

Trailer-Like Leader Following Trajectory Planning

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Abstract—In this paper a trajectory planner for n autonomous vehicles following a common leader is presented, with the planning being accomplished in real time and in a three dimensional setting. The trajectory planner is designed such that n follower vehicles behave as n distinct points of a unique two dimensional trailer attached to a single leader vehicle. We prove that for a wide range of initial conditions the trailer reference frame converges to a unique solution, thus guaranteeing that each follower can plan its trajectory independently from its peers, thereby reducing the need for communications among vehicles. Additionally, convergence to a fixed formation of $n+1$ vehicles with respect to the trailer reference frame is also guaranteed. Finally, we present bounds on the planned velocity and acceleration, which provide conditions for the feasibility of the planned trajectory. An experimental validation of the planner's behavior is presented with quadrotor vehicles, demonstrating the richness of the planned trajectories.

I. INTRODUCTION

Robot coordination has been the scope of a significant amount of research over the last decade. Coordination of multiple vehicles is particularly useful in a variety of applications, such as mapping, coverage, and surveillance of large areas like the sea floor [1], [2], providing results in a faster and more efficient manner. Coordinated motion is also required in transportation via multiple vehicles, in cases when the loading capacity of an individual vehicle is surpassed [3]. Sensing robots moving in a coordinated manner can also be perceived as a distributed network of sensors, altogether accomplishing a larger sensing task or alternatively providing robustness to sensor loss in critical environments [4].

Different approaches to formation control have been proposed in the literature. In a behavior based approach, a desired behavior results from a weighting between different goal oriented behaviors but guaranteeing convergence to a desired configuration proves difficult [5], [6]. In a virtual

structure approach [7], the vehicles move as points of a virtual rigid body, whose motion is prescribed in a global manner. Another approach is leader following, which is the one we tackle in this paper.

In the leader-follower approach to formation control, the objective is for a follower vehicle to remain at a fixed relative position, in a given reference frame, w.r.t. the leader vehicle. The problem is completely characterized by the relative position vector and the reference frame where it is defined. The choice of reference frame plays an important role in the definition of the follower's trajectory and in the sensory information necessary for computing such trajectory.

Several leader following strategies have been proposed in the literature. The simplest approach to leader following is to specify the relative position vector in the inertial reference frame as in [8]. The planned trajectory is simple in the sense that both the leader and the follower describe an identical path, apart from a translation. Because the follower's path can overlap with the leader's path, a reduced efficiency gain when using multiple vehicles is expected.

Defining the relative position vector in a reference frame attached to the leader, such as the leader's Serret-Frenet frame, results in follower trajectories with more complex behavior. This approach has been proposed by several authors [10], [11], [12] and because the chosen reference frame is uniquely defined, no convergence analysis is required. However, its implementation requires more information than should be expected, since vehicle control at the velocity level requires the knowledge of the leader's velocity and acceleration (in the form of the angular velocity of the Serret-Frenet reference frame).

In this paper, we design an intuitive leader following strategy where the relative position vector between leader and follower is specified in the reference frame of a *virtual* trailer attached to the leader. The work shares similarities with those in [15], [16], however we provide new and rigorous proofs on the convergence to a unique formation for n followers and additionally bounds that can be used in studying the feasibility of the planned trajectory.

A two part solution to the leader-follower formation problem is here proposed. First, a desired trajectory is computed for an ideal follower vehicle, hereafter called *virtual* follower, with the help of the proposed trajectory planner. The planned trajectory is then used as a reference to a trajectory tracking controller that drives the *real* follower vehicle to the *virtual* follower. In this paper, we focus on the problem of generating the follower trajectory and providing conditions for its feasibility. We also present experimental results for the complete planning/tracking problem using quadrotor vehicles.

This work was partially supported by the Fundação para a Ciência e a Tecnologia (FCT) under the project PEst-OE/EEI/LA0009/2013 and by Project MYRG118(Y1-L3)-FST12-SSW of the University of Macau. The work of Pedro Pereira was supported by a Research Student Grant under the FCT project PEst- OE/EEI/LA0009/2011. The work of Rita Cunha was supported by the FCT Investigator Programme (IF/00921/2013).

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Quadrotors are aerial vehicles ideal for testing algorithms, due to their simplicity, high maneuverability, VTOL/hover capability and ability to track any trajectory within the limits of their actuation dynamics. Tracking controllers for quadrotor vehicles have been extensively studied in the literature, c.f. the survey article [17], and the *virtual* follower trajectory generated by our planner can be used as reference for any generic tracking controller applied to the follower vehicle.

The remainder of this paper is structured as follows. Section II presents some mathematical notation used throughout the paper. Section III describes the leader-follower problem. Section IV derives the trajectory planner and studies its properties. Section V presents and examines the obtained experimental results.

