2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

Globally Convergent Relative Attitude Observers for Three-Platform Formations

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Abstract— **This paper presents relative attitude filters for a formation of three mobile platforms based on direction measurements between the platforms, in addition to rate gyro readings. In short, relative attitude estimates are initially obtained from the raw direction measurements using a computationally efficient algebraic method, which has no ambiguities. Afterwards, an attitude and rate gyro bias observer is proposed, which is also computationally efficient and that provides filtered estimates of both the relative attitude between the platforms and the rate gyro bias. The stability is analyzed and it is shown that the error converges to zero for all initial conditions under some mild constraints on the filter gains. Simulation results are presented, in the presence of sensor noise, that illustrate the achievable performance of the proposed solutions.**

I. Introduction

In recent years, given the multitude of robotic platforms and sensors available and new low power embedded computation units, the scientific community embraced complex challenges involving the operation of formations of autonomous vehicles. Operational formations of such robotic platforms can tackle a wider range of potential problems. In land robotics, platoons of self-driving automobiles were studied as a viable solution to optimize the use of motorways, e.g. project PATH [1]. Rich underwater scenarios demand the use of multiple autonomous underwater and surface vehicles, as in the European project TRIDENT, where intervention operations by autonomous underwater vehicles were possible in infrastructures installed at the sea bottom [2], or the MORPH European project devoted to novel underwater survey techniques in complex scenarios [3]. The trend on the use of formations of multi-spacecraft vehicles is also observed in aerospace applications. Examples of successful projects are the NAVSTAR-GPS and GALILEO systems, and also missions for space and Earth observation [4]. All these applications share the requirement that each robotic platform needs to estimate its attitude and position, in absolute coordinates or relative to other vehicles of the formation in the vicinity, to optimize the performance of the tasks to be performed.

The by now classic solution to pose estimation is the extended Kalman filter (EKF), see [5] and [6]. Recently several alternative solutions to avoid the drawbacks and limitations of EKF have been proposed in [7], [8], [9], and [10] formulated resorting to differential geometry tools and exploiting the manifold properties, namely $SO(3)$ and/or $SE(3)$.

The problem of relative attitude determination of formation of spacecrafts was addressed in [11] based on line of sight data for a formation of three vehicles. Deterministic solutions were proposed and an error covariance analysis was presented. Solutions for the same problem, in a stochastic setting, were also proposed by some of the authors of the previous paper in [12], resorting to an EKF, leading to a solution where quaternions were used to express the relative attitude. No stability on the estimation errors is guaranteed and the attained performance was studied in simulation.

Recently, the authors of this work proposed a solution for relative position and linear velocity estimation for formations of vehicles, with fixed topologies. The decentralized solution proposed in [13] can be applied in marine robotic scenarios, where each vehicle in the formation estimates its own state relying only on locally available measurements and data communicated by neighboring agents, requiring lower computational and communication loads than centralized solutions. The solution that was proposed features error dynamics that converge globally asymptotically to zero. The proposed algorithm minimizes the $\mathcal{H}_2$ norm of the global estimation error dynamics, expressed as an optimization problem subject to bilinear matrix inequality (BMI) constraints. Thus, to complement those solutions, the problem addressed in this paper is the design of novel relative attitude filters for a formation of three mobile platforms based on direction measurements, in addition to rate gyro readings. The solution exploited avoids the use of EKFs and does not involve any linearization or approximation. The initial attitude estimates are obtained from the raw direction measurements using a computationally efficient algebraic method, with no ambiguities. Afterwards, attitude and rate gyro bias observers are proposed, which are also computationally efficient and that provide filtered estimates of both the relative attitude between the platforms and the rate gyro bias. The stability analyzes is carried out and it is shown that, under appropriate conditions, the observer error converges to zero for all initial conditions.

This work was supported by the Fundação para a Ciência e a Tecnologia (FCT) through ISR under LARSyS UID/EEA/50009/2013, and through IDMEC, under LAETA UID/EMS/50022/2013 contracts, by the University of Macau Project MYRG2015-00127-FST, and by the Macau Science and Technology Development Fund under Grant FDCT/065/2014/A.

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A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented by $\mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_\mathrm{n})$. For $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \times \mathbf{y}$ represents the inner and cross product. The set of unit vectors in $\mathbb{R}^3$ is denoted by $S(2) := \left\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\right\}$ and the Special Orthogonal Group is denoted by $SO(3) := \left\{\mathbf{X} \in \mathbb{R}^{3\times 3} : \mathbf{X}\mathbf{X}^T = \mathbf{X}^T\mathbf{X} = \mathbf{I} \wedge \det(\mathbf{X}) = 1\right\}$.

II. Problem statement

Consider three mobile platforms with 3 body-fixed frames associated to each of them, as depicted in Fig. 1. Each platform measures the directions to the other two, expressed in its own coordinates. Additionally, all platforms have angular velocity measurements provided by rate gyros. The problem considered here, loosely speaking, is that of estimating and filtering the relative attitudes between the platforms, considering also rate gyro bias.

