# Nonlinear observability and observer design through state augmentation

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*Abstract*— **This paper addresses the problems of observability analysis and state observer design for nonlinear systems based on the study of linear systems which mimic their dynamics. It is shown that, if a nonlinear system can be related to a linear system through an appropriate transformation of the state variables, the observability analysis of the linear system can determine observability or non-observability of the nonlinear system, depending on the properties of the state transformation function. Conditions are also derived for state observers for the linear systems to double as state observers for the original nonlinear systems, as well as retaining exponential stability properties. To illustrate the usefulness of those results, several application examples are detailed.**

## I. Introduction

The subject of nonlinear observer design has been extensively studied by the research community over the past few decades, resulting in a myriad of different approaches and compelling contributions. While the same problems have been definitely answered for the linear case with the introduction of the observability Gramian and the Luenberger observer, the overwhelming diversity and complexity found in the world of nonlinear systems has not yet allowed for a definitive, unified solution. On the topic of nonlinear observer design, in [9] a transformation of the state variables is used to achieve a linear, time-invariant equivalent system, and [5] addressed the problem for a certain class of uniformly observable systems. Other notable works include [10] and [1], which consider globally Lipschitz and output-to-state stable systems, respectively. Similarly, the definition of observability for nonlinear systems has seen several different approaches, from the early concepts of weak observability [6] to the more recent Z-observability [7]. For a recent, detailed survey on the matter, see [8].

The problem addressed in this paper is the observability analysis of nonlinear systems through the study of linear systems which mimic their dynamics. It is shown that, if a nonlinear system can be related to a linear system through a transformation of the state variables (not necessarily of the same dimensionality), observability of the linear system can translate into observability or non-observability of the nonlinear system, depending on the injectivity or lack thereof of the state transformation function. Then, it is further shown that a state observer for the linear system with globally exponentially stable error dynamics can double as a state observer for the nonlinear system, retaining the exponential convergence properties if the pseudo-inverse of the state transformation verifies the Lipschitz condition. These results are then applied to several examples, ranging from real-life application scenarios to pertinent toy examples to show their usefulness and versatility. This work is motivated by previous research by the authors such as in [11], [3], and [12], in which nonlinear estimation problems are addressed via introduction of new state variables, resulting in linear systems which mimic the dynamics of the original nonlinear systems. The aim here is to consolidate the theoretical basis behind those results, and at the same time to facilitate the observability analysis and observer design processes for future work on nonlinear estimation problems.

The rest of the paper is organized as follows. Section II formally details the problem at hand and introduces several key definitions which are used throughout the rest of the paper. In Section III, the main results of the paper are presented, supported by several application examples in Section IV. Finally, Section V summarizes the main conclusions of the paper.

### A. Notation

Throughout the paper, $\mathbf{I}$ and $\mathbf{0}$ denote the identity matrix and a vector or matrix of zeros, respectively, both of appropriate dimensions. When the size of the matrix cannot be inferred from the context, $\mathbf{I}_n$ is used to denote the $n$ by $n$ identity matrix. For two vectors $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$, $\mathbf{a} \cdot \mathbf{b}$ denotes the inner product and $\|\mathbf{a}\|$ is the Euclidian norm of $\mathbf{a}$.

## II. Problem Statement

Consider a continuous-time, nonlinear, nonautonomous dynamic system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) \end{cases}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1}$$

in which $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, and $\mathbf{y}(t) \in \mathbb{R}^o$ are the state, input, and output of the system, respectively, $\mathbf{x}_0 \in \mathbb{R}^n$ is the initial condition of the system at $t = t_0$, and $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\mathbf{h} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^o$ are known functions of time and the state and input of the system.

In the context of this paper, nonlinear observability is defined as follows.

*Definition 1 (Observability for a given $\mathbf{u}$):* The nonlinear system (1) is observable on $[t_0, t_f]$ for a given $\mathbf{u}(t)$ if and only if for that input $\mathbf{u}(t)$ the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ is uniquely determined by the response $\mathbf{y}(t)$ of the system for $t \in [t_0, t_f]$.

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*Definition 2 (Observability):* The nonlinear system (1) is observable on $[t_0, t_f]$ if and only if it is observable for any given $\mathbf{u} : [t_0, t_f] \to \mathbb{R}^m$.

While for linear systems observability does not depend on the input of the system, it is necessary to factor in the input $\mathbf{u}$ when studying nonlinear systems. For example, consider a system with output $\mathbf{y}(t) = \mathbf{x}(t) \cdot \mathbf{u}(t)$ (assuming $\mathbf{x}$ and $\mathbf{u}$ have the same dimensionality). In this case, the unforced system ($\mathbf{u}(t) = \mathbf{0}$ for all $t \in [t_0, t_f]$) is not observable, since the output is always zero. However, for a nonzero input the system might be observable, depending on the dynamics of the system.

