Attitude and Earth Velocity Estimation - Part II: Observer on the Special Orthogonal Group

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Abstract— **This paper presents a novel cascade attitude observer based on measurements of a single body-fixed vector, in addition to high-grade rate gyro readings. First, a second body-fixed vector, related to the angular velocity of the Earth around its own axis, is dynamically estimated. Then, an attitude observer is built based upon the measured body-fixed vector, the estimated one, and the angular velocity provided by the rate gyros. The topological characteristics of the attitude estimates are preserved by construction and the estimation error is shown to converge to zero. The region of convergence is characterized and the result is best described as semi-global. Simulation results are included that show the achievable performance of the proposed solution. Finally, the results and characteristics of the observer are compared with that proposed in a companion paper, which considers attitude estimates on $\mathbb{R}^{3\times 3}$.**

I. Introduction

Attitude estimation has been a hot topic of research for the past decades, and not by mere chance. Indeed, the attitude of an autonomous robotic platform is usually a key requirement for its successful operation. Moreover, the problem is both interesting and challenging. Algebraic solutions, using body-fixed vector measurements of at least two known non-parallel inertial vectors, have long existed, see e.g. the Wahba's problem [1]. However, this solution does not incorporate any filtering process and all the imperfections of the vector measurements are reflected in the resulting algebraic estimates. This can be highly mitigated using angular velocity measurements, readily available from rate gyros, and designing appropriate filtering processes. Many different filtering alternatives have been reviewed in the survey paper [2]. As in many nonlinear problems, the extended Kalman filter (EKF) provided an earlier solution, see e.g. [3], [4], and [5]. However, the divergence due to the linearization errors of the EKF has compelled the research community to pursue different solutions with convergence guarantees, such as those presented in [6], [7], [8], [9], and references therein. In [10] a deterministic attitude estimator is presented based on uncertainty ellipsoids. Previous work by the authors includes [11] and [12], where two different solutions for attitude estimation are presented with global convergence results.

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For attitude estimation it is usually considered that there are available, at least, two body-fixed measurements of corresponding known constant vectors in inertial coordinates. Recent exceptions to this approach are [13], [14], and [15], where time-varying reference vectors are considered. In this paper, a single direct measurement, in body-fixed coordinates, of a constant inertial vector is assumed available. Yet, a second vector is dynamically computed from the measurements of high-grade rate gyros, which ultimately allow one to estimate the angular velocity of the Earth around its own axis.

In the companion paper [16] a novel cascade attitude observer is proposed that is based on high-grade rate gyros and single body-fixed measurements of a constant inertial vector. In short, estimates of direction vectors are considered, in the first stage of the observer, in $\mathbb{R}^3$, while the attitude estimates are obtained on $\mathbb{R}^{3\times 3}$, and solutions on how to later obtained attitude estimates from the observer estimates are later mentioned. This paper presents an alternative cascade attitude observer for the same problem. However, the structures of both the first and the second observer in the present approach are such that the topological characteristics of the quantities are preserved. First, an estimate of a second body-fixed vector is obtained, related to the angular velocity of the Earth, whose counterpart in inertial coordinates is known. Then, a second observer fuses the measurements of the high-grade rate gyros and the two vectors available to obtain a filtered version of the attitude of the platform. The error of the estimate of the second vector required for attitude estimation, provided by the first observer, is shown to converge to zero for all initial conditions. The attitude error of the second observer, considering the cascade structure, is shown to converge to zero, with a large region of convergence. As previously mentioned, the characteristics of the quantities are preserved. In particular, for the first observer, the norm of the estimate of the measured vector is constant by construction, while for the attitude observer the estimates provided by the observer belong, by construction, to the special orthogonal group. The caveat of this approach, as of those presented in [8] and [9], is that it is not possible to achieve global asymptotic stability by continuous feecback, see [17]. The advantage is that the attitude estimates belong to the special orthogonal group and no additional projections or other computations are applied to the estimates provided by the observer.

The paper is organized as follows. The problem statement and the nominal system dynamics are introduced in Section

II, while the observer design and stability analysis are detailed in Section III. Simulation results are presented in Section IV and Section V summarizes the main results of the paper.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. For $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ represents the inner and cross product, respectively. The special orthogonal group is denoted by $SO(3) := \left\{\mathbf{X} \in \mathbb{R}^{3\times 3} : \mathbf{X}\mathbf{X}^T = \mathbf{X}^T\mathbf{X} = \mathbf{I} \wedge \det(\mathbf{X}) = 1\right\}$. For convenience, define also the transpose operator $(.)^T$, and notice that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

II. Problem statement

For the purpose of attitude estimation, assume that a platform is equipped with a set of three, high-grade, orthogonally mounted rate gyros, which are sensitive to the angular velocity of the Earth around its own axis. Further assume that measurements of a known constant inertial vector are also available, in body-fixed coordinates. Roughly speaking, this paper aims at determining the attitude of the platform with these sensor measurements.

