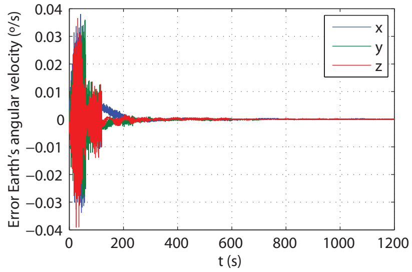


Fig. 2. Initial convergence of the error $\tilde{\boldsymbol{\omega}}_E(t)$

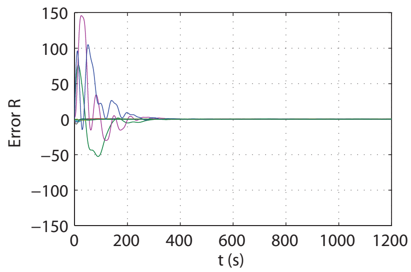


Fig. 3. Initial convergence of the error $\tilde{\mathbf{z}}_2(t)$

to the magnitude of the corresponding variables, which is roughly 4.5×10^4 nT for $\mathbf{x}_1(t)$, 3.27 nT/s for $\mathbf{x}_2(t)$, and 15°/hour for $\boldsymbol{\omega}_E(t)$. Evidently, the observer achieves very good results. Finally, an additional error variable is defined as $\tilde{\mathbf{R}}(t) = \mathbf{R}^T(t)\hat{\mathbf{R}}_e(t)$, which corresponds to the rotation matrix error. Here, $\hat{\mathbf{R}}_e(t)$ corresponds to the projection of the obtained attitude estimate on $SO(3)$. Using the Euler angle-axis representation for this new error variable,

$$\begin{aligned}\tilde{\mathbf{R}}(t) &= \mathbf{I}\cos\left(\tilde{\theta}(t)\right) + \left[1 - \cos\left(\tilde{\theta}(t)\right)\right]\tilde{\mathbf{d}}(t)\tilde{\mathbf{d}}^T(t) \\ &\quad -\mathbf{S}\left(\tilde{\mathbf{d}}(t)\right)\sin\left(\tilde{\theta}(t)\right),\end{aligned}$$

where $0 \leq \tilde{\theta}(t) \leq \pi$ and $\tilde{\mathbf{d}}(t) \in \mathbb{R}^3, \left\|\tilde{\mathbf{d}}(t)\right\| = 1$, are the angle and axis that represent the rotation error, the performance of the filter is easily identified from the evolution of $\tilde{\theta}$. The mean angle error, computed for $t \geq 2400$ s, is $0.586°$, which is a very good result.

V. CONCLUSIONS

This paper presented a novel attitude observer based on measurements of a single body-fixed vector, in addition to angular velocity measurements provided by a triaxial high-grade rate gyro, that has as key feature the explicit estimation of the angular velocity of the Earth around its own axis. By explicitly estimating this vector, one is endowed in the end with two body-fixed vector observations, whose counterparts in inertial coordinates are known, which thus allows to estimate the attitude of a platform. The proposed observer has a cascade structure, it is computationally efficient, and the overall error dynamics were shown to be globally exponentially stable. The extension to more than one vector observation can be envisioned and simulations results were presented that illustrate the achievable performance. Future work will include extensive Monte Carlo simulations prior to experimental evaluation.

TABLE II
STANDARD DEVIATION OF THE STEADY-STATE ERROR

Variable	Standard deviation
$\tilde{\mathbf{x}}_1(t)$ (nT)	17
$\tilde{\mathbf{x}}_2(t)$ (nT/s)	0.044
$\tilde{\boldsymbol{\omega}}_E(t)$ (°/hour)	0.18
$\tilde{\mathbf{z}}_2(t)$	0.044

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