Now, consider the following dynamic system, affine in its state,

$$\begin{cases} \dot{\mathbf{w}}(t) = \mathbf{A}(t, \mathbf{u}(t), \mathbf{y}(t))\mathbf{w}(t) + \mathbf{B}(t, \mathbf{u}(t), \mathbf{y}(t))\mathbf{v}(t) \\ \mathbf{z}(t) = \mathbf{C}(t, \mathbf{u}(t), \mathbf{y}(t))\mathbf{w}(t) \\ \mathbf{w}(t_0) = \mathbf{w}_0 \end{cases}, \tag{2}$$

in which $\mathbf{w}(t) \in \mathbb{R}^p$, $\mathbf{w}_0 \in \mathbb{R}^p$, $\mathbf{v}(t) \in \mathbb{R}^m$, and $\mathbf{z}(t) \in \mathbb{R}^o$ are the state, initial condition, input, and output of the system, and $\mathbf{u}(t)$ and $\mathbf{y}(t)$ are the input and output of the nonlinear system (1). $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are matrix-valued functions of appropriate dimensions. Note that, as is usual in observability problems, both $\mathbf{u}(t)$ and $\mathbf{y}(t)$ are assumed to be known functions of time, and therefore the system (2) can be regarded as a linear time-varying (LTV) system for the sake of observability analysis, see [3, Lemma 1].

In some cases, the nonlinear system (1) and the LTV system (2) can be related through finding an appropriate nonlinear state transformation such that the system (2) effectively replicates the behavior of the nonlinear system (1), which motivates the introduction of the following definition.

*Definition 3 (Mimicking):* Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^p$. If setting the initial condition of (2) to $\mathbf{w}_0 = \mathbf{T}(\mathbf{x}_0)$ and its input to $\mathbf{v}(t) = \mathbf{u}(t)$ for all $t \in [t_0, t_f]$ makes it so that

$$\begin{cases} \mathbf{w}(t) = \mathbf{T}(\mathbf{x}(t)) \\ \mathbf{z}(t) = \mathbf{y}(t) \end{cases}$$

holds for all $t \in [t_0, t_f]$, the system (2) is said to mimic the dynamics of the system (1) on $[t_0, t_f]$.

The two main problems addressed in this paper are the following:

- If a linear system mimics the dynamics of a nonlinear system, does observability of the linear system also imply observability of the nonlinear system?
- Can a state observer for a linear system double as a state observer for a nonlinear system which it mimics?

## III. Observability and State Observer Design

The following results establish sufficient conditions for observability of the nonlinear system (1).

*Theorem 1:* Consider the nonlinear system (1) with a given input $\mathbf{u}(t)$, and assume that there exists a function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^p$ such that the system (2) mimics the dynamics of (1). Suppose that the system (2) is observable on $[t_0, t_f]$. Then, the nonlinear system (1) is observable on $[t_0, t_f]$ for the given input $\mathbf{u}(t)$ if and only if the function $\mathbf{T}$ is injective.

*Proof:* Let $\mathbf{x}_A \in \mathbb{R}^n$ and $\mathbf{x}_B \in \mathbb{R}^n$, with $\mathbf{x}_A \neq \mathbf{x}_B$, and denote by $\mathbf{y}_A(t)$ and $\mathbf{y}_B(t)$ the outputs of the nonlinear system (1) with input $\mathbf{u}(t)$ and initial conditions $\mathbf{x}_A$ and $\mathbf{x}_B$, respectively. Define $\mathbf{w}_A := \mathbf{T}(\mathbf{x}_A)$ and $\mathbf{w}_B := \mathbf{T}(\mathbf{x}_B)$, and denote by $\mathbf{z}_A(t)$ and $\mathbf{z}_B(t)$ the outputs of the system (2) with input $\mathbf{v}(t) = \mathbf{u}(t)$ and initial conditions $\mathbf{w}_A$ and $\mathbf{w}_B$, respectively. Since the system (2) mimics the dynamics of (1), it follows that $\mathbf{y}_A(t) = \mathbf{z}_A(t)$ and $\mathbf{y}_B(t) = \mathbf{z}_B(t)$ for all $t \in [t_0, t_f]$.

Now, suppose that $\mathbf{T}$ is injective. Then, as $\mathbf{x}_a \neq \mathbf{x}_b$ it follows that $\mathbf{w}_A \neq \mathbf{w}_B$. Since the system (2) is observable, it follows that there exists a $t^* \in [t_0, t_f]$ such that $\mathbf{z}_A(t^*) \neq \mathbf{z}_B(t^*)$ or, equivalently, $\mathbf{y}_A(t^*) \neq \mathbf{y}_B(t^*)$. Thus, for the given input $\mathbf{u}(t)$ of the nonlinear system (1), any two different initial conditions $\mathbf{x}_A$ and $\mathbf{x}_B$ yield distinct responses $\mathbf{y}_A : [t_0, t_f] \to \mathbb{R}^o$ and $\mathbf{y}_B(t) : [t_0, t_f] \to \mathbb{R}^o$ or, in other words, the output $\mathbf{y} : [t_0, t_f] \to \mathbb{R}^o$ uniquely determines the initial condition $\mathbf{x}(t_0)$ of the system. Therefore, the nonlinear system (1) is observable on $[t_0, t_f]$ for the input $\mathbf{u}(t)$.