In order to clearly describe the problem framework, let $\{I\}$ denote a local inertial coordinate reference frame, e.g. the North-East-Down (NED) coordinate frame with origin fixed to some point of the Earth, and denote by $\{B\}$ the so-called body-fixed frame, attached to the platform. It is assumed, without loss of generality, that this is also the frame of the sensors. Let $\mathbf{R}(t) \in SO(3)$ denote the rotation matrix from $\{B\}$ to $\{I\}$ and $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ be the angular velocity of $\{B\}$ relative to $\{I\}$, expressed in $\{B\}$. Then,

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left[\boldsymbol{\omega}(t)\right], \tag{1}$$

where $\mathbf{S}\left[\boldsymbol{\omega}(t)\right] \in \mathbb{R}^{3\times 3}$ is the skew-symmetric matrix that encodes the cross product, i.e., $\mathbf{S}(\mathbf{x})\,\mathbf{y} = \mathbf{x} \times \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

The measurements of the high-grade set of rate gyros are given by

$$\boldsymbol{\omega}_m(t) = \boldsymbol{\omega}(t) + \boldsymbol{\omega}_E(t), \tag{2}$$

where $\boldsymbol{\omega}_E(t) \in \mathbb{R}^3$ is the angular velocity of the Earth around its own axis, expressed in $\{B\}$. Denote by ${}^I\boldsymbol{\omega}_E \in \mathbb{R}^3$ the angular velocity of the Earth around its own axis expressed in $\{I\}$. Then,

$${}^I\boldsymbol{\omega}_E = \mathbf{R}(t)\boldsymbol{\omega}_E(t) \tag{3}$$

for all time.

Let $\mathbf{m}(t) \in \mathbb{R}^3$ denote the measurements of the second sensor, which measures, in body-fixed coordinates, a vector that, when expressed in inertial coordinates, is assumed known and constant. Let ${}^I\mathbf{m} \in \mathbb{R}^3$ denote the inertial vector corresponding to $\mathbf{m}(t)$. Then, similarly to the angular velocity of the Earth around its own axis, one can write

$${}^I\mathbf{m} = \mathbf{R}(t)\mathbf{m}(t) \tag{4}$$

for all time.

The following assumptions are considered throughout the paper.

Assumption 1: The inertial vector ${}^I\mathbf{m}$ is not parallel to the angular velocity of the Earth ${}^I\boldsymbol{\omega}_E$, i.e., there exists a constant $c_v > 0$ such that $\left\|{}^I\boldsymbol{\omega}_E \times {}^I\mathbf{m}\right\|^2 \geq c_v$.

Assumption 2: The signal $\boldsymbol{\omega}_m(t)$ and its derivative $\dot{\boldsymbol{\omega}}_m(t)$ are bounded for all time.

The first assumption is mild and it is rather standard as most attitude estimation solutions assume the existence of at least two known non-parallel inertial vectors, which are measured in body-fixed coordinates. Yet, in this paper, of the two vectors, $\boldsymbol{\omega}_E(t)$ is not even measured. Instead, it is also explicitly estimated. The second is a technical assumption that is evidently verified for all systems in practice, as one cannot have arbitrarily large angular velocities or angular accelerations.

The problem addressed in the paper is that of designing an observer, with convergence guarantees, for the rotation matrix $\mathbf{R}(t)$ from $\{B\}$ to $\{I\}$ based on the rate gyro measurements $\boldsymbol{\omega}_m(t)$, the additional measurements $\mathbf{m}(t)$ and the knowledge of both ${}^I\boldsymbol{\omega}_E$ and ${}^I\mathbf{m}$.

III. Observer design

A. Auxiliary observer

As briefly discussed in the introduction, the idea of the paper is to obtain an estimate of a second vector, in body-fixed coordinates, whose counterpart, in inertial coordinates, is constant and known, so that two vectors are available for attitude estimation purposes. The observer for this second, auxiliary vector, is detailed in this section.

First, take the time derivative of the vector measurement $\mathbf{m}(t)$, which from (1), (2), and (4) can be written as

$$\dot{\mathbf{m}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t) - \boldsymbol{\omega}_E(t)\right]\mathbf{m}(t). \tag{5}$$