II. NOTATION

The configuration of a reference frame $\{\mathcal{B}\}$ w.r.t. a frame $\{\mathcal{A}\}$ is represented as an element of the Special Euclidean group of order n , $({}^{\mathcal{A}}\mathcal{R}, {}^{\mathcal{A}}\mathbf{p}_{\mathcal{B}}) \in \mathcal{SE}(n)$, where ${}^{\mathcal{A}}\mathbf{p}_{\mathcal{B}} \in \mathbb{R}^n$ is the position, ${}^{\mathcal{A}}\mathcal{R} \in \mathcal{SO}(n)$ is the rotation matrix, and n is either 2 or 3. For points in the inertial frame $\{\mathcal{I}\}$, the superscript frame letter is often omitted, i.e. $\mathbf{p}_{\mathcal{B}} := {}^{\mathcal{I}}\mathbf{p}_{\mathcal{B}}$. The vectors $\mathbf{e}_i \in \mathbb{R}^n$ with $i = \{1, \dots, n\}$ are used to denote the unit vectors from the canonical basis for $\mathbb{R}^n$. The matrix $\mathcal{S}(x) = x [\mathbf{e}_2 \ -\mathbf{e}_1] \in \mathbb{R}^{2 \times 2}$ is a cross product skew-symmetric matrix and it satisfies $\mathbf{b}^T \mathcal{S}(x) \mathbf{b} = 0$. The map $\Pi(\mathbf{x}) : \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x}^T \mathbf{x} = 1\} \mapsto \mathbb{R}^{3 \times 3}$ yields a matrix that represents the orthogonal projection operator onto the subspace perpendicular to $\mathbf{x}$. We write as $f^{(i)}(t)$ the i^{th} time derivative of function $f(t)$ for $i = \{1, 2, \dots\}$.

III. PROBLEM STATEMENT

In a leader following problem, a leader vehicle moves freely and it is the *goal* of one or more followers to proceed so as to *see* the leader vehicle at a constant relative position. This simple problem cannot be accomplished by most *real* vehicles as full position control is not available, i.e. a *real* follower vehicle cannot follow a leader at all times. This inspires the introduction of a *virtual* follower vehicle as one with full position control and consequently one which can meet the leader following *goal* at all times.

The goal of the *virtual* follower is to remain at a fixed relative position w.r.t. the leader from its own point of view, such that

$${}^{\mathcal{F}}\mathbf{p}_{\mathcal{L}|\mathcal{F}} \equiv {}^{\mathcal{F}}\mathcal{R}(\mathbf{p}_{\mathcal{L}} - \mathbf{p}_{\mathcal{F}}) \equiv \mathbf{d}, \quad (1)$$

with $\mathbf{d} \in \mathbb{R}^3$ as a constant vector. According to (1), once the kinematics of ${}^{\mathcal{I}}\mathcal{R}$ are established then the *virtual* follower's reference frame, $\{{}^{\mathcal{I}}\mathcal{R}, \mathbf{p}_{\mathcal{F}}\}$, becomes completely defined. Thus, the design challenge lies on finding an appropriate kinematic behavior.

The *virtual* follower's position can then be considered as the desired position for a *real* follower. As such, the problem of leader following can be addressed in two steps. The primary step is that of trajectory planning which is vehicle independent and is solved with the help of a *virtual* follower.

A secondary step is that of trajectory tracking which depends on the selected vehicle and its dynamics and is solved with the help of a trajectory tracking controller. Here, we focus on the problem of trajectory planning but we present bounds on the planned velocity and acceleration that can be used to guarantee that the planned trajectory is indeed tractable.

In order to generate three dimensional trajectories, consider the standard inertial reference frame $\{\mathcal{I}\}$ defined in 3D space, such that the third axis is aligned with the acceleration due to gravity. Additionally, consider a second inertial reference frame $\{\mathcal{I}^*\}$ that coincides with $\{\mathcal{I}\}$ apart from a constant rotation, so that its configuration w.r.t. $\{\mathcal{I}\}$ is given by $({}^{\mathcal{I}^*}\mathcal{R}, \mathbf{0}) \in \mathcal{SE}(3)$. The unit vector ${}^{\mathcal{I}^*}\mathcal{R} \mathbf{e}_3 \equiv \mathbf{n} \in \mathbb{R}^3$ shall be selected and considered the preferred vertical direction (for example, if a leader is scanning a vertical wall, $\mathbf{n}$ should be selected as the normal to such wall). With that in mind, the leader's motion is decomposed into a planar motion living in the plane orthogonal to $\mathbf{n}$ and a motion along the direction $\mathbf{n}$, i.e. $\mathbf{p}_{\mathcal{L}}^{(i)} = \Pi(\mathbf{n}) \mathbf{p}_{\mathcal{L}}^{(i)} + (\mathbf{n}^T \mathbf{p}_{\mathcal{L}}^{(i)}) \mathbf{n}$, where the leader's Frenet reference frame $\{\mathcal{L}\} \in \mathcal{SE}(2)$ is associated to $\Pi(\mathbf{n}) \mathbf{p}_{\mathcal{L}}^{(i)}$. Two separate leader following strategies will be accomplished, one along the plane orthogonal to $\mathbf{n}$ and one along the direction $\mathbf{n}$.