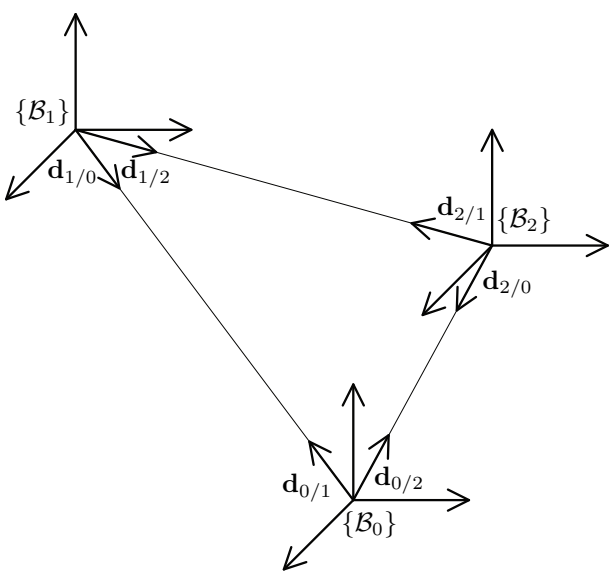


Fig. 1. Formation of three platforms: body-fixed frames and direction measurements

In order to properly set the problem framework, let $\{\mathcal{I}\}$ denote an inertial reference frame and $\{\mathcal{B}_i\}$ denote the body-fixed frame associated with the i-th platform, $i = 0, 1, 2$. The rotation matrix $\mathbf{R}_i^j(t) \in SO(3)$ represents the rotation from $\{\mathcal{B}_i\}$ to $\{\mathcal{B}_j\}$, $i, j = 0, 1, 2$, whereas $\mathbf{R}_i^I(t) \in SO(3)$ denotes the rotation matrix from $\{\mathcal{B}_i\}$ to $\{\mathcal{I}\}$, $i = 0, 1, 2$. The angular velocity of $\{\mathcal{B}_i\}$ with respect to $\{\mathcal{I}\}$ and expressed in $\{\mathcal{B}_i\}$ is denoted by $\boldsymbol{\omega}_i(t)$, $i = 0, 1, 2$, and hence

$$\dot{\mathbf{R}}_i^I(t) = \mathbf{R}_i^I(t)\mathbf{S}\left(\boldsymbol{\omega}_i(t)\right), \; i = 0, 1, 2, \tag{1}$$

where $\mathbf{S}\left(\boldsymbol{\omega}_i(t)\right) \in \mathbb{R}^{3\times 3}$ is the skew-symmetric matrix that encodes the cross product, i.e., $\mathbf{S}\left(\mathbf{x}\right)\mathbf{y} = \mathbf{x} \times \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

As previously mentioned, each platform measures, in its own reference frame, the directions to the other two. Let $\mathbf{d}_{i/j}(t) \in S(2)$ denote the direction from the i-th platform to the j-th platform, expressed in $\{\mathcal{B}_i\}$, i.e.,

$$\mathbf{d}_{i/j}(t) = \left[\mathbf{R}_i^I(t)\right]^T \frac{\mathbf{p}_j(t) - \mathbf{p}_i(t)}{\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\|}, \; i, j = 0, 1, 2,$$

where $\mathbf{p}_i(t)$ is the inertial position of the i-th platform. Finally, let the rate gyro measurements be given by

$$\boldsymbol{\omega}_{0m}(t) = \boldsymbol{\omega}_0(t)$$

and

$$\boldsymbol{\omega}_{im}(t) = \boldsymbol{\omega}_i(t) + \mathbf{b}_i(t), \; i = 1, 2,$$

where $\mathbf{b}_i(t)$ corresponds to the rate gyro bias, which is assumed constant, i.e.

$$\dot{\mathbf{b}}_i(t) = \mathbf{0}, \; i = 1, 2.$$

The following assumptions are considered throughout the paper.

Assumption 1: All angular velocities and accelerations are bounded.

Assumption 2: The positions of the three platforms define a plane.

The first is a common technical assumption that is evidently verified for all systems in practice, as one cannot have arbitrarily large angular velocities or accelerations [7]. The second is a necessary condition without which it is impossible to determine the relative attitudes between the platforms with the set of measurements that is here considered.

The problem addressed in the paper is that of designing observers, with globally convergent error dynamics, for the relative rotation matrices $\mathbf{R}_1^0(t)$, $\mathbf{R}_2^0(t)$, and $\mathbf{R}_1^2(t)$, as well as for the rate gyro biases $\mathbf{b}_1(t)$ and $\mathbf{b}_2(t)$, based on the rate gyro measurements $\boldsymbol{\omega}_i(t)$, $i = 0, 1, 2$, and the direction measurements $\mathbf{d}_{i/j}(t)$, $i, j = 0, 1, 2$, $i \neq j$.