Next, notice that, under Assumption 1, the set

$$\left\{\frac{\mathbf{m}(t)}{\|\mathbf{m}(t)\|}, \frac{\mathbf{m}(t) \times \boldsymbol{\omega}_E(t)}{\|\mathbf{m}(t) \times \boldsymbol{\omega}_E(t)\|}, \frac{\mathbf{m}(t) \times \left[\mathbf{m}(t) \times \boldsymbol{\omega}_E(t)\right]}{\|\mathbf{m}(t) \times \left[\mathbf{m}(t) \times \boldsymbol{\omega}_E(t)\right]\|}\right\}.$$

provides an orthonormal basis in $\mathbb{R}^3$, which can be used to write the angular velocity of the Earth as

$$\boldsymbol{\omega}_E(t) = c_1\mathbf{m}(t) - c_2\mathbf{v}(t), \tag{6}$$

with

$$c_1 := \frac{{}^I\boldsymbol{\omega}_E \cdot {}^I\mathbf{m}}{\|{}^I\mathbf{m}\|^2}$$

and

$$c_2 := \frac{\left\|{}^I\mathbf{m} \times {}^I\boldsymbol{\omega}_E\right\|^2}{\left\|{}^I\mathbf{m} \times ({}^I\mathbf{m} \times {}^I\boldsymbol{\omega}_E)\right\|^2},$$

and where

$$\mathbf{v}(t) := \mathbf{m}(t) \times \left[\mathbf{m}(t) \times \boldsymbol{\omega}_E(t)\right]$$

is an auxiliary vector. Here, several norm, rotation, and cross product properties were employed. Notice that, using (3), (4), and some rotation and cross product properties, one can establish that

$$\mathbf{v}(t) = \mathbf{R}^T(t){}^I\mathbf{v}, \tag{7}$$

with

$$ {}^{I}\mathbf{v} := {}^{I}\mathbf{m} \times \left({}^{I}\mathbf{m} \times {}^{I}\boldsymbol{\omega}_E\right). $$

Taking the time derivative of (7), and using (1) and (2), yields

$$ \dot{\mathbf{v}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t) - \boldsymbol{\omega}_E(t)\right]\mathbf{v}(t). \tag{8} $$

This is an undesirable expression as the angular velocity of the Earth is not estimated directly. Instead, consider the decomposition (6) in (8), as well as in (5), which allows to write the nominal system dynamics

$$ \begin{cases} \dot{\mathbf{m}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t) + c_2\mathbf{v}(t)\right]\mathbf{m}(t) \\ \dot{\mathbf{v}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t) - c_1\mathbf{m}(t)\right]\mathbf{v}(t) \end{cases}. \tag{9} $$

Consider the observer for (9) given by

$$ \begin{cases} \dot{\hat{\mathbf{m}}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t) + c_2\hat{\mathbf{v}}(t) + \alpha_1\mathbf{m}(t)\times\hat{\mathbf{m}}(t)\right]\hat{\mathbf{m}}(t) \\ \dot{\hat{\mathbf{v}}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}_m(t) - c_1\mathbf{m}(t)\right]\hat{\mathbf{v}}(t) + \alpha_2\mathbf{m}(t)\times\hat{\mathbf{m}}(t) \end{cases}, \tag{10} $$

where $\hat{\mathbf{m}}(t) \in \mathbb{R}^3$ and $\hat{\mathbf{v}}(t) \in \mathbb{R}^3$ correspond to the estimates of $\mathbf{m}(t)$ and $\mathbf{v}(t)$, respectively, and $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$ are positive observer gains. Let $\tilde{\mathbf{m}}(t) := \mathbf{m}(t) - \hat{\mathbf{m}}(t)$ and $\tilde{\mathbf{v}}(t) := \mathbf{v}(t) - \hat{\mathbf{v}}(t)$ denote the estimation errors. Then, from (9) and (10) one may write the error dynamics

$$ \begin{cases} \dot{\tilde{\mathbf{m}}}(t) = & -\mathbf{S}\left[\boldsymbol{\omega}_m(t) + c_2\mathbf{v}(t) - c_2\tilde{\mathbf{v}}(t)\right]\tilde{\mathbf{m}}(t) \\ & +\mathbf{S}\left[\alpha_1\mathbf{m}(t)\times\tilde{\mathbf{m}}(t)\right]\tilde{\mathbf{m}}(t) \\ & +\alpha_1\left[\mathbf{m}(t)\times\hat{\mathbf{m}}(t)\right]\times\mathbf{m}(t) \\ & -c_2\tilde{\mathbf{v}}(t)\times\mathbf{m}(t) \\ \dot{\tilde{\mathbf{v}}}(t) = & -\mathbf{S}\left[\boldsymbol{\omega}_m(t) - c_1\mathbf{m}(t)\right]\tilde{\mathbf{v}}(t) + \alpha_2\mathbf{m}(t)\times\tilde{\mathbf{m}}(t) \end{cases}. \tag{11} $$

Before stating the main result of this section, compute the time derivative of $\mathbf{m}(t)\times\hat{\mathbf{m}}(t)$, which will be useful in the proof. This is given, using (9), (10), and the vector triple product by

$$ \begin{aligned} \tfrac{d}{dt}\mathbf{m}(t)\times\hat{\mathbf{m}}(t) = & -\left[\boldsymbol{\omega}_m(t) + c_2\mathbf{v}(t)\right]\times\left[\mathbf{m}(t)\times\hat{\mathbf{m}}(t)\right] \\ & -\alpha_1\left[\mathbf{m}(t)\cdot\hat{\mathbf{m}}(t)\right]\left[\mathbf{m}(t)\times\hat{\mathbf{m}}(t)\right] \\ & +c_2\left[\mathbf{m}(t)\cdot\hat{\mathbf{m}}(t)\right]\tilde{\mathbf{v}}(t) - c_2\left[\mathbf{m}(t)\cdot\tilde{\mathbf{v}}(t)\right]\hat{\mathbf{m}}(t). \end{aligned} \tag{12} $$

The following theorem addresses the stability and convergence properties of (11).