On the other hand, suppose that $\mathbf{T}$ is not injective. Then, it follows that there exist $\mathbf{x}_C \in \mathbb{R}^n$ and $\mathbf{x}_D \in \mathbb{R}^n$, with $\mathbf{x}_C \neq \mathbf{x}_D$, such that $\mathbf{w}_C := \mathbf{T}(\mathbf{x}_C) = \mathbf{T}(\mathbf{x}_D)$. Denote by $\mathbf{y}_C(t)$ and $\mathbf{y}_D(t)$ the outputs of the nonlinear system (1) with input $\mathbf{u}(t)$ and initial conditions $\mathbf{x}_C$ and $\mathbf{x}_D$, respectively, and by $\mathbf{z}_C(t)$ the output of the system (2) with input $\mathbf{v}(t) = \mathbf{u}(t)$ and initial condition $\mathbf{w}_C$. Since the system (2) mimics the dynamics of (1), this implies that $\mathbf{z}_C(t) = \mathbf{y}_C(t) = \mathbf{y}_D(t)$ for all $t \in [t_0, t_f]$. Thus, the initial conditions $\mathbf{x}_C$ and $\mathbf{x}_D$ of the system (1) yield identical outputs for $t \in [t_0, t_f]$, which means the nonlinear system (1) is not observable. ∎

*Corollary 1:* Consider the nonlinear system (1), and assume that there exists a function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^p$ such that the system (2) mimics the dynamics of (1). Suppose that the system (2) is observable on $[t_0, t_f]$ for any $\mathbf{u} : [t_0, t_f] \to \mathbb{R}^m$. Then, the nonlinear system (1) is observable on $[t_0, t_f]$ if and only if the function $\mathbf{T}$ is injective.

The following technical result follows directly from Theorem 1, and will be useful in the derivation of the other results detailed in this section.

*Corollary 2:* Consider the nonlinear system (1) with a given input $\mathbf{u}(t)$ and the LTV system (2) with $\mathbf{v}(t) = \mathbf{u}(t)$. Assume that there exists an injective state transformation function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^p$ such that the system (2) mimics the dynamics of (1), and that the system (2) is observable on $[t_0, t_f]$. If $\mathbf{z}(t) = \mathbf{y}(t)$ for all $t \in [t_0, t_f]$, then $\mathbf{w}_0 = \mathbf{T}(\mathbf{x}_0)$.

*Proof:* Since the LTV system (2) mimics the dynamics of the nonlinear system (1) and the transformation $\mathbf{T}$ is injective, it follows from Theorem 1 that the nonlinear system (1) is observable and thus, for the given $\mathbf{u}(t)$, the initial condition $\mathbf{x}_0$ of the nonlinear system (1) is uniquely determined by the output $\mathbf{y}(t)$. On the other hand, as the LTV system (2) is observable, for the input $\mathbf{v}(t) = \mathbf{u}(t)$ and the output $\mathbf{z}(t) = \mathbf{y}(t)$, its initial condition $\mathbf{w}_0$ is also uniquely determined.

Now, consider the LTV system (2) with $\mathbf{w}(t_0) = \mathbf{T}(\mathbf{x}_0)$ and $\mathbf{v}(t) = \mathbf{u}(t)$. As the LTV system (2) mimics the dynamics of (1), it follows that $\mathbf{z}(t) = \mathbf{y}(t)$ for all $t \in [t_0, t_f]$. Then, as the initial condition $\mathbf{w}_0$ of the LTV system (2) is uniquely determined by the input and output of the system, it must be $\mathbf{w}_0 = \mathbf{T}(\mathbf{x}_0)$. ∎

While the previous results established conditions for observability of the nonlinear system (1), the transformed LTV system (2) can be also used to design state observers for the nonlinear system (1). In fact, if $\mathbf{T}(\mathbf{x})$ is injective, $\mathbf{x}$ is always uniquely determined by the value of $\mathbf{T}(\mathbf{x})$, and a function with the following property exists:

*Definition 4 (Left inverse state transformation):* A function $\mathbf{T}' : \mathbb{R}^p \to \mathbb{R}^n$ is a left inverse of the state transformation $\mathbf{T}(\mathbf{x})$ if and only if $\mathbf{T}'(\mathbf{T}(\mathbf{x})) = \mathbf{x}$ holds for any $\mathbf{x} \in \mathbb{R}^n$.

If such a left inverse transformation is known, the observer design process follows naturally: design a state observer for the linear system (2) applying classical linear systems theory, and denote its state estimate by $\hat{\mathbf{w}}(t)$. Then, a state estimate for the nonlinear system (1) is simply $\hat{\mathbf{x}}(t) := \mathbf{T}'(\hat{\mathbf{w}}(t))$. However, in doing so, the convergence properties of the observer for the linear system (2) might not apply to the new state estimate for the nonlinear system (1), mainly because the state estimate $\hat{\mathbf{w}}(t)$ might not be meaningful with respect to the nonlinear system (1) for the cases in which $\mathbf{T}(\hat{\mathbf{x}}(t)) \neq \hat{\mathbf{w}}(t)$. The results that follow detail sufficient conditions for the state estimate $\hat{\mathbf{x}}(t)$ to retain the convergence properties of the state estimate $\hat{\mathbf{w}}(t)$ of the linear observer.

*Lemma 1:* Consider the nonlinear system (1) with a given input $\mathbf{u}(t)$ and the LTV system (2) with $\mathbf{v}(t) = \mathbf{u}(t)$ and $\mathbf{z}(t) = \mathbf{y}(t)$. Assume that there exists an injective state transformation function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^p$ such that the system (2) mimics the dynamics of (1), with a left inverse transformation $\mathbf{T}' : \mathbb{R}^p \to \mathbb{R}^n$. Suppose that the system (2) is observable on $[t_0, t_f]$, and that there exists a positive scalar constant $\alpha$ such that

$$\|\mathbf{T}'(\mathbf{w}_a) - \mathbf{T}'(\mathbf{w}_b)\| \leq \alpha \|\mathbf{w}_a - \mathbf{w}_b\| \tag{3}$$

holds for any $\mathbf{w}_a \in \mathbb{R}^p$ and $\mathbf{w}_b \in \mathbb{R}^p$. Finally, assume that there exists a state observer for the transformed LTV system (2) with globally exponentially stable error dynamics, and denote by $\hat{\mathbf{w}}(t) \in \mathbb{R}^p$ its state estimate. Then, for the given input $\mathbf{u}(t)$, the state estimate

$$\hat{\mathbf{x}}(t) := \mathbf{T}'(\hat{\mathbf{w}}(t)) \tag{4}$$

converges exponentially fast to the state $\mathbf{x}(t)$ of the nonlinear system (1), in the sense that for any initial $\mathbf{x}(t_0)$ and $\hat{\mathbf{x}}(t_0)$ there exist positive scalar constants $\gamma \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| \leq \gamma e^{-\lambda(t-t_0)}$$

for all $t \in [t_0, t_f]$.