III. Observer design

A. Deterministic determination of the relative attitude

The deterministic algorithm that is used in this paper to computed algebraic estimates of the rotation matrices is briefly presented in this section. The algorithm was proposed in [14] and it is essentially based on a triangle inequality.

Consider Fig. 2, where two reference frames are depicted, $\{\mathcal{V}\}$ and $\{\mathcal{W}\}$, and a third object is also included, visible from both reference frames. Denote by $\mathbf{R}_V^W \in SO(3)$ the rotation matrix from $\{\mathcal{V}\}$ to $\{\mathcal{W}\}$. In the figure, $\mathbf{w}_1$ corresponds to the direction from $\{\mathcal{W}\}$ to $\{\mathcal{V}\}$, expressed in $\{\mathcal{W}\}$, whereas $\mathbf{v}_1$ corresponds to the same direction but expressed in $\{\mathcal{V}\}$, i.e.,

$$\mathbf{w}_1 = \mathbf{R}_V^W \mathbf{v}_1. \tag{2}$$

The direction $\mathbf{w}_2$, from $\{\mathcal{W}\}$ to the third object, is expressed in $\{\mathcal{W}\}$, whereas the direction $\mathbf{v}_2$, from $\{\mathcal{V}\}$ to the third object, is expressed in $\{\mathcal{V}\}$. The triangle inequality is obtained by noticing that the inner angles of the triangle depicted in Fig. 2 must add up to π rad, which allows to write

$$\theta_3 = \pi - \theta_1 - \theta_2. \tag{3}$$

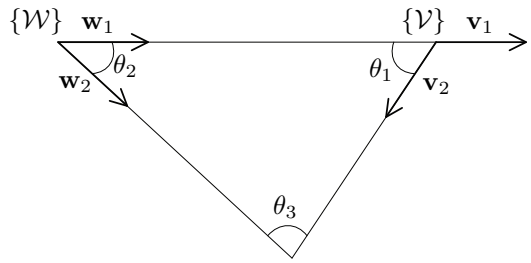


Fig. 2. Geometry for deterministic determination of relative attitude

Taking the cosine of both sides of (3), using $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$, with $\alpha = \pi-\theta_1$ and $\beta = \theta_2$, and noticing that

$$\begin{cases} \cos(\pi-\theta_1) = \mathbf{v}_1^T\mathbf{v}_2 \\ \cos(\theta_2) = \mathbf{w}_1^T\mathbf{w}_2 \\ \cos(\theta_3) = \mathbf{w}_2^T\mathbf{R}_V^W\mathbf{v}_2 \\ \sin(\pi-\theta_1) = \|\mathbf{v}_1\times\mathbf{v}_2\| \\ \sin(\theta_2) = \|\mathbf{w}_1\times\mathbf{w}_2\| \end{cases}$$

allows to write

$$\mathbf{w}_2^T\mathbf{R}_V^W\mathbf{v}_2 = \mathbf{w}_1^T\mathbf{w}_2\mathbf{v}_1^T\mathbf{v}_2 + \|\mathbf{w}_1\times\mathbf{w}_2\|\,\|\mathbf{v}_1\times\mathbf{v}_2\|. \quad (4)$$

The deterministic method that is used in this paper to obtain the relative attitude consists in determining the rotation matrix $\mathbf{R}_V^W$ that satisfies both (2) and (4), which is given by

$$\mathbf{R}_V^W = \mathbf{R}_2\mathbf{R}_1,$$

where

$$\mathbf{R}_1 := \frac{(\mathbf{w}_1+\mathbf{v}_1)(\mathbf{w}_1+\mathbf{v}_1)^T}{1+\mathbf{v}_1^T\mathbf{w}_1} - \mathbf{I}$$

and

$$\mathbf{R}_2 := \cos(\gamma)\,\mathbf{I} + [1+\cos(\gamma)]\,\mathbf{w}_1\mathbf{w}_1^T - \sin(\gamma)\,\mathbf{S}(\mathbf{w}_1),$$

with

$$\gamma := \operatorname{atan2}\left(\mathbf{w}_2\mathbf{S}(\mathbf{w}_1)\mathbf{R}_1\mathbf{v}_2, \mathbf{w}_2\mathbf{S}(\mathbf{w}_1)^2\mathbf{R}_1\mathbf{v}_2\right) + \pi,$$

where atan2 corresponds to the four-quadrant inverse tangent function. The idea behind this solution is to first obtain a rotation matrix that satisfies (2). It is possible to see, by direct substitution, that $\mathbf{R}_1$ satisfies (2). Then, the idea is to find the angle of the rotation $\mathbf{R}_2$ about the direction $\mathbf{w}_1$ such that (3) is satisfied, see [14] for further details. It is important to stress that this solution has no ambiguities and is computationally inexpensive.