Theorem 1: Consider the state observer (10), with positive observer gains α_1 and α_2. Further assume that Assumptions 1 and 2 hold. Then, for all initial conditions such that $\|\hat{\mathbf{m}}(t_0)\| = \|{}^{I}\mathbf{m}\|$, it is true that:

i) the error variables $\tilde{\mathbf{m}}(t)$ and $\tilde{\mathbf{v}}(t)$ are bounded;

ii)
$$ \lim_{t\to\infty}\mathbf{m}(t)\times\hat{\mathbf{m}}(t) = \mathbf{0}; $$
and

iii)
$$ \lim_{t\to\infty}\frac{\mathbf{m}(t)}{\|{}^{I}\mathbf{m}\|}\times\left(\tilde{\mathbf{v}}(t)\times\frac{\mathbf{m}(t)}{\|{}^{I}\mathbf{m}\|}\right) = \mathbf{0}. $$

Proof: Consider the Lyapunov candidate function

$$ V(t) := \frac{1}{2}\|\tilde{\mathbf{m}}(t)\|^2 + \frac{1}{2}\frac{c_2}{\alpha_2}\|\tilde{\mathbf{v}}(t)\|^2, $$

which is positive definite as both c_2 and α_2 are positive scalars. Its time derivative can be shown to be given by

$$ \dot{V}(t) = -\alpha_1\|\mathbf{m}(t)\times\hat{\mathbf{m}}(t)\|^2. $$

As $\dot{V}(t) \le 0$ and $V(t) \ge 0$, it follows that $V(t)$ is bounded, which in turn implies the first statement of the theorem, i.e., the error variables $\tilde{\mathbf{m}}(t)$ and $\tilde{\mathbf{v}}(t)$ are bounded. Next, compute the time derivative of $\dot{V}(t)$, which is given by

$$ \ddot{V}(t) = -2\alpha_1\left[\mathbf{m}(t)\times\hat{\mathbf{m}}(t)\right]\cdot\frac{d}{dt}\mathbf{m}(t)\times\hat{\mathbf{m}}(t). $$

Notice that, by definition, $\mathbf{m}(t)$ has constant norm. As $\tilde{\mathbf{m}}(t)$ is bounded, it follows that so are the estimates $\hat{\mathbf{m}}(t)$. In addition, $\mathbf{v}(t)$ also has constant norm and, by assumption, $\boldsymbol{\omega}_m(t)$ is bounded. Also, it has already been shown that $\tilde{\mathbf{v}}(t)$ is bounded. Therefore, one can conclude that all terms in (12) are bounded and, consequently, $\ddot{V}(t)$ is also bounded. Thus, one further concludes that $\dot{V}(t)$ is uniformly continuous. Moreover, as $\dot{V}(t) \le 0$ and $V(t) \ge 0$, it follows that $V(t)$ converges to a limit. Now, using Barbalat's lemma, one concludes that $\dot{V}(t)$ converges to zero, which establishes the second statement of the theorem. Next, notice that the time derivative of (12) is, under the assumptions of the theorem, bounded, which means that (12) is uniformly continuous. Moreover, from the second statement of the theorem, $\mathbf{m}(t)\times\hat{\mathbf{m}}(t)$ converges to a limit. Therefore, invoking the Barbalat's lemma again, one can conclude that

$$ \lim_{t\to\infty}\frac{d}{dt}\mathbf{m}(t)\times\hat{\mathbf{m}}(t) = \mathbf{0}. \tag{13} $$

Now, taking the limit of both sides of (12) and noticing that all quantities therein are bounded, and using (13) as well as the second statement of the theorem, allows one to conclude that

$$ \lim_{t\to\infty}\left(\left[\mathbf{m}(t)\cdot\hat{\mathbf{m}}(t)\right]\tilde{\mathbf{v}}(t) - \left[\mathbf{m}(t)\cdot\tilde{\mathbf{v}}(t)\right]\hat{\mathbf{m}}(t)\right) = \mathbf{0} $$

or, equivalently,

$$ \lim_{t\to\infty}\left(\left[\frac{\mathbf{m}(t)}{\|{}^{I}\mathbf{m}\|}\cdot\frac{\hat{\mathbf{m}}(t)}{\|{}^{I}\mathbf{m}\|}\right]\tilde{\mathbf{v}}(t) - \left[\frac{\mathbf{m}(t)}{\|{}^{I}\mathbf{m}\|}\cdot\tilde{\mathbf{v}}(t)\right]\frac{\hat{\mathbf{m}}(t)}{\|{}^{I}\mathbf{m}\|}\right) = \mathbf{0}. \tag{14} $$