*Proof:* As the state observer for the transformed system (2) is assumed to have globally exponentially stable error dynamics, there exist positive scalar constants $\beta \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$\|\mathbf{w}(t) - \hat{\mathbf{w}}(t)\| \leq \beta \|\mathbf{w}(t_0) - \hat{\mathbf{w}}(t_0)\| e^{-\lambda(t-t_0)} \tag{5}$$

for all $t \in [t_0, t_f]$. On the other hand, as it is assumed that $\mathbf{v}(t) = \mathbf{u}(t)$ and $\mathbf{z}(t) = \mathbf{y}(t)$ and that the LTV system (2) is observable for the given $\mathbf{u}(t)$ and mimics the dynamics of the nonlinear system (1), it follows from Corollary 2 that $\mathbf{w}_0 = \mathbf{T}(\mathbf{x}_0)$. Then, it follows from the definition of mimicking that

$$\mathbf{w}(t) = \mathbf{T}(\mathbf{x}(t)) \tag{6}$$

for all $t \in [t_o, t_f]$. Applying the left inverse transformation to both sides of (6) yields

$$\mathbf{T}'(\mathbf{w}(t)) = \mathbf{T}'(\mathbf{T}(\mathbf{x}(t))) = \mathbf{x}(t). \tag{7}$$

From (4) and (7), the norm of the estimation error for the nonlinear system (1) can be expressed as

$$\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| = \|\mathbf{T}'(\mathbf{w}(t)) - \mathbf{T}'(\hat{\mathbf{w}}(t))\|. \tag{8}$$

Then, using the bounds in (3) and (5), it follows from (8) that

$$\begin{aligned}\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| &= \|\mathbf{T}'(\mathbf{w}(t)) - \mathbf{T}'(\hat{\mathbf{w}}(t))\| \leq \\ &\leq \alpha\|\mathbf{w}(t) - \hat{\mathbf{w}}(t)\| \leq \alpha\beta\|\mathbf{w}(t_0) - \hat{\mathbf{w}}(t_0)\| e^{-\lambda(t-t_0)},\end{aligned}$$

which implies that, for any initial condition there exists a positive scalar constant $\gamma \in \mathbb{R}$ such that

$$\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| \leq \gamma e^{-\lambda(t-t_0)}$$

for all $t \in [t_0, t_f]$. ∎

*Corollary 3:* Suppose that the conditions of Lemma 1 hold for any $\mathbf{u} : [t_0, t_f] \to \mathbb{R}^m$. Then, the state estimate (4) converges exponentially fast to the state $\mathbf{x}(t)$ of the nonlinear system (1) for any $\mathbf{u} : [t_0, t_f] \to \mathbb{R}^m$.

*Lemma 2:* Consider the nonlinear system (1) with a given input $\mathbf{u}(t)$ and the LTV system (2) with $\mathbf{v}(t) = \mathbf{u}(t)$ and $\mathbf{z}(t) = \mathbf{y}(t)$. Assume that there exists an injective state transformation function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^p$ such that the system (2) mimics the dynamics of (1), with a left inverse transformation $\mathbf{T}' : \mathbb{R}^p \to \mathbb{R}^n$. Suppose that the system (2) is observable on $[t_0, t_f]$, and that there exist positive scalar constants $\alpha$ and $\beta$ such that

$$\alpha\|\mathbf{w}_a - \mathbf{w}_b\| \leq \|\mathbf{T}'(\mathbf{w}_a) - \mathbf{T}'(\mathbf{w}_b)\| \leq \beta\|\mathbf{w}_a - \mathbf{w}_b\| \tag{9}$$

holds for any $\mathbf{w}_a \in \mathbb{R}^p$ and $\mathbf{w}_b \in \mathbb{R}^p$. Finally, assume that there exists a state observer for the transformed LTV system (2) with globally exponentially stable error dynamics, and denote by $\hat{\mathbf{w}}(t) \in \mathbb{R}^p$ its state estimate. Then, for the given input $\mathbf{u}(t)$, the state estimate

$$\hat{\mathbf{x}}(t) := \mathbf{T}'(\hat{\mathbf{w}}(t)) \tag{10}$$

converges globally exponentially fast to the state $\mathbf{x}(t)$ of the nonlinear system (1).