Next, focus on the two dimensional Leader following strategy. With an obvious abuse of notation, consider the inertial reference frame $\{\mathcal{I}\} \in \mathcal{SE}(2)$, the leader's Frenet reference frame $\{\mathcal{L}\}$ defined by the pair $({}^{\mathcal{I}}\mathcal{R}, \mathbf{p}_{\mathcal{L}}) \in \mathcal{SE}(2)$ and the *virtual* follower reference frame $\{\mathcal{F}\}$ defined by the pair $({}^{\mathcal{I}}\mathcal{R}, \mathbf{p}_{\mathcal{F}}) \in \mathcal{SE}(2)$. The kinematics of the leader are given by $\dot{\mathbf{p}}_{\mathcal{L}} = \mathbf{v}_{\mathcal{L}} \equiv \|\mathbf{v}_{\mathcal{L}}\| {}^{\mathcal{I}}\mathcal{R} \mathbf{e}_1$ and ${}^{\mathcal{I}}\dot{\mathcal{R}} = {}^{\mathcal{I}}\mathcal{R} \mathcal{S}(\omega_{\mathcal{L}})$, where $\mathbf{v}_{\mathcal{L}} \in \mathbb{R}^2$ is the linear velocity expressed in inertial coordinates and $\omega_{\mathcal{L}} \in \mathbb{R}$ is the angular velocity expressed in body coordinates. Using the path curvature $\kappa_{\mathcal{L}}(t)$ the leader's angular velocity can be written as

$$\omega_{\mathcal{L}}(t) = \|\mathbf{v}_{\mathcal{L}}(t)\| \kappa_{\mathcal{L}}(t). \quad (2)$$

The kinematics of the follower are similarly defined.

The trajectory planning here proposed is to be implemented for n followers with one common leader. The objective is for the $n + 1$ vehicles to move in a cohesive manner, i.e. to move in a fixed configuration w.r.t. to some known reference frame. However, in order to minimize communications among vehicles, we require each follower to move independently from its peers. Under certain conditions, we guarantee the $n + 1$ vehicles asymptotically move in a fixed formation requiring at the kinematic level solely information about the leader's position and velocity expressed in their reference frame.

IV. TRAJECTORY PLANNER

Two *virtual* followers will be designed, one in a one dimensional setting, $\mathbf{p}_{\mathcal{F}}^{1D} \in \mathbb{R}$, and another in a two dimensional setting $\mathbf{p}_{\mathcal{F}}^{2D} \in \mathbb{R}^2$. The desired position for the *real* follower is the combined position of both previous *virtual* followers, i.e. $\mathbf{p}_{\mathcal{F}} = {}^{\mathcal{I}}\mathcal{R} [(\mathbf{p}_{\mathcal{F}}^{2D})^T \ \mathbf{p}_{\mathcal{F}}^{1D}]^T$, with the leader's motion also being decomposed in a one and two dimensional motions, i.e. $\mathbf{p}_{\mathcal{L}} = {}^{\mathcal{I}}\mathcal{R} [(\mathbf{p}_{\mathcal{L}}^{2D})^T \ \mathbf{p}_{\mathcal{L}}^{1D}]^T$.

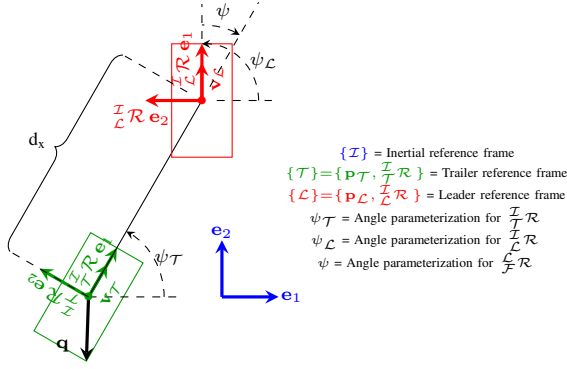


Fig. 1: Standard 1-Trailer System

A. Motion along $\mathbf{n}$

For the one dimensional *virtual* follower, consider the leader $\mathbf{p}_L^{1D} = \mathbf{n}^T \mathbf{p}_L$. The *virtual* follower is constrained to be at a constant distance $d_x \in \mathbb{R}$ from the leader along the direction $\mathbf{n}$, i.e. $\mathbf{p}_F^{1D} = \mathbf{p}_L^{1D} - d_x$. This planning is a static one (given $\mathbf{p}_L^{(i)}$ for $i = \{0, 1, 2, \dots\}$ as inputs) thus n followers can independently apply this algorithm.

B. Motion along the plane orthogonal to $\mathbf{n}$

For the two dimensional *virtual* follower, consider the leader defined by $\mathbf{p}_L^{2D}$, which, from the previous definitions, satisfies $[(\mathbf{p}_L^{2D})^T \ 0]^T = \Pi(\mathbf{e}_3) \frac{x}{\mathcal{L}} \mathcal{R} \mathbf{p}_L$ (the superscript $2D$ will hereafter be dropped). As explained in Section III, the design problem lies in finding kinematics that produce an intuitive leader following behavior.