With this algorithm it is possible to obtain the rotation matrices $\mathbf{R}_1^0(t)$, $\mathbf{R}_2^0(t)$, and $\mathbf{R}_1^2(t)$ based on the direction measurements $\mathbf{d}_{i/j}(t)$, $i,\, j = 0,\, 1,\, 2,\, i \neq j$. As an example, to obtain $\mathbf{R}_1^0$ one considers $\{\mathcal{V}\} = \{\mathcal{B}_1\}$, $\{\mathcal{W}\} = \{\mathcal{B}_0\}$, and

$$\begin{cases} \mathbf{v}_1 = -\mathbf{d}_{1/0} \\ \mathbf{v}_2 = \mathbf{d}_{1/2} \\ \mathbf{w}_1 = \mathbf{d}_{0/1} \\ \mathbf{w}_2 = \mathbf{d}_{0/2} \end{cases}.$$

The other two relative rotation matrices can be computed in a similar way and therefore the details are omitted.

B. Relative Attitude Observers

This section presents the design of observers for the relative attitude between the platforms, as well as for the rate gyro bias. First, the kinematics of the relative attitudes are derived.

The rotation matrix $\mathbf{R}_i^j(t)$ can be written as

$$\mathbf{R}_i^j(t) = \left[\mathbf{R}_j^I(t)\right]^T\mathbf{R}_i^I(t), \quad (5)$$

$i,\, j = 0,\, 1,\, 2$. Taking the time derivative of (5) and using (1) gives

$$\begin{aligned} \dot{\mathbf{R}}_i^j(t) &= \left[\mathbf{R}_j^I(t)\mathbf{S}(\boldsymbol{\omega}_j(t))\right]^T\mathbf{R}_i^I(t) \\ &\quad + \left[\mathbf{R}_j^I(t)\right]^T\mathbf{R}_i^I(t)\mathbf{S}(\boldsymbol{\omega}_i(t)). \end{aligned} \quad (6)$$

Substituting (5) in (6) and using the fact that $\mathbf{S}(.)$ is a skew-symmetric matrix allows to rewrite (6) as

$$\dot{\mathbf{R}}_i^j(t) = -\mathbf{S}(\boldsymbol{\omega}_j(t))\,\mathbf{R}_i^j(t) + \mathbf{R}_i^j(t)\mathbf{S}(\boldsymbol{\omega}_i(t)). \quad (7)$$

Now, using the property $\mathbf{R}\mathbf{S}(\mathbf{x})\mathbf{R}^T = \mathbf{S}(\mathbf{R}\mathbf{x})$ for all $\mathbf{R} \in SO(3)$, $\mathbf{x} \in \mathbb{R}^3$, allows to write the relative attitude kinematics (7) as

$$\dot{\mathbf{R}}_i^j(t) = \mathbf{R}_i^j(t)\mathbf{S}\left(\boldsymbol{\omega}_i(t) - \left[\mathbf{R}_i^j(t)\right]^T\boldsymbol{\omega}_j(t)\right). \quad (8)$$

Next, an observer for $\mathbf{R}_1^0(t)$ and $\mathbf{b}_1(t)$ is proposed and its stability is analyzed. The design follows the same basic idea of the attitude observer proposed in [15] but differs due to the fact that an additional nonlinear term must be considered. This is so because one is estimating the relative attitude and hence the angular velocity of the other platform also comes into play.

Consider a column representation of the rotation matrix $\mathbf{R}_1^0(t)$ given by

$$\boldsymbol{x}_1(t) = \begin{bmatrix} \mathbf{r}_1^T(t) & \mathbf{r}_2^T(t) & \mathbf{r}_3^T(t) \end{bmatrix}^T \in \mathbb{R}^9,$$

where

$$\mathbf{R}_1^0(t) = \begin{bmatrix} \mathbf{r}_1^T(t) \\ \mathbf{r}_2^T(t) \\ \mathbf{r}_3^T(t) \end{bmatrix},\ \mathbf{r}_i(t) \in \mathbb{R}^3,\ i = 1,\, 2,\, 3.$$

Then, it follows from (8) that

$$\dot{\boldsymbol{x}}_1(t) = -\mathbf{S}_3\left[\boldsymbol{\omega}_1(t) - \left[\mathbf{R}_1^0(t)\right]^T\boldsymbol{\omega}_0(t)\right]\boldsymbol{x}_1(t), \quad (9)$$

where

$$\mathbf{S}_3(\mathbf{x}) := \operatorname{diag}(\mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{x})) \in \mathbb{R}^{9\times 9},\ \mathbf{x} \in \mathbb{R}^3.$$

Let

$$\boldsymbol{\omega}_0(t) = \begin{bmatrix} \omega_{0x}(t) & \omega_{0y}(t) & \omega_{0z}(t) \end{bmatrix}.$$