*Proof:* Following the same steps as in the proof for Lemma 2, there exist a positive scalar constants $\gamma$ and $\lambda$ such that

$$\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| \leq \gamma\|\mathbf{w}(t_0) - \hat{\mathbf{w}}(t_0)\| e^{-\lambda(t-t_0)}. \tag{11}$$

On the other hand, since $\mathbf{T}'(\mathbf{w}(t_0)) = \mathbf{x}(t_0)$ and $\mathbf{T}'(\hat{\mathbf{w}}(t_0)) = \hat{\mathbf{x}}(t_0)$, the left hand side of (9) implies that

$$\begin{aligned}\|\mathbf{w}(t_0) - \hat{\mathbf{w}}(t_0)\| &\leq \frac{1}{\alpha}\|\mathbf{T}'(\mathbf{w}(t_0)) - \mathbf{T}'(\hat{\mathbf{w}}(t_0))\| = \\ &= \frac{1}{\alpha}\|\mathbf{x}(t_0) - \hat{\mathbf{x}}(t_0)\|.\end{aligned} \tag{12}$$

Finally, substituting (12) in (11) yields

$$\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| \leq \frac{\gamma}{\alpha}\|\mathbf{x}(t_0) - \hat{\mathbf{x}}(t_0)\| e^{-\lambda(t-t_0)},$$

which concludes the proof. ∎

*Corollary 4:* Suppose that the conditions of Lemma 2 hold for any $\mathbf{u} : [t_0, t_f] \to \mathbb{R}^m$. Then, the state estimate (10) converges globally exponentially fast to the state $\mathbf{x}(t)$ of the nonlinear system (1) for any $\mathbf{u} : [t_0, t_f] \to \mathbb{R}^m$.

One particularly useful type of state transformation functions is the kind built by adding new state variables to the original state of the nonlinear system (1):

*Definition 5 (State augmentation function):* Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^p$, with $p > n$. The transformation $\mathbf{T}$ is said to be a state augmentation function if it is of the form

$$\mathbf{T}(\mathbf{x}) = \begin{bmatrix} \mathbf{x} \\ \mathbf{T}_a(\mathbf{x}) \end{bmatrix}$$

for some $\mathbf{T}_a : \mathbb{R}^n \to \mathbb{R}^{(p-n)}$.

State augmentation functions can be applied in several relevant estimation problems, as it is discussed in more detail in the next section, and are very easy to use with state observers, as a left inverse which verifies the conditions of Lemma 1 (but not Lemma 2) is given by $\mathbf{T}'(\mathbf{w}) = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix} \mathbf{w}$.

*Remark 1:* The proofs for Corollaries 1, 3, and 4 were omitted as they follow naturally from Theorem 1, Lemma 1, and Lemma 2, respectively.

## IV. Application examples

This section details examples of applications of the results of the previous section to different nonlinear systems. The first two examples focus on localization problems based on range measurements, as those types of problems were the main motivation behind the development of this framework. To show that the usefulness of these results is not limited to this specific type of problems, a toy example with different dynamics is also presented.

The following notation is used in this section for integrals of functions of time: for a vector-valued function of time $\mathbf{a}(t) \in \mathbb{R}^s$,

$$\mathbf{a}^{[1]}(t_f, t_i) := \int_{t_i}^{t_f} \mathbf{a}(\sigma) d\sigma = \begin{bmatrix} a_1^{[1]}(t_f, t_i) \\ a_2^{[1]}(t_f, t_i) \\ \vdots \\ a_s^{[1]}(t_f, t_i) \end{bmatrix} \tag{13}$$

and

$$\mathbf{a}^{[2]}(t_f, t_i) := \int_{t_i}^{t_f} \int_{t_i}^{\sigma_1} \mathbf{a}(\sigma_1) d\sigma_1 d\sigma_2 = \begin{bmatrix} a_1^{[2]}(t_f, t_i) \\ a_2^{[2]}(t_f, t_i) \\ \vdots \\ a_s^{[2]}(t_f, t_i) \end{bmatrix}. \tag{14}$$

### A. Single range localization with velocity measurements

Consider the following nonlinear system:

$$\begin{cases} \dot{\mathbf{x}}_1(t) = \mathbf{x}_2(t) + \mathbf{u}(t) \\ \dot{\mathbf{x}}_2(t) = \mathbf{0} \\ y(t) = \|\mathbf{x}_1(t)\| \end{cases}, \tag{15}$$

with $\mathbf{x}_1(t) \in \mathbb{R}^3$, $\mathbf{x}_2(t) \in \mathbb{R}^3$, $\mathbf{u}(t) \in \mathbb{R}^3$, and $y(t) \in \mathbb{R}$. This nonlinear system can be used to model the problem of estimating the position and velocity of a vehicle subject to a constant unknown current based on range measurements to a single source, see [3] for a comprehensive treatment of the problem. In this context, $\mathbf{x}_1(t)$ is the position of the vehicle relative to the source, $\mathbf{x}_2(t)$ is the velocity of the current, $\mathbf{u}(t)$ is the velocity of the vehicle relative to the fluid in which it operates, which is assumed to be measured, and $y(t)$ is the available range measurement.