To establish the final result of the theorem, is is important to notice that the observer (10) preserves the norm of the estimates $\hat{\mathbf{m}}(t)$, which can be verified by computing its time derivative, which gives

$$ \frac{d}{dt}\|\hat{\mathbf{m}}(t)\| = 0. $$

Therefore, from the second statement of the theorem, one can conclude that, in the limit, either i) $\hat{\mathbf{m}}(t) = \mathbf{m}(t)$ or ii) $\hat{\mathbf{m}}(t) = -\mathbf{m}(t)$. Either way, one can conclude, from (14), that

$$ \lim_{t\to\infty}\left(\left[\frac{\mathbf{m}(t)}{\|{}^{I}\mathbf{m}\|}\cdot\frac{\mathbf{m}(t)}{\|{}^{I}\mathbf{m}\|}\right]\tilde{\mathbf{v}}(t) - \left[\frac{\mathbf{m}(t)}{\|{}^{I}\mathbf{m}\|}\cdot\tilde{\mathbf{v}}(t)\right]\frac{\mathbf{m}(t)}{\|{}^{I}\mathbf{m}\|}\right) = \mathbf{0}, $$

which using the vector triple product property finally allows to demonstrate the third statement of the theorem. ■

At this point, it is important to explain that one already has all the required information for attitude estimation, even though it was not shown that the error of (11) converges to zero. Indeed, it can be shown that

$$\frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|} \times \left(\mathbf{v}(t) \times \frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|}\right) = \mathbf{v}(t)$$

and hence

$$\frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|} \times \left(\tilde{\mathbf{v}}(t) \times \frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|}\right) = \mathbf{v}(t) - \frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|} \times \left(\hat{\mathbf{v}}(t) \times \frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|}\right). \tag{15}$$

An estimate of $\mathbf{v}(t)$, whose error converges to zero under the conditions of Theorem 1, is effectively provided by

$$\frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|} \times \left(\hat{\mathbf{v}}(t) \times \frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|}\right).$$

This estimate provides not only the second body-fixed vector required for attitude estimation but it also allows to obtain an estimate of the angular velocity of the Earth.

B. Attitude observer

This section presents the design and stability analysis of an attitude observer based upon the estimates of $\mathbf{v}(t)$, provided by the observer (10), the rate gyro measurements $\boldsymbol{\omega}_m(t)$, and the body-fixed vector measurements $\mathbf{m}(t)$, in addition to the information of the corresponding vectors in inertial coordinates, ${}^I\mathbf{v}$ and ${}^I\mathbf{m}$. First, substitute (2) in (1), which gives

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left[\boldsymbol{\omega}_m(t) - \boldsymbol{\omega}_E(t)\right].$$

Further using the decomposition of the angular velocity of the Earth (6), one can write the nominal rotation dynamics as

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left[\boldsymbol{\omega}_m(t) - c_1\mathbf{m}(t) + c_2\mathbf{v}(t)\right]. \tag{16}$$

Consider the attitude observer given by

$$\dot{\hat{\mathbf{R}}}(t) = \hat{\mathbf{R}}(t)\mathbf{S}\left[\boldsymbol{\omega}_o(t)\right], \tag{17}$$

where $\hat{\mathbf{R}}(t)$ is the attitude estimate, with

$$\begin{aligned}\boldsymbol{\omega}_o(t) := {} & \boldsymbol{\omega}_m(t) - c_1\mathbf{m}(t) + c_2\frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|} \times \left(\hat{\mathbf{v}}(t) \times \frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|}\right) \\ & +\alpha_3\mathbf{m}(t) \times \left[\hat{\mathbf{R}}^T(t){}^I\mathbf{m}\right] \\ & +\alpha_4\left[\frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|} \times \left(\hat{\mathbf{v}}(t) \times \frac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|}\right)\right] \times \left[\hat{\mathbf{R}}^T(t){}^I\mathbf{v}\right],\end{aligned}$$

where α_3 and α_4 are positive scalar observer gains and $\hat{\mathbf{v}}(t)$ is an estimate provided by (10). Define the error variable

$$\tilde{\mathbf{R}}(t) = \mathbf{R}(t)\hat{\mathbf{R}}^T(t). \tag{18}$$

Notice that, by construction, $\tilde{\mathbf{R}}(t) \in SO(3)$ and the estimation error converges to zero if an only if $\tilde{\mathbf{R}}(t)$ converges to an identity matrix. Its time derivative can be written, after some computations and simplifications, as