The proposed strategy is simple and intuitive. We model each *virtual* follower as a point of a trailer attached to the leader vehicle, as illustrated in Figure 1. In this case, the *virtual* follower reference frame is identical to the trailer reference frame, apart from a constant position offset $\mathbf{q}$. Equivalently¹,

$$\mathbf{p}_F = \mathbf{p}_T + \frac{x}{\mathcal{L}} \mathcal{R} \mathbf{q} \equiv \mathbf{p}_T + \frac{x}{\mathcal{L}} \mathcal{R} \mathbf{q}, \quad (3)$$

where $\mathbf{p}_T$ is the trailer hinge rigidly connected to the leader, $\mathbf{q}$ represents a point in the trailer rigid body (specified in the trailer reference frames) and by definition $\frac{x}{\mathcal{L}} \mathcal{R} \equiv \frac{x}{\mathcal{L}} \mathcal{R}$.

It should be clear now, that if n followers, each one with a different $\mathbf{q}$, follow a common leader, then they behave as n distinct points of a trailer rigid body as long as they share the same trailer reference frame $\{\mathcal{T}\}$. Later, we will show that, under certain conditions, there is a unique trailer reference frame to which all other reference frames converge to. As a consequence, n followers can independently plan their trajectories and they will asymptotically behave as n points of a common trailer rigid body.

A trailer vehicle is one that can only move along the axis that rigidly connects its hinge to the leader. Without loss of generality, the axis of motion is assumed to be the trailer's first axis (see Figure 1) so that (1) takes the form

$$\mathbf{p}_L = \mathbf{p}_T + d_x \frac{x}{\mathcal{L}} \mathcal{R} \mathbf{e}_1, \quad (4)$$

¹In (3), the notation $\frac{x}{\mathcal{L}} \mathcal{R}$ should be used instead of $\frac{x}{\mathcal{L}} \mathcal{R}$ because the trailer is defined in the x - y plane of $\{\mathcal{I}^*\}$. However and without hindering comprehension, the notation $\frac{x}{\mathcal{L}} \mathcal{R}$ is used throughout this Section.

and $\mathbf{v}_T = v_T \frac{x}{\mathcal{L}} \mathcal{R} \mathbf{e}_1$, where $v_T \equiv \|\mathbf{v}_T\|$. Notice that with this definition, the trailer's reference frame $\{\mathcal{T}\}$, like the leader's, becomes a Frenet reference frame. Taking the time derivative of (4) yields

$$\mathbf{v}_L = \mathbf{v}_T + d_x \frac{x}{\mathcal{L}} \mathcal{R} \mathcal{S}(\omega_T) \mathbf{e}_1 \Leftrightarrow \frac{x}{\mathcal{L}} \mathcal{R} [v_T \ \omega_T d_x]^T = \mathbf{v}_L$$

The trailer's speed is then $v_T = \mathbf{e}_1^T \frac{x}{\mathcal{L}} \mathcal{R} \mathbf{v}_L$ while the trailer's angular velocity is

$$\omega_T = \mathbf{e}_2^T \frac{x}{\mathcal{L}} \mathcal{R} \mathbf{v}_L d_x^{-1}. \quad (5)$$

As explained in Section III, the kinematics of $\frac{x}{\mathcal{L}} \mathcal{R}$ (which are those of $\frac{x}{\mathcal{L}} \mathcal{R}$) defines the leader following behavior, thus (5) completes our planning strategy.

Hereafter, we will dedicate efforts to studying the planner's behavior, more specifically we will show under what conditions there is a unique nominal trailer reference frame. Additionally, we will also present bounds on the follower's speed and acceleration which are necessary in guaranteeing the planned trajectory is indeed feasible.

C. Trajectory Planner Properties for the 2D Setting

The planning along the direction $\mathbf{n}$ is a static one. However, the planning in the 2 dimensional space orthogonal to $\mathbf{n}$ is a dynamic one. Notice the trailer configuration with respect to $\{\mathcal{I}\}$ belongs to the domain

$$\{(\mathbf{p}_T, \frac{x}{\mathcal{L}} \mathcal{R}) : \mathbf{p}_T = \mathbf{p}_L - d_x \mathcal{R} \mathbf{e}_1, \frac{x}{\mathcal{L}} \mathcal{R} = \mathcal{R} \in \mathcal{SO}(2)\}. \quad (6)$$

The problem at hand is simple: can we guarantee that for a wide range of initial conditions the trailer reference frame converges to a unique (possibly time varying) reference frame. This problem shares similarities with contraction analysis, whose objective is to determine convergence to a nominal solution which is independent of initial conditions [18]. We will provide conditions under which such nominal solution $\{\mathcal{T}\}$ exists. As a result, if those conditions are met, we guarantee n followers can independently perform their trajectory planning while asymptotically behaving as n points of a common rigid trailer. In that case, the leader and n followers form a fixed configuration that rotates in space with the trailer reference frame.