Now, consider the following system, linear in its state:

$$\begin{cases} \dot{\mathbf{w}}(t) = \mathbf{A}(t, \mathbf{u}(t), \mathbf{y}(t))\mathbf{w}(t) + \mathbf{B}\mathbf{v}(t) \\ z(t) = \mathbf{C}\mathbf{w}(t) \end{cases}, \tag{16}$$

with $\mathbf{w}(t) \in \mathbb{R}^9$, $\mathbf{v}(t) \in \mathbb{R}^3$, $z(t) \in \mathbb{R}$,

$$\mathbf{A}(t, \mathbf{u}(t), \mathbf{y}(t)) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{1}{y(t)}\mathbf{u}^\mathsf{T}(t) & \mathbf{0} & 0 & \frac{1}{y(t)} & 0 \\ \mathbf{0} & \mathbf{u}^\mathsf{T}(t) & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^\mathsf{T},$$

and

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \end{bmatrix}.$$

It is straightforward to verify that the system (16) mimics the dynamics of (15), with associated state augmentation function

$$\mathbf{T}(\mathbf{x}_1, \mathbf{x}_2) = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \|\mathbf{x}_1\| \\ \mathbf{x}_1 \cdot \mathbf{x}_2 \\ \mathbf{x}_2 \cdot \mathbf{x}_2 \end{bmatrix} \in \mathbb{R}^9.$$

As the transformation is injective, the observability and observer design problem for the nonlinear system (15) can be solved by applying linear systems theory to the system (16). As shown in [3], the system (16) is observable on $[t_0, t_f]$ if the functions in the set

$$\mathcal{F} = \Big\{(t - t_0), (t - t_0)^2, u_1^{[1]}(t, t_0), u_2^{[1]}(t, t_0), u_3^{[1]}(t, t_0), (t - t_0)u_1^{[1]}(t, t_0), (t - t_0)u_2^{[1]}(t, t_0), (t - t_0)u_3^{[1]}(t, t_0)\Big\}$$

are linearly independent on $[t_0, t_f]$, in which the $u_i^{[1]}(t, t_0)$ are computed from the input $\mathbf{u}(t)$ following (13).

Going back to the practical problem of single range localization, in which $\mathbf{u}$ is built from velocity measurements, this condition means that the vehicle must describe rich trajectories in order for the system to be observable. Without additional information, a single range measurement is obviously not sufficient to recover the position of the vehicle. However, if the condition is met, the trajectory of the vehicle will pass through several non-coplanar points, allowing for an implicit trilateration of the position over time, as well as isolating the current velocity $\mathbf{x}_2(t)$.

The filter design process for the nonlinear system (15) follows naturally from the observability analysis, through the design of a state observer for the augmented LTV system (16). A state estimate for the nonlinear system (15) can be easily recovered from the estimate $\hat{\mathbf{w}}(t)$ of the linear state observer using the left inverse transformation $\hat{\mathbf{x}}(t) := \begin{bmatrix} \mathbf{I}_6 & \mathbf{0} \end{bmatrix} \hat{\mathbf{w}}(t)$. Furthermore, if the error dynamics of the state observer are globally exponentially stable (which is easily achieved using a Kalman filter, for example), it follows from Lemma 1 that $\hat{\mathbf{x}}(t)$ converges exponentially fast to the state $\mathbf{x}(t)$ of the nonlinear system (15).

### B. Single range localization with acceleration measurements

Consider the following nonlinear system:

$$\begin{cases} \dot{\mathbf{x}}_1(t) = -\mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) = \mathbf{x}_3(t) + \mathbf{u}(t) \\ \dot{\mathbf{x}}_3(t) = \mathbf{0} \\ y(t) = \|\mathbf{x}_1(t)\| \end{cases}, \tag{17}$$

with $\mathbf{x}_1(t) \in \mathbb{R}^3$, $\mathbf{x}_2(t) \in \mathbb{R}^3$, $\mathbf{x}_3(t) \in \mathbb{R}^3$, $\mathbf{u}(t) \in \mathbb{R}^3$, and $y(t) \in \mathbb{R}$. This nonlinear system can be used to model the problem of estimating the position and velocity of a vehicle based on accelerometer readings and range measurements to a single source, see [2]. In this context, $\mathbf{x}_1$ is the position of the vehicle relative to the source, $\mathbf{x}_2$ is the velocity of the vehicle, $\mathbf{u}$ is the acceleration measurement, $\mathbf{x}_3$ is the acceleration of gravity, estimated due to performance concerns, and $y$ is the available range measurement.

Now, consider the following system, linear in its state:

$$\begin{cases} \dot{\mathbf{w}}(t) = \mathbf{A}(t, \mathbf{u}(t), \mathbf{y}(t))\mathbf{w}(t) + \mathbf{B}\mathbf{v}(t) \\ z(t) = \mathbf{C}\mathbf{w}(t) \end{cases}, \tag{18}$$

with $\mathbf{w}(t) \in \mathbb{R}^{14}$, $\mathbf{v}(t) \in \mathbb{R}^3$, $z(t) \in \mathbb{R}$,

$$\mathbf{A}(t, \mathbf{u}(t), \mathbf{y}(t)) = \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & -\frac{1}{y(t)} & 0 & 0 & 0 \\ \mathbf{u}^\mathsf{T}(t) & \mathbf{0} & \mathbf{0} & 0 & 0 & 1 & 0 & 0 \\ \mathbf{0} & -2\mathbf{u}^\mathsf{T}(t) & \mathbf{0} & 0 & 0 & 0 & -3 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{u}^\mathsf{T}(t) & 0 & 0 & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^\mathsf{T},$$

and

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be shown that the system (18) mimics the dynamics of (17), with associated state augmentation function

$$\mathbf{T}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \|\mathbf{x}_1\| \\ \mathbf{x}_1 \cdot \mathbf{x}_2 \\ \mathbf{x}_1 \cdot \mathbf{x}_3 - \mathbf{x}_2 \cdot \mathbf{x}_2 \\ \mathbf{x}_2 \cdot \mathbf{x}_3 \\ \mathbf{x}_3 \cdot \mathbf{x}_3 \end{bmatrix} \in \mathbb{R}^{14}.$$