$$\dot{\tilde{\mathbf{R}}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}_f(t) + \tilde{\boldsymbol{\omega}}_f(t)\right]\tilde{\mathbf{R}}(t),$$

with

$$\boldsymbol{\omega}_f(t) := \alpha_3{}^I\mathbf{m} \times \left[\tilde{\mathbf{R}}(t){}^I\mathbf{m}\right] + \alpha_4{}^I\mathbf{v} \times \left[\tilde{\mathbf{R}}(t){}^I\mathbf{v}\right]$$

and

$$\begin{aligned}\tilde{\boldsymbol{\omega}}_f(t) := {} & -c_2\mathbf{R}(t)\left[\tfrac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|} \times \left(\tilde{\mathbf{v}}(t) \times \tfrac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|}\right)\right] \\ & -\alpha_4\left(\mathbf{R}(t)\left[\tfrac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|} \times \left(\tilde{\mathbf{v}}(t) \times \tfrac{\mathbf{m}(t)}{\|{}^I\mathbf{m}\|}\right)\right]\right) \times \left[\tilde{\mathbf{R}}^T(t){}^I\mathbf{v}\right].\end{aligned}$$

Before proceeding to the main result of the paper, it is convenient to introduce the equivalent error in terms of quaternions [18]. Let $((\tilde{s}(t), \tilde{\mathbf{r}}(t))$ denote the unit quaternion corresponding to the rotation error $\tilde{\mathbf{R}}(t)$, where $\tilde{s}(t)$ and $\tilde{\mathbf{r}}(t)$ are the so-called scalar and vector parts. The equivalence can be expressed by

$$\tilde{\mathbf{R}}(t) = \mathbf{I} + 2\tilde{s}(t)\mathbf{S}\left(\tilde{\mathbf{r}}(t)\right) + 2\left[\mathbf{S}\left(\tilde{\mathbf{r}}(t)\right)\right]^2 \tag{19}$$

and the corresponding quaternion dynamics are given by

$$\begin{cases} \dot{\tilde{s}}(t) = \frac{1}{2}\left[\boldsymbol{\omega}_f(t) + \tilde{\boldsymbol{\omega}}_f(t)\right] \cdot \tilde{\mathbf{r}}(t) \\ \dot{\tilde{\mathbf{r}}}(t) = -\frac{1}{2}\left[\tilde{s}(t)\mathbf{I} - \mathbf{S}\left(\tilde{\mathbf{r}}(t)\right)\right]\left[\boldsymbol{\omega}_f(t) + \tilde{\boldsymbol{\omega}}_f(t)\right] \end{cases}. \tag{20}$$

The following theorem is the main result of the paper.

Theorem 2: Define the parameterized set

$$\mathcal{R}(\epsilon) := \left\{\tilde{\mathbf{R}}\left(\tilde{s}, \tilde{\mathbf{r}}\right) \in SO(3) : |\tilde{s}| \geq \epsilon\right\}$$

and consider the attitude observer (17), where the estimates $\hat{\mathbf{v}}(t)$ are obtained using the state observer (10). Further assume that the conditions of Theorem 1 hold and suppose that both α_3 and α_4 are positive gains. Fix $0 < \epsilon < 1$. Then, for all initial conditions such that $\tilde{\mathbf{R}}(t) \in \mathcal{R}(\epsilon)$, the attitude error $\tilde{\mathbf{R}}(t)$ converges to the identity, i.e., $\hat{\mathbf{R}}(t)$ converges to $\mathbf{R}(t)$.

Proof: The proof consists essentially in showing that the error dynamics of $\tilde{\mathbf{r}}(t)$ are locally input-to-state stable with $\tilde{\mathbf{v}}(t)$ as input. As under the conditions of the theorem $\tilde{\mathbf{v}}(t)$ converges to zero, it follows that, for all initial conditions such that $\tilde{\mathbf{R}}\left(t_0\right) \in \mathcal{R}(\epsilon)$, $\tilde{\mathbf{r}}(t)$ converges to zero, which is equivalent to say that $\tilde{\mathbf{R}}(t)$ converges to the identity matrix, or $\hat{\mathbf{R}}(t)$ converges to $\mathbf{R}(t)$, therefore concluding the proof. The demonstration is done considering the Lyapunov-like function

$$V(t) := \frac{1}{2}\left\|\tilde{\mathbf{r}}(t)\right\|^2.$$

■

Notice that, in the definition of the region of convergence $\mathcal{R}(\epsilon)$ in Theorem 2, ϵ can be chosen as an arbitrarily small positive constant, which is equivalent to say that the initial error rotation angle must be smaller than π by an arbitrarily small positive margin. Therefore, the result is better characterized as semi-global.

Finally, in this paper only one body-fixed vector measurement was considered, as it is the most demanding case from the theoretical point of view. Nevertheless, the design can be extended to include additional body-fixed vector measurements.