The trailer reference frame is not unique (see (6)), but the leader reference frame is, which is the reason for conducting all analysis w.r.t. $\{\mathcal{L}\}$. Having said this, consider $\frac{x}{\mathcal{L}} \mathcal{R} \in \mathcal{SO}(2)$ as the rotation matrix from $\{\mathcal{T}\}$ to $\{\mathcal{L}\}$ given by $\frac{x}{\mathcal{L}} \mathcal{R} \equiv \frac{x}{\mathcal{L}} \mathcal{R} \frac{x}{\mathcal{L}} \mathcal{R}$ and let $\frac{x}{\mathcal{L}} \mathcal{R}$ be parametrized by the angle ψ , presented in Figure 1. In that case, the leader position w.r.t. the trailer position specified in $\{\mathcal{L}\}$ can be rewritten as

$${}^L \mathbf{p}_{L|T} = d_x \frac{x}{\mathcal{L}} \mathcal{R} \mathbf{e}_1 = d_x [\cos(\psi) \ \sin(\psi)]^T. \quad (7)$$

and, using (2) and (5), the kinematics of ψ can be written as

$$\dot{\psi} = \omega_T - \omega_L = -\|\mathbf{v}_L\| d_x^{-1} (\sin(\psi) + \kappa_L d_x). \quad (8)$$

We will show the non-autonomous system (8) (with exogenous terms $\|\mathbf{v}_L(t)\|$ and $\kappa_L(t)$) converges to a unique nominal solution for a wide range of initial conditions. This answers the problem posed in the beginning of this

Subsection.

Remark 1: In order to guarantee the leader Frenet reference frame is always well defined, we require $\mathbf{v}_\mathcal{L}(t) \in \mathcal{C}(\mathbb{R}^2)$ with $\|\mathbf{v}_\mathcal{L}\| \geq v_\mathcal{L}^{\min} > 0$. Hereafter and for obvious reasons, we shall assume those conditions are always met.

1) *Pulled vs Pushed Trailer:* Before we study the existence of a nominal solution $\psi(t)$, we present a result which is of extreme importance in later proofs and also of interesting physical interpretation.

Lemma 2: Consider a leader with bounded curvature satisfying $|\kappa_\mathcal{L}(t)d_x|_\infty \leq 1 - \epsilon^2$ (for $\epsilon \in (0, 1]$) and a trailer attached to such leader with kinematics described by (8). Then the set $\Omega = \{\psi : \cos(\psi) \geq \epsilon\}$ is positively invariant with respect to (8) and, for all initial conditions such that $|\cos(\psi(0))| < \epsilon$, the solutions of the system will enter Ω in finite time.

Proof: Consider the positive definite Lyapunov function $V = 1 - \cos(\psi)$, which decreases with $\cos(\psi)$ and whose time derivative, with the help of (8), yields

$$\dot{V} = -\|\mathbf{v}_\mathcal{L}\| d_x^{-1} (\sin^2(\psi) + \sin(\psi) \kappa_\mathcal{L}(t) d_x).$$

Consider $-\epsilon < \cos(\psi(t)) < \epsilon$ for $t \in [0, T]$, which implies $\sin^2(\psi(t)) > 1 - \epsilon^2$ for that same time interval. Given the condition $|\kappa_\mathcal{L}(t)d_x|_\infty \leq 1 - \epsilon^2$ it then follows

$$\dot{V} < -\|\mathbf{v}_\mathcal{L}\| d_x^{-1} (1 - \epsilon)(1 - \sqrt{1 - \epsilon^2}) V \quad \forall t \in [0, T],$$

which means there exists a finite time T such that $\cos(\psi(t)) \geq \epsilon$ for all $t > T$. ■

Notice, from (7), that $\mathbf{e}_1^T \mathbf{p}_{\mathcal{L}|T} = \cos(\psi)$. Then Lemma 2 has a very interesting physical interpretation. It says that given an upper bound on the curvature (whose interpretation will be provided later) and a proper initialization, the trailer will be forever pulled after a finite time, i.e. $\mathbf{e}_1^T \mathbf{p}_{\mathcal{L}|T}(t) > 0$ for $t > T$. In a weaker form, it says that if a trailer is initially being pulled then it will be forever pulled.

2) *Leader path with constant curvature:* We now consider the simplest case where a leader describes a circular or rectilinear path. For these paths the curvature is constant and we prove that ψ has a unique stable equilibrium point ψ^* for $\kappa_\mathcal{L} d_x < 1$, a unique equilibrium point ψ^* for $\kappa_\mathcal{L} d_x = 1$ and no equilibrium points for $\kappa_\mathcal{L} d_x > 1$. A quick analysis of equation (8) reveals that if $\kappa_\mathcal{L}$ is a constant satisfying $\kappa_\mathcal{L} d_x < 1$ then two equilibrium solutions exist,

$$\cos(\psi^*) = +\sqrt{1 - (\kappa_\mathcal{L} d_x)^2} \wedge \sin(\psi^*) = -\kappa_\mathcal{L} d_x, \quad (9a)$$

$$\cos(\psi^\dagger) = -\sqrt{1 - (\kappa_\mathcal{L} d_x)^2} \wedge \sin(\psi^\dagger) = -\kappa_\mathcal{L} d_x. \quad (9b)$$

Lemma 2 suggests that an equilibrium point can only be stable if it satisfies $\cos(\psi) > 0$. If $\kappa_\mathcal{L} d_x = 1$ then *only one* equilibrium solution exists ($\cos(\psi^*) = 0 \wedge \sin(\psi^*) = -1$) and it lacks asymptotic stability (which we do not prove in this paper).