As the state transformation is injective, the observability and observer design problem for the nonlinear system (17) can be solved by applying linear systems theory to the system (18). As shown in [2], the system (18) is observable on $[t_0, t_f]$ if the functions in the set

$$\begin{aligned} \mathcal{F} = \big\{ &(t - t_0), (t - t_0)^2, (t - t_0)^3, (t - t_0)^4, \\ &u_1^{[2]}(t, t_0), u_2^{[2]}(t, t_0), u_3^{[2]}(t, t_0), \\ &(t - t_0)u_1^{[2]}(t, t_0), (t - t_0)u_2^{[2]}(t, t_0), \\ &(t - t_0)u_3^{[2]}(t, t_0), (t - t_0)^2 u_1^{[2]}(t, t_0), \\ &(t - t_0)^2 u_2^{[2]}(t, t_0), (t - t_0)^2 u_3^{[2]}(t, t_0) \big\} \end{aligned}$$

are linearly independent on $[t_0, t_f]$, in which the $u_j^{[2]}(t, t_0)$ are computed from the input $\mathbf{u}(t)$ following (14).

This observability condition is similar to the one in the previous example, although a bit more restrictive due to the higher complexity of the system (17) in comparison with (15). Once again, the vehicle must perform rich trajectories in order to be able to determine its position and velocity from the available measurements.

The nonlinear system (17) can also be used to show another particularity of the results of the previous section, namely, that non-observability of the augmented system does not necessarily imply non-observability of the original nonlinear system. Consider the following system:

$$\begin{cases} \dot{\mathbf{w}}(t) = \mathbf{A}(t, \mathbf{u}(t), \mathbf{y}(t))\mathbf{w}(t) + \mathbf{B}\mathbf{v}(t) \\ z(t) = \mathbf{C}\mathbf{w}(t) \end{cases}, \tag{19}$$

with $\mathbf{w}(t) \in \mathbb{R}^{15}$, $\mathbf{v}(t) \in \mathbb{R}^3$, $z(t) \in \mathbb{R}$,

$$\mathbf{A}(t, \mathbf{u}(t), \mathbf{y}(t)) = \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & -\frac{1}{y(t)} & 0 & 0 & 0 & 0 \\ \mathbf{u}^\mathsf{T}(t) & \mathbf{0} & \mathbf{0} & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbf{0} & -2\mathbf{u}^\mathsf{T}(t) & \mathbf{0} & 0 & 0 & 0 & 0 & -2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 & -1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{u}^\mathsf{T}(t) & 0 & 0 & 0 & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is straightforward to verify that the system (19) mimics the dynamics of (17), with associated state augmentation function

$$\mathbf{T}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \|\mathbf{x}_1\| \\ \mathbf{x}_1 \cdot \mathbf{x}_2 \\ \mathbf{x}_1 \cdot \mathbf{x}_3 \\ -\mathbf{x}_2 \cdot \mathbf{x}_2 \\ \mathbf{x}_2 \cdot \mathbf{x}_3 \\ \mathbf{x}_3 \cdot \mathbf{x}_3 \end{bmatrix} \in \mathbb{R}^{15}.$$

This augmented system is not observable since the states corresponding to $\mathbf{x}_1 \cdot \mathbf{x}_3$ and $-\mathbf{x}_2 \cdot \mathbf{x}_2$ are indistinguishable. Denoting by $w_6(t)$ and $w_7(t)$ the state variables of (19) associated with $\mathbf{x}_1 \cdot \mathbf{x}_3$ and $-\mathbf{x}_2 \cdot \mathbf{x}_2$, respectively, it can be shown that, by fixing the input and all other initial conditions, for any $k \in \mathbb{R}$ all solutions of the system with $w_6(t_0) + w_7(t_0) = k$ will result in the same output $z(t)$. However, it was seen previously that the nonlinear system (17) is observable for a certain class of input functions $\mathbf{u}(t)$. Thus, a non-observable augmented system should not be taken as an indication that the original system is also non-observable, but rather that the state augmentation process must be carried out carefully in order to avoid such pitfalls.

### C. Nonlinearity in the dynamics

Applications of the results of the previous section are not limited to systems with nonlinearities in the output. For example, consider the nonlinear system

$$\begin{cases} \dot{x}_1(t) = x_2^2(t) \\ \dot{x}_2(t) = c \\ y(t) = x_1(t) \end{cases}, \tag{20}$$

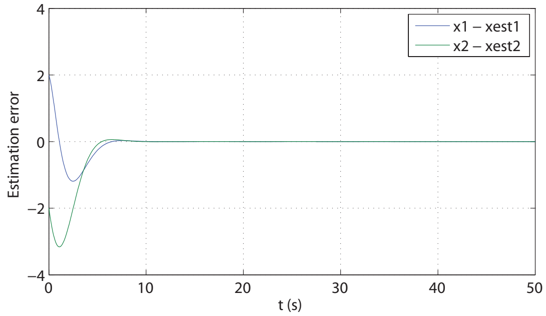


Fig. 1. Evolution of the error on the state estimates for the nonlinear system (20), recovered using the left inverse transformation (22)

where $x_1(t) \in \mathbb{R}$, $x_2(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and $c \in \mathbb{R}$ is a known real constant, regarded in this case as the input of the system. Now, consider the LTI system

$$\begin{cases} \dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{w}(t) + \mathbf{B}v(t) \\ z(t) = \mathbf{C}\mathbf{w}(t) \end{cases} \tag{21}$$

with $w(t) \in \mathbb{R}^3$, $v(t) \in \mathbb{R}$, $z(t) \in \mathbb{R}$,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2c & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

It can be verified that, with $v(t) = c$, the LTI system (21) mimics the dynamics of the nonlinear system (20), with associated state augmentation function

$$\mathbf{T}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ x_2^2 \end{bmatrix},$$

which is injective. The observability matrix associated with the LTI system (21) is given by

$$\boldsymbol{\mathcal{O}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2c & 0 \end{bmatrix},$$

which allows to conclude that the LTI system (21) is observable for any $c \neq 0$, which in turn implies that the nonlinear system (20) is observable for any $c \neq 0$.