IV. Simulation results

This section presents simulation results in order to show the performance achieved with the proposed solution. The local inertial frame was considered as the NED frame, centered at a latitude of $\varphi = 38.7138°$, a longitude of

TABLE I
OBSERVER GAINS

Time interval (s)	$\alpha_1 \left\|{}^I\mathbf{m}\right\|^2$	α_2
$[0, 300[$	10	10×10^{-1}
$[300, 420[$	10	5×10^{-2}
$[420, 600[$	5	2.5×10^{-2}
$[600, 720[$	2.5	1×10^{-2}
$[720, +\infty[$	2.5	5×10^{-3}

$\psi = 9.1394°$, and at sea level. The norm of the angular velocity of the Earth was set to $\left\|{}^I\boldsymbol{\omega}_E\right\| = 7.2921150 \times 10^{-5}$ rad/s, which is about 15 degrees per hour. In the NED frame, one has ${}^I\boldsymbol{\omega}_E = \left\|{}^I\boldsymbol{\omega}_E\right\| \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \end{bmatrix}^T$. As for the vector measurement $\mathbf{m}(t)$, it is assumed that magnetic field measurements are available. However, any other inertial vector could have been considered. In this case, ${}^I\mathbf{m}$ was set according to the 11^{th} generation of International Geomagnetic Reference Field for the latitude, longitude, and altitude previously described. Notice that, with this choice, Assumption 1 is satisfied.

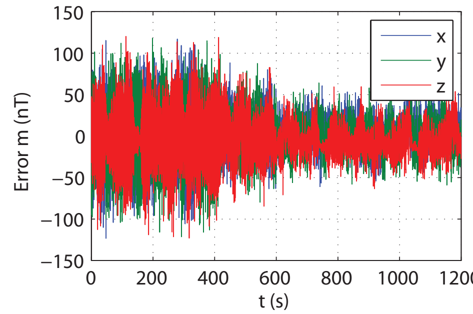
The initial attitude of the platform was set to $\mathbf{R}(0) = \mathbf{I}$ and the evolution of the angular velocity is given by

$$\boldsymbol{\omega}(t) = \begin{bmatrix} 5\frac{\pi}{180}\sin\left(\frac{2\pi}{60}t\right) \\ \frac{\pi}{180}\sin\left(\frac{2\pi}{180}t\right) \\ -2\frac{\pi}{180}\sin\left(\frac{2\pi}{300}t\right) \end{bmatrix} \text{ (rad/s)}.$$

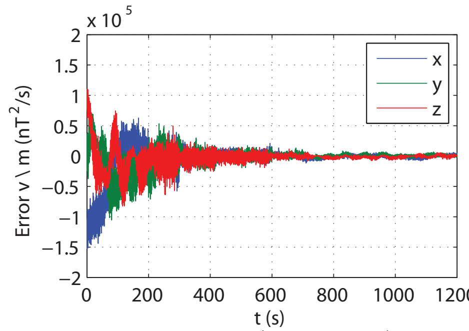
In the simulations, the measurements of the magnetometer were assumed to be corrupted by zero-mean white Gaussian noise, with standard deviation of 150 nT, which corresponds to the worst case specification of the triaxial magnetometer of the nanoIMU NA02-0150F50. The rate gyro measurements noise is characterized by an angle random walk of $4°/\text{hr}/\sqrt{\text{Hz}}$, which corresponds to the KVH DSP-3000 fiber optic gyro. A sampling frequency of 100 Hz was considered and the fourth-order Runge-Kutta method was employed in the simulations.

In order to ensure both fast convergence speed and good steady-state performance of the first observer, a set of piecewise constant gains was chosen. Notice that this does not impact the stability of the error dynamics as a finite set of transitions is considered. Indeed, for each particular choice of observer gains, the error dynamics are stable, and once the final gain is set, the results previously derived apply. These gains are described in Table I. As for the second observer, the gains were set to $\alpha_3 = 0.02/\left\|{}^I\mathbf{m}\right\|^2$ and $\alpha_4 = 0.4/\left\|{}^I\mathbf{v}\right\|^2$. Additional tweaking could have ensured faster convergence rate with two sets of gains. As in most nonlinear observers, these gains were chosen empirically, although the relative gains of the second observer are related to the error noise of its observations. The initial estimate of the first observer $\hat{\mathbf{m}}(0)$ was set identical to the first measurement, while $\hat{\mathbf{v}}(0)$ was set to zero. The initial attitude estimate was set such that the initial angle rotation error is very close to 180 degrees.