System (8) also reveals that no equilibrium solution exists for $\kappa_\mathcal{L} d_x > 1$, with a very clear interpretation. According to (5), ω_τ is bounded by $\frac{\|\mathbf{v}_\mathcal{L}\|}{d_x}$ whereas $\omega_\mathcal{L} = \|\mathbf{v}_\mathcal{L}\| \kappa_\mathcal{L}$; this means that if d_x is too large (compared with the leader path radius or $\kappa_\mathcal{L}^{-1}$) the trailer will not have enough angular

velocity to keep up with the leader's rotation. In that case, $\frac{\epsilon}{\tau} \mathcal{R}$ can never stabilize.

Remark 3: The absence of an equilibrium solution for $\kappa_\mathcal{L} d_x > 1$ should not be interpreted as a disadvantage but rather as an advantage. A leader describing a path with $\kappa_\mathcal{L} d_x \gg 1$ is a leader which is almost at rest w.r.t. to the trailer and we do not want the *virtual* follower to rotate with the leader but rather to stay at rest (in which case ψ has no equilibrium solution).

Theorem 4: Consider a leader describing a path with constant curvature satisfying $\kappa_\mathcal{L} d_x < 1$ and a trailer attached to such leader with kinematics described by (8). Let ψ^* and $\psi^\dagger$ be given by (9a) and (9b), respectively. If $\psi(0) \neq \psi^\dagger$ then $\psi(t)$ converges exponentially fast to the stable equilibrium point ψ^* , i.e. (8) has an almost globally exponentially stable (AGES) equilibrium point at $\psi = \psi^*$.

Proof: Consider the positive semi-definite Lyapunov function $V = \frac{1}{2} (\sin(\psi(t)) + \kappa_\mathcal{L} d_x)^2$, which is positive every where expect for the two equilibrium points ψ^* and $\psi^\dagger$ and bounded, more specifically $V < 2$. From (8) and recalling that $\dot{\kappa}_\mathcal{L} = 0$, the Lyapunov time derivative yields

$$\dot{V} = -2 \|\mathbf{v}_\mathcal{L}\| d_x^{-1} \cos(\psi(t)) V. \quad (10)$$

where $V(0) \neq 0$ from the conditions of the Theorem. Assume $\cos(\psi(t)) \leq -\epsilon \equiv -\sqrt{1 - \kappa_\mathcal{L} d_x}$ for all t . Then from (10), $V(t) > V(0) \exp\left(2 \frac{v_\mathcal{L}^{\min}}{d_x} \epsilon t\right)$, which means the Lyapunov function grows unbounded with time. However, this scenario is not possible since that same Lyapunov function is upper bounded (by 2). The logical conclusion is that $\cos(\psi(t)) > -\epsilon \equiv -\sqrt{1 - \kappa_\mathcal{L} d_x}$ will have to be verified after some finite time at which point we recall Lemma 2 to conclude that $\cos(\psi(t))$ will be forever positive after some finite time. From that point on, (10) is definite negative which means V converges exponentially fast to origin for the equilibrium point in (9a). ■

Theorem 4 says there are two equilibrium solutions, one unstable and the other AGES. The stable solution corresponds to a trailer being pulled while the unstable solution corresponds to a trailer being pushed.

Thus for a leader describing a circular or a rectilinear path and given the conditions mentioned in Theorem 4, the trailer will converge to a unique solution for all initial conditions excluding a single unstable equilibrium point. Naturally, a question arises whether a similar result can be obtained for arbitrary paths.

3) *Leader describing an arbitrary path:* Previously, for a leader describing a circular path, a solution corresponding to an equilibrium point was found and its stability was proven. For a leader describing an arbitrary path, and under some conditions, we can also prove the existence of a unique attracting solution. The analytic expression for this solution (which is not an equilibrium point) will not be provided (as it is unknown) but, later on, we will be able to determine what this solution *resembles* or looks like.

Theorem 5: Consider a leader describing a path with a time-varying curvature satisfying $|\kappa_\mathcal{L}(t)d_x|_\infty \leq \kappa_\mathcal{L}^{\max} d_x < 1$

and a trailer attached to such leader, initialized such that $\cos(\psi(0)) \geq -\sqrt{1 - \kappa_{\mathcal{L}}^{\max} d_x}$. Then, $\cos(\psi(t))$ has a unique attracting solution.

Proof: Under the conditions of the Theorem, Lemma 2 may be used to conclude that after a finite time $\cos(\psi(t))$ will enter and not leave the interval $[\sqrt{1 - \kappa_{\mathcal{L}}^{\max} d_x}, 1]$. Consider now $\psi_1(t)$ and $\psi_2(t)$ as two solutions to (8) with different initial conditions. Without loss of generality, consider $\psi_1(0), \psi_2(0) \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ which is a positively invariant set as a consequence of Lemma 2. Now define the error angle, $\psi_e(t) = \psi_2(t) - \psi_1(t)$ (with $\psi_e(0) \neq 0$) whose dynamics, with the help of the mean value Theorem, can be written as

$$\dot{\psi}_e = -\|\mathbf{v}_{\mathcal{L}}\| d_x^{-1} (\sin(\psi_2) - \sin(\psi_1)) = -\|\mathbf{v}_{\mathcal{L}}\| d_x^{-1} \cos(\psi') \psi_e,$$

where $\psi' \in [\min(\psi_2, \psi_1), \max(\psi_2, \psi_1)] \subset]-\frac{\pi}{2}, \frac{\pi}{2}[$. Thus $\psi_e = 0$ is an exponentially stable equilibrium point. ■

From Theorem 5 one may conclude that the initialization of the trailer reference frame (under the upper mentioned conditions) does not influence the final trailer reference frame, which means n followers can independently plan their trajectories as they will asymptotically behave as n points of a common trailer body. However, this is only valid for an upper bounded leader path curvature (for reasons already discussed) and the trailer must be properly initialized (from Lemma 2, it is enough to initialize the trailer in a pulling position, which provides a wide range of initializations).