Then, it is a matter of computation to show that

$$\left[\mathbf{R}_1^0(t)\right]^T\boldsymbol{\omega}_0(t) = \mathbf{W}(\boldsymbol{\omega}_0(t))\,\boldsymbol{x}_1(t),$$

with

$$\mathbf{W}(\boldsymbol{\omega}_0(t)) = \begin{bmatrix} \omega_{0x}(t)\mathbf{I} & \omega_{0y}(t)\mathbf{I} & \omega_{0z}(t)\mathbf{I} \end{bmatrix} \in \mathbb{R}^{3\times 9},$$

and hence (9) can be rewritten as

$$\dot{\boldsymbol{x}}_1(t) = -\mathbf{S}_3\left[\boldsymbol{\omega}_1(t) - \mathbf{W}(\boldsymbol{\omega}_0(t))\,\boldsymbol{x}_1(t)\right]\boldsymbol{x}_1(t).$$

Consider the attitude observer given by

$$\dot{\hat{\boldsymbol{x}}}_1(t) = -\mathbf{S}_3\left[\boldsymbol{\omega}_{1m}(t) - \hat{\mathbf{b}}_1(t) - \mathbf{W}\left(\boldsymbol{\omega}_0(t)\right)\hat{\boldsymbol{x}}_1(t)\right]\hat{\boldsymbol{x}}_1(t) + k_1(t)\left[\boldsymbol{y}_1(t) - \hat{\boldsymbol{x}}_1(t)\right] \quad (10)$$

and

$$\begin{aligned}\dot{\hat{\mathbf{b}}}_1(t) &= k_2\mathbf{S}\left(\left[\mathbf{I}\ \mathbf{0}\ \mathbf{0}\right]\hat{\boldsymbol{x}}_1(t)\right)\left[\mathbf{I}\ \mathbf{0}\ \mathbf{0}\right]\mathbf{y}_1(t) \\ &\quad + k_2\mathbf{S}\left(\left[\mathbf{0}\ \mathbf{I}\ \mathbf{0}\right]\hat{\boldsymbol{x}}_1(t)\right)\left[\mathbf{0}\ \mathbf{I}\ \mathbf{0}\right]\mathbf{y}_1(t) \\ &\quad + k_2\mathbf{S}\left(\left[\mathbf{0}\ \mathbf{0}\ \mathbf{I}\right]\hat{\boldsymbol{x}}_1(t)\right)\left[\mathbf{0}\ \mathbf{0}\ \mathbf{I}\right]\mathbf{y}_1(t), \quad (11)\end{aligned}$$

where $k_1(t)$ and k_2 are observer gains, $\hat{\boldsymbol{x}}_1(t)$ and $\hat{\mathbf{b}}_1(t)$ are the observer estimates, and $\mathbf{y}_1(t)$ corresponds to the deterministic attitude $\mathbf{R}_1^0(t)$ in vector form, i.e., $y_1(t) = \boldsymbol{x}_1(t)$, obtained as detailed in Section III-A. Define the error variables

$$\begin{cases} \tilde{\mathbf{x}}_1(t) := \boldsymbol{x}_1(t) - \hat{\boldsymbol{x}}_1(t) \\ \tilde{\mathbf{b}}_1(t) := \mathbf{b}_1(t) - \hat{\mathbf{b}}_1(t) \end{cases}.$$

Then, the observer error dynamics are given by

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}_1(t) &= -k_1(t)\tilde{\mathbf{x}}_1(t) + \mathbf{S}_3\left(\tilde{\mathbf{b}}_1(t)\right)\boldsymbol{x}_1(t) \\ &\quad + \mathbf{S}_3\left(\mathbf{W}\left(\boldsymbol{\omega}_0(t)\right)\tilde{\boldsymbol{x}}_1(t)\right)\boldsymbol{x}_1(t) \\ &\quad - \mathbf{S}_3\left(\boldsymbol{\omega}_1(t) + \tilde{\boldsymbol{b}}_1(t) - \mathbf{W}\left(\boldsymbol{\omega}_0(t)\right)\left[\boldsymbol{x}_1(t) - \tilde{\boldsymbol{x}}_1(t)\right]\right)\tilde{\boldsymbol{x}}_1(t) \quad (12)\end{aligned}$$

and

$$\begin{aligned}\dot{\tilde{\mathbf{b}}}_1(t) &= k_2\mathbf{S}\left(\left[\mathbf{I}\ \mathbf{0}\ \mathbf{0}\right]\tilde{\boldsymbol{x}}_1(t)\right)\left[\mathbf{I}\ \mathbf{0}\ \mathbf{0}\right]\boldsymbol{x}_1(t) \\ &\quad + k_2\mathbf{S}\left(\left[\mathbf{0}\ \mathbf{I}\ \mathbf{0}\right]\tilde{\boldsymbol{x}}_1(t)\right)\left[\mathbf{0}\ \mathbf{I}\ \mathbf{0}\right]\boldsymbol{x}_1(t) \\ &\quad + k_2\mathbf{S}\left(\left[\mathbf{0}\ \mathbf{0}\ \mathbf{I}\right]\tilde{\boldsymbol{x}}_1(t)\right)\left[\mathbf{0}\ \mathbf{0}\ \mathbf{I}\right]\boldsymbol{x}_1(t). \quad (13)\end{aligned}$$

The following theorem is the main result of the paper.