The initial convergence of the errors $\tilde{\mathbf{m}}(t)$ and $\tilde{\mathbf{v}}(t)$ is depicted in Fig. 1. While the convergence of the observer is fast, different gains are required, as detailed in Table I, in order to ensure an adequate steady-state level of error,



(a) Error $\tilde{\mathbf{m}}(t)$



(b) Error $\frac{\mathbf{m}(t)}{\left\|{}^I\mathbf{m}\right\|} \times \left(\tilde{\mathbf{v}}(t) \times \frac{\mathbf{m}(t)}{\left\|{}^I\mathbf{m}\right\|}\right)$

Fig. 1. Initial convergence of the errors of the first observer

as it will be detailed shortly. The initial convergence of the attitude error $\tilde{\mathbf{R}}(t) - \mathbf{I}$ is shown in Fig. 2. These plots show that the error converges to a neighborhood of zero. In the absence of noise, the errors converge to zero.

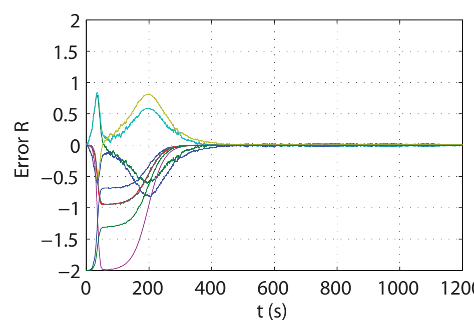


Fig. 2. Initial convergence of the error $\tilde{\mathbf{R}}(t) - \mathbf{I}$

In order to evaluate the performance of the attitude observer, the steady-state standard deviation of the errors is depicted in Table II. These values should be compared to the magnitude of the corresponding variables, which is roughly 4.5×10^4 nT for $\mathbf{m}(t)$ and 1.5×10^5 nT^2/s for $\mathbf{v}(t)$. Evidently, the observer achieves very good results. Using the Euler angle-axis representation for the rotation error,

$$\begin{aligned} \tilde{\mathbf{R}}(t) &= \mathbf{I}\cos\left(\tilde{\theta}(t)\right) + \left[1 - \cos\left(\tilde{\theta}(t)\right)\right]\tilde{\mathbf{d}}(t)\tilde{\mathbf{d}}^T(t) \\ &\quad -\mathbf{S}\left(\tilde{\mathbf{d}}(t)\right)\sin\left(\tilde{\theta}(t)\right), \end{aligned}$$

where $0 \leq \tilde{\theta}(t) \leq \pi$ and $\tilde{\mathbf{d}}(t) \in \mathbb{R}^3$, $\left\|\tilde{\mathbf{d}}(t)\right\| = 1$, are

TABLE II
STANDARD DEVIATION OF THE STEADY-STATE ERROR

Variable	Standard deviation
$\tilde{\mathbf{m}}(t)$ (nT)	15
$\tilde{\mathbf{v}}(t)$ (nT^2/s)	1700
$\tilde{\mathbf{R}}(t) - \mathbf{I}$	0.01

the angle and axis that represent the rotation error, the performance of the proposed solution is identified from the evolution of $\tilde{\theta}$. The mean angle error, computed for $t \geq 2400$ s, is $0.636°$, which is a very good result.

Finally, a brief comparison between the observer presented in this paper and the one proposed in the companion paper [16] is offered. Both observers have cascade structures and they are both computationally efficient and equivalent, with the same number of states. In the first stage of both observers one of the states corresponds to the measured body-fixed vector, with both observers achieving very similar levels of performance. The only difference is that in this paper the norm of the vector is preserved, while in the companion paper [16] the norm characteristics are not imposed by the observer, although it converges exponentially to the true value. The other state of the first stage of both observers is not comparable as it corresponds to different quantities. In terms of attitude estimation performance, both observers achieve very similar results, with a mean angle error of $0.636°$ for the observer proposed herein, against $0.586°$ obtained in the companion paper [16]. This is so in spite of very different approaches to the problem: in the companion paper the topological characteristics are discarded (but verified asymptotically) and globally exponential stability is obtained; in the solution proposed herein, the design is explicitly made on $SO(3)$ and as a consequence only semi-global asymptotic stability was shown.

V. CONCLUSIONS

The problem of attitude estimation has long received the attention of the research community, as it is vital to the successful operation of robotic platforms. This paper proposes a novel attitude estimation solution that is based solely on measurements of a single body-fixed vector and the angular velocity provided by a set of three high-grade rate gyros, sensitive to the angular velocity of the Earth around its own axis. In short, an estimate of a second body-fixed vector is obtained, with an auxiliary observer that preserves the norm of the estimates of the measured body-fixed vector. This second body-fixed vector, together with the first, allows not only to obtain an estimate of the angular velocity of the Earth, required for attitude filtering purposes, but it also works as a second vector for attitude estimation. An attitude observer that preserves the topological properties of the attitude estimates is then proposed and the overall stability of the system was detailed, where it was shown that, for all initial errors such that the error rotation angle is smaller than π by an arbitrarily small positive margin, the attitude error converges to zero. Simulations results were presented that illustrate the achievable performance. Future work will include extensive Monte Carlo simulations prior to experimental evaluation.

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