The next Theorem, whose proof is omitted due to paper length limitations, provides useful insight into the *closeness* of ψ to ψ^* .

Theorem 6: Consider a leader describing a path with a time-varying curvature satisfying $-1 \leq \kappa_{\mathcal{L}}^{\min} d_x \leq \kappa_{\mathcal{L}}(t) d_x \leq \kappa_{\mathcal{L}}^{\max} d_x < 1$, and $|\kappa'_{\mathcal{L}}| \leq \kappa'_{\mathcal{L}}^{\infty}$ and a trailer attached to the leader with kinematics described by (8). Then,

$$\cos(\psi(t) - \psi^*(t)) \geq (1 - C^2)(1 + C^2)^{-1}, \quad (11)$$

where $C = \frac{1}{2} \frac{\kappa'_{\mathcal{L}}^{\infty} d_x^2}{1 - (\kappa_{\mathcal{L}}^{\max} d_x)^2}$ and $\psi^*(t)$ is given by (9a), defines an invariant set. Additionally, the conditions $-\kappa_{\mathcal{L}}^{\max} d_x \leq \sin(\psi(t)) \leq -\kappa_{\mathcal{L}}^{\min} d_x$ also define an invariant set.

D. Bounds on Velocity and Acceleration

As explained in Section III, the planned trajectory must be fed to a trajectory tracking controller that guarantees a *real* follower asymptotically tracks the *virtual* follower. However, the planning is vehicle independent, as such it is not guaranteed that the planned trajectory is indeed feasible.

The proposed planner does not mimic the leader's motion, thus higher velocities and accelerations than the leader's may be required from the *real* follower. Hence, for vehicles of the same family, for example quadrotors, a logical conclusion is that if a leader describes trajectories close to the limits of its flight envelope, then the *real* follower's flight envelope must be broader than the leader's. In general and for heterogeneous vehicles, the restrictions in a follower's envelope will impose some upper bounds on the leader's trajectories.

Here, we provide bounds on the planned velocity, $\bar{\mathbf{v}}_{\mathcal{F}}^{\max}$, and planned acceleration, $\bar{\mathbf{a}}_{\mathcal{F}}^{\max}$, for the complete 3D plan-

ning, which are given by

$$\bar{\mathbf{v}}_{\mathcal{F}}^{\max} \leq \sqrt{\mathbf{v}_{\mathcal{F}}^{\max 2} + (\mathbf{n}^T \mathbf{v}_{\mathcal{L}})^2_{\max}}, \quad (12a)$$

$$\bar{\mathbf{a}}_{\mathcal{F}}^{\max} \leq \sqrt{\mathbf{a}_{\mathcal{F}}^{\max 2} + (\mathbf{n}^T \mathbf{a}_{\mathcal{L}})^2_{\max}}, \quad (12b)$$

where $\mathbf{n}$ is the preferred vertical direction and $\mathbf{v}_{\mathcal{F}}^{\max}$ and $\mathbf{a}_{\mathcal{F}}^{\max}$ are the maximum velocity and acceleration of the trailer,

$$\mathbf{v}_{\mathcal{F}}^{\max} \leq \mathbf{v}_{\mathcal{L}}^{\max} \sqrt{1 + (\|\mathbf{q}\| d_x^{-1} (\kappa_{\mathcal{L}}^{\max} d_x))^2}, \quad (13a)$$

$$\mathbf{a}_{\mathcal{F}}^{\max} \leq \mathbf{a}_{\mathcal{L}}^{\max} C_2 + \mathbf{v}_{\mathcal{L}}^{\max 2} d_x^{-1} C_3 (\kappa_{\mathcal{L}}^{\max} d_x), \quad (13b)$$

which can be found by differentiating (3) once and twice, respectively. (C_2 and C_3 are bounded coefficients that depend exclusively on d_x and $\|\mathbf{q}\|$ - which we omit - and the term $(\kappa_{\mathcal{L}}^{\max} d_x)$ in (13) must be replaced by 1 if the trailer is not properly initialized).

By imposing the bounds defined in (12), we extract bounds to impose on the leader's velocity and acceleration.

V. EXPERIMENTS

A. Experimental Set-Up

The proposed trajectory planner was tested with two radio controlled Blade mQX quadrotor vehicles [19]. A VICON Bonita motion capture system [20], composed of 12 cameras and markers attached to the quadrotors, provides highly accurate position and orientation measurements for the leader and follower at a rate of 100Hz. The trajectory planner is implemented in a Matlab/Simulink model, which computes the follower's position reference and feeds it to the quadrotors's trajectory tracker controller developed in [21] which requires a time-parametrized position reference of class $\mathcal{C}^3$. Consequently, the trajectory planner requires the knowledge of $\mathbf{p}_{\mathcal{L}}^{(i)}(t)$ for $i = \{1, 2, 3\}$, which are obtained from the raw position measurements by means of dynamic differentiators.