Theorem 1: Suppose that Assumptions 1 and 2 hold and consider the observer given by (10)-(11), where the measurements $\mathbf{y}_1(t)$, which correspond to the vector form of the relative attitude $\mathbf{R}_1^0(t)$, are obtained as detailed in Section III-A from the direction measurements. Suppose that $k_1(t) > 0$ is a positive bounded function of time, with bounded time derivative, such that

$$k_1(t) \geq \lambda + 3\sqrt{3}\left\|\boldsymbol{\omega}_0(t)\right\| \quad (14)$$

for all t, where $\lambda > 0$ is an arbitrarily small positive constant. Suppose also that $k_2 > 0$. Then, the errors $\tilde{\boldsymbol{x}}_1(t)$ and $\tilde{\mathbf{b}}_1(t)$ converge to zero for all initial conditions.

Proof: Due to the lack of space, only a sketch of the proof is presented. Using the Lyapunov candidate function

$$V(t) := \frac{1}{2}\left\|\tilde{\boldsymbol{x}}_1(t)\right\|^2 + \frac{1}{2k_2}\left\|\tilde{\mathbf{b}}_1(t)\right\|^2.$$

is can be shown that, in the conditions of the theorem, $\dot{V}(t) \leq -\lambda\left\|\tilde{\mathbf{x}}_1(t)\right\|^2$. After careful verification, one can invoke Barbalat's lemma to show that

$$\lim_{t\to\infty}\tilde{\mathbf{x}}_1(t) = \mathbf{0}. \quad (15)$$

Additionally, it is possible to show that $\dot{\tilde{\mathbf{x}}}_1(t)$ is uniformly continuous. Therefore, invoking Barbalat's lemma again, one concludes that

$$\lim_{t\to\infty}\dot{\tilde{\mathbf{x}}}_1(t) = \mathbf{0}. \quad (16)$$

Finally, taking the limit of both sides of (12) and using both (15) and (16) allows to conclude that

$$\lim_{t\to\infty}\mathbf{S}_3\left(\tilde{\mathbf{b}}_1(t)\right)\boldsymbol{x}_1(t) = \mathbf{0}. \quad (17)$$

Now, recalling that $\boldsymbol{x}_1(t)$ corresponds to a column representation of the rotation matrix $\mathbf{R}_1^0(t)$, it is possible to rewrite (17) as

$$\lim_{t\to\infty}\mathbf{R}_1^0(t)\mathbf{S}\left(\tilde{\mathbf{b}}_1(t)\right) = \mathbf{0}. \quad (18)$$

As $\mathbf{R}_1^0(t)$ is a rotation matrix, it follows from (18) that

$$\lim_{t\to\infty}\tilde{\mathbf{b}}_1(t) = \mathbf{0}.$$

■

At this point, one already has all it needs to estimate the three relative attitude rotation matrices. Indeed, an estimate of $\mathbf{R}_1^0(t)$ is given by

$$\hat{\mathbf{R}}_1^0(t) = \begin{bmatrix} \hat{\mathbf{r}}_1^T(t) \\ \hat{\mathbf{r}}_2^T(t) \\ \hat{\mathbf{r}}_3^T(t) \end{bmatrix}, \quad (19)$$

with

$$\hat{\boldsymbol{x}}_1(t) = \begin{bmatrix} \hat{\mathbf{r}}_1(t) \\ \hat{\mathbf{r}}_2(t) \\ \hat{\mathbf{r}}_3(t) \end{bmatrix} \in \mathbb{R}^9,\ \hat{\mathbf{r}}_i(t) \in \mathbb{R}^3,\ i = 1,\ 2,\ 3.$$

Next, notice that the structure of the dynamics of $\mathbf{R}_2^0(t)$ is identical to that of $\mathbf{R}_1^0(t)$. Hence, an observer with a similar structure to (10)-(11) can provide an estimate $\hat{\boldsymbol{x}}_2(t)$, which corresponds to an estimate $\hat{\mathbf{R}}_2^0(t)$ of $\mathbf{R}_2^0(t)$, in a similar way to (19). Finally, an estimate of $\mathbf{R}_1^2(t)$ is readily given by

$$\hat{\mathbf{R}}_1^2(t) = \left[\hat{\mathbf{R}}_2^0(t)\right]^T\hat{\mathbf{R}}_1^0(t).$$

C. Further discussion

1) Estimates on $SO(3)$: The observer design that is proposed in this paper does not explicitly take into account the fact that the rotation matrices belong to $SO(3)$. This is a choice of design that allows to obtain global convergence results, which is not possible by continuous feedback if the design is explicitly made on $SO(3)$. Nevertheless, as the estimation error converges to zero for all initial conditions, the attitude estimates converge asymptotically to elements of $SO(3)$. Additionally, there exist several computationally inexpensive methods in the literature that can be used to refine the estimates provided by the observers proposed in this paper, yielding estimates on $SO(3)$. As this has been previously discussed, the reader is referred to [15] and [16] for such alternatives.