To validate this result, the dynamics of the nonlinear system (20) were simulated with $c = 0.5$ and initial conditions $x_1(t_0) = 2$ and $x_2(t_0) = -2$, as well as a Luenberger observer for the augmented system (21). The dynamics of the observer follow

$$\dot{\hat{\mathbf{w}}}(t) = \mathbf{A}\hat{\mathbf{w}}(t) + \mathbf{B}c + \mathbf{L}\left(z(t) - \mathbf{C}\hat{\mathbf{w}}(t)\right),$$

where $\hat{\mathbf{w}}(t) \in \mathbb{R}^3$ is the state estimate. The matrix $\mathbf{L}$ of output injection gains was set to $\begin{bmatrix}(1+\sqrt{2}) & 1 & (1+\sqrt{2})\end{bmatrix}^\mathsf{T}$ in order to make the closed loop matrix $(\mathbf{A} - \mathbf{L}\mathbf{C})$ Hurwitz stable, with eigenvalues at $\lambda_{1,2} = -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}$ and $\lambda_3 = -1$.

The results of the simulation are depicted in Fig. 1, in which the estimates for $x_1(t)$ and $x_2(t)$ were recovered using the left inverse transformation

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} := \begin{bmatrix} \mathbf{I}_2 & 0 \end{bmatrix} \hat{\mathbf{w}}(t). \tag{22}$$

As it can be seen, the error on both estimates converge to zero at an exponential rate, which was expected since the systems and the left inverse transformation (22) verify the conditions of Lemma 1.

## V. Conclusions

This paper addressed the problems of observability analysis and state observer design for nonlinear systems based on the study of linear systems which mimic their dynamics. It was shown that, if a nonlinear system can be related to a linear system through an appropriate transformation of the state variables, the observability analysis of the linear system can determine observability or non-observability of the nonlinear system, depending on the properties of the state transformation. Conditions were also derived for state observers for the linear systems to double as state observers for the original nonlinear systems, as well as retaining exponential stability properties. To illustrate the usefulness of those results, several application examples were detailed.

Related future work will focus mainly on three aspects: apply the results derived here to different, relevant nonlinear observer design problems; extend the observability results to cases in which the output can be augmented or transformed to aid the observability analysis process, such as in [4]; and find methods to systematize the state augmentation process used in the first two application examples of Section IV.

## References


[1] Alessandro Astolfi and Laurent Praly. Global complete observability and output-to-state stability imply the existence of a globally convergent observer. *Mathematics of Control, Signals and Systems*, 18(1):32–65, February 2006.

[2] P. Batista, C. Silvestre, and P. Oliveira. Single beacon navigation: Observability analysis and filter design. In *American Control Conference (ACC), 2010*, pages 6191–6196, June 2010.

[3] P. Batista, C. Silvestre, and P. Oliveira. Single range aided navigation and source localization: Observability and filter design. *Systems & Control Letters*, 60(8):665–673, August 2011.

[4] P. Batista, C. Silvestre, and P. Oliveira. Sensor-based Long Baseline Navigation: observability analysis and filter design. *Asian Journal of Control*, 16(4):974–994, July 2014.

[5] Hassan Hammouri, Boubekeur Targui, and Frederic Armanet. High gain observer based on a triangular structure. *International Journal of Robust and Nonlinear Control*, 12(6):497518, 2002.

[6] R. Hermann and Arthur J. Krener. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22(5):728–740, October 1977.

[7] Wei Kang and Jean-Pierre Barbot. Discussions on observability and invertibility. In *NOLCOS 2007*, Pretoria, South Africa, 2007.

[8] Wei Kang, Arthur J. Krener, Mingqing Xiao, and Liang Xu. A survey of observers for nonlinear dynamical systems. In *Data Assimilation for Atmospheric, Oceanic and Hydrologic Applications (Vol. II)*, pages 1–25. Springer Berlin Heidelberg, January 2013.

[9] Nikolaos Kazantzis and Costas Kravaris. Nonlinear observer design using lyapunovs auxiliary theorem. *Systems & Control Letters*, 34(5):241–247, July 1998.

[10] G. Kreisselmeier and R. Engel. Nonlinear observers for autonomous lipschitz continuous systems. *IEEE Transactions on Automatic Control*, 48(3):451–464, March 2003.

[11] D. Viegas, P. Batista, P. Oliveira, and C. Silvestre. Linear motion observers for asc/auv tandems based on single range readings. In *2011 American Control Conference*, pages 3966–3971, June 2011.

[12] D. Viegas, P. Batista, P. Oliveira, and C. Silvestre. Decentralized range-based linear motion estimation in acyclic vehicle formations with fixed topologies. In *Proceedings of the 2012 American Control Conference*, pages 6575–6580, Montreal, Canada, June 2012.