B. Experiments

One experiment is presented, where the leader's path is depicted in red, the *real* follower's path in blue and the *virtual* follower's path in magenta, with the magenta reference frame being that of the virtual follower, i.e. $\mathcal{F}_{\mathcal{R}}$. In Figures 2(a)-2(c), the vehicles' positions are shown five times, using the symbol $\diamond$ for the initial position and the symbol $\bullet$ for the other positions, equally spaced in time (elapsed time divided by four). The leader's velocity was set at 0.5ms^{-1} .

The angles ψ and ψ^* are presented in dashed and full lines, in Figure 2(e), with ψ^* computed from (9a) and ψ computed from $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{L}} \mathcal{F}_{\mathcal{R}}^T$, where $\mathcal{F}_{\mathcal{R}}$ is the leader frenet reference frame for the motion along the plane orthogonal to $\mathbf{n}$.

Figures 2(a)-2(c) depict the paths for $d_x = 0.4 \text{m}$, $\mathbf{q} = [0 \ 0.4]^T \text{m}$, $d_z = 0 \text{m}$ and $\mathbf{n} = \mathbf{e}_3$. The leader quadrotor describes a path composed of two circular paths: one in a horizontal plane and the other in a plane tilted 45° . As a consequence, for the circular path in the horizontal plane, the leader pulling the trailer is describing a circle (thus

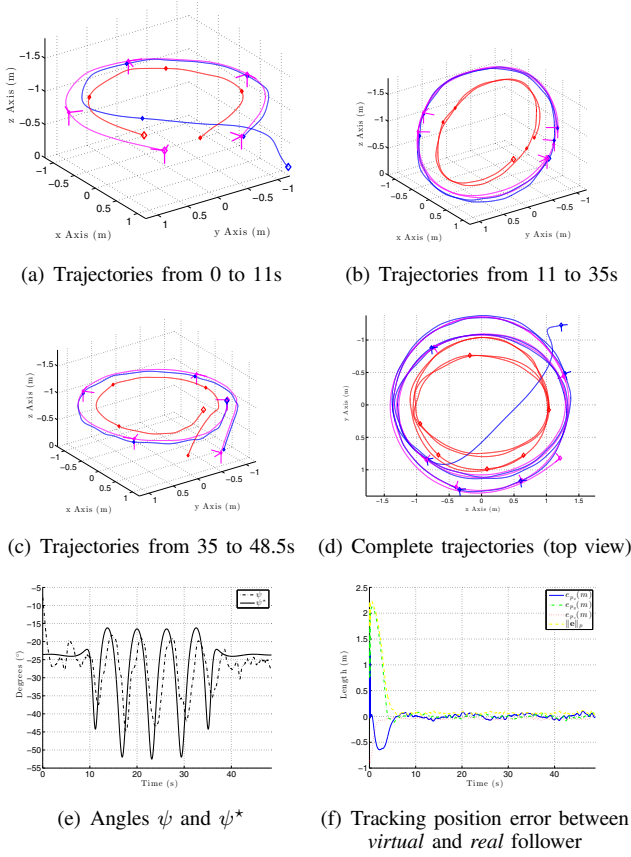


Fig. 2: Leader describing (in red) circular path in planes tilted 0° and 45°

Theorem 4 applies). However, for the circular path in the tilted plane, the leader pulling the trailer is describing an ellipse, which is better seen in Figure 2(d). For the first case, the *virtual* follower converges to a circular path as expected and ψ^* is a constant ($\approx -23.6^\circ$) to which ψ converges to, as can be seen in Figure 2(e). For the second case, ψ^* is not a constant (in an ellipse, the path curvature changes) and ψ tries to follow it, which can also be verified in Figure 2(e). The planning along $\mathbf{n}$ is very trivial, with the *virtual* follower keeping the same altitude as the leader's, clearly perceptible in Figures 2(a)-2(c). In Figure 2(f), the position error between *virtual* and *real* follower is seen converging to zero corresponding to the convergence of the blue line to the magenta line.

VI. CONCLUSIONS

In this paper, a real-time three dimensional trajectory planner for leader following is presented. The proposed trajectory planner is intuitive and can be implemented independently by each follower, thus reducing the need for communications among vehicles. Each follower vehicle behaves as a point of a two dimensional rigid body trailer rigidly attached to a leader vehicle. We prove that under a wide range of conditions, there is a unique attractive solution for the trailer reference frame to which all solutions converge, which demonstrates the robustness of the planning to disturbances and perturbations. Additionally, we provide bounds on the

planned velocity and acceleration, which can be used in limiting the leader's velocity and acceleration with the intent of guaranteeing the planned trajectories are feasible for all followers. Experiments performed with quadrotor vehicles were conducted that demonstrate the richness and suitability of the generated trajectories. Directions for future work include studying a sequence of n-trailers and incorporating a collision avoidance strategy.

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