2) Estimate of $\boldsymbol{R}_1^2(t)$: In this paper the observers that are proposed are only for $\mathbf{R}_1^0(t)$ and $\mathbf{R}_2^0(t)$ and the estimates of $\mathbf{R}_1^2(t)$ are obtained from these. This is so because all three rotation matrices satisfy the relation $\mathbf{R}_1^2(t) = \left[\mathbf{R}_2^0(t)\right]^T\mathbf{R}_1^0(t)$ for all t and as such there is no need to include a third observer. Moreover, as it will be shown in the simulations, the performance that is achieved in this way is very good. Additionally, if an estimate of $\mathbf{R}_1^2(t)$ is obtained from refined estimates (on $SO(3)$) of $\mathbf{R}_1^0(t)$ and $\mathbf{R}_2^0(t)$, the relation between the three is satisfied by construction.

IV. Simulation results

Simulation results are presented in this section to illustrate the achievable performance with the proposed solution. In the simulations the platforms are assumed, without loss of generality, to be positioned in the plane $z = 0$, in the inertial frame. The evolution of the positions of the three platforms, in the xOy plane, is depicted in Fig. 3, where the starting positions are marked with a cross. In short, the position of platform 0 is constant, while platforms 1 and 2 move along straight paths with constant velocity. The initial inertial

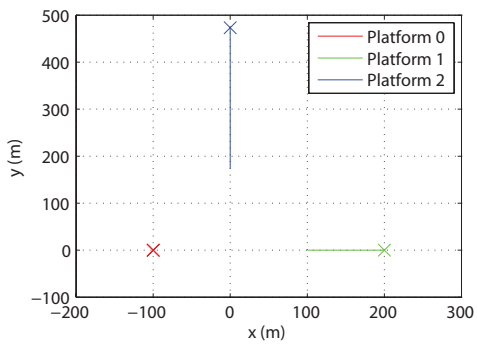


Fig. 3. Trajectories of the formation of three platforms

attitude of all platforms was set to the identity matrix, i.e., $\mathbf{R}_0^I(0) = \mathbf{R}_1^I(0) = \mathbf{R}_2^I(0) = \mathbf{I}$, and the angular velocities of the platforms with respect to the inertial frame follow

$$\boldsymbol{\omega}_0(t) = \frac{\pi}{180}\begin{bmatrix} 15\sin\left(\frac{2\pi}{60}t\right) \\ \sin\left(\frac{2\pi}{180}t\right) \\ -5\sin\left(\frac{2\pi}{300}t\right) \end{bmatrix} \text{(rad/s)},$$

$$\boldsymbol{\omega}_1(t) = \frac{\pi}{180}\begin{bmatrix} 2\sin\left(\frac{2\pi}{60}t\right) \\ 10\sin\left(\frac{2\pi}{180}t\right) \\ -5\sin\left(\frac{2\pi}{300}t\right) \end{bmatrix} \text{(rad/s)},$$

and

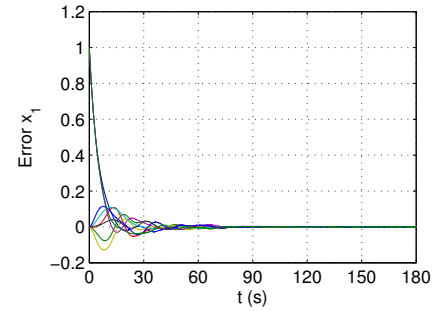
$$\boldsymbol{\omega}_2(t) = \frac{\pi}{180}\begin{bmatrix} 5\sin\left(\frac{2\pi}{60}t\right) \\ 10\sin\left(\frac{2\pi}{180}t\right) \\ -2\sin\left(\frac{2\pi}{300}t\right) \end{bmatrix} \text{(rad/s)}.$$

These have no specific meaning, in fact any profile for the angular velocity could have been chosen.

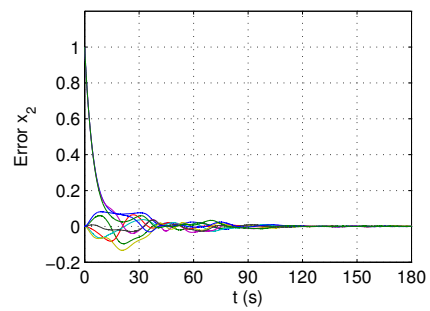
In the simulations the measurements of the directions are assumed to be corrupted by zero-mean white Gaussian noise, with standard deviation of 0.01 on each axis. Afterwards, the directions are normalized again so that they correspond to unit vectors. The rate gyro measurements are also corrupted by additive, zero-mean, white Gaussian noise, with standard deviation of 0.1 $^\circ/s$ on each axis. The rate gyro biases were set to $\mathbf{b}_1 = \begin{bmatrix} -0.5 & 1 & 2 \end{bmatrix}$ $(^\circ/s)$ and $\mathbf{b}_2 = \begin{bmatrix} -1 & 0.5 & -2 \end{bmatrix}$ $(^\circ/s)$. A sampling frequency of 100 Hz was considered and the fourth-order Runge-Kutta method was employed in the simulations.

The observer gains were chosen empirically, as in most nonlinear estimation problems, to ensure both fast convergence and good steady-state filtering performance. The gains were set to $k_1 = 0.2$ and $k_2 = 0.01$. The initial estimates were set to zero.

The initial convergence of the errors $\tilde{\boldsymbol{x}}_1(t)$ and $\tilde{\boldsymbol{x}}_2(t)$ is depicted in Fig. 4, whereas the initial convergence of the errors $\tilde{\mathbf{b}}_1(t)$ and $\tilde{\mathbf{b}}_2(t)$ is shown in Fig. 5. It is possible to



(a) Error $\tilde{\boldsymbol{x}}_1(t)$



(b) Error $\tilde{\boldsymbol{x}}_2(t)$

Fig. 4. Initial convergence of the errors $\tilde{\boldsymbol{x}}_1(t)$ and $\tilde{\boldsymbol{x}}_2(t)$

observe that the errors converge to a neighborhood of zero very fast. In the absence of noise, the errors converge to zero.

In order to evaluate the steady-state filtering performance, additional error variables are defined as

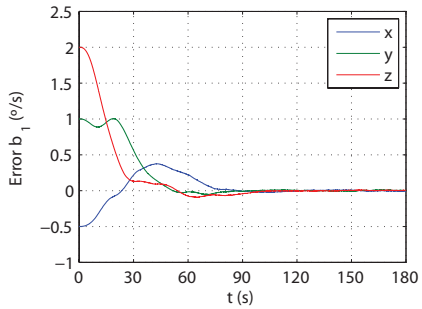
$$\begin{cases} \tilde{\mathbf{R}}_1^0(t) = \left[\mathbf{R}_1^0\right]^T(t)\hat{\mathbf{R}}_1^0(t) \\ \tilde{\mathbf{R}}_2^0(t) = \left[\mathbf{R}_2^0\right]^T(t)\hat{\mathbf{R}}_2^0(t) \\ \tilde{\mathbf{R}}_1^2(t) = \left[\mathbf{R}_1^2\right]^T(t)\hat{\mathbf{R}}_1^2(t) \end{cases},$$

which corresponds to the rotation matrix errors. Here, $\hat{\mathbf{R}}_1^0(t)$ and $\hat{\mathbf{R}}_2^0(t)$ correspond to refined estimates of the rotation matrices, obtained by projecting the estimates provided by the observers on $SO(3)$, while $\hat{\mathbf{R}}_1^2(t)$ was computed based on the other two refined estimates. Using the Euler angle-axis representation for these new error variables,

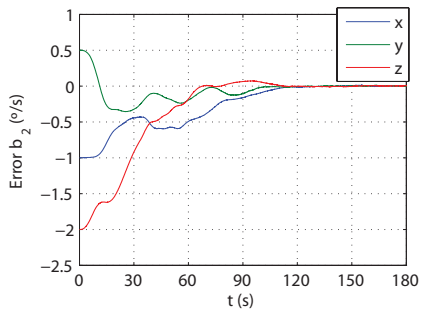
$$\begin{aligned} \tilde{\mathbf{R}}_1^0(t) &= \mathbf{I}\cos\left(\tilde{\theta}_1(t)\right) + \left[1-\cos\left(\tilde{\theta}_1(t)\right)\right]\tilde{\mathbf{d}}_1(t)\tilde{\mathbf{d}}_1^T(t) \\ &\quad -\mathbf{S}\left(\tilde{\mathbf{d}}_1(t)\right)\sin\left(\tilde{\theta}_1(t)\right), \\ \tilde{\mathbf{R}}_2^0(t) &= \mathbf{I}\cos\left(\tilde{\theta}_2(t)\right) + \left[1-\cos\left(\tilde{\theta}_2(t)\right)\right]\tilde{\mathbf{d}}_2(t)\tilde{\mathbf{d}}_2^T(t) \\ &\quad -\mathbf{S}\left(\tilde{\mathbf{d}}_2(t)\right)\sin\left(\tilde{\theta}_2(t)\right), \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathbf{R}}_1^2(t) &= \mathbf{I}\cos\left(\tilde{\theta}_3(t)\right) + \left[1-\cos\left(\tilde{\theta}_3(t)\right)\right]\tilde{\mathbf{d}}_3(t)\tilde{\mathbf{d}}_3^T(t) \\ &\quad -\mathbf{S}\left(\tilde{\mathbf{d}}_3(t)\right)\sin\left(\tilde{\theta}_3(t)\right), \end{aligned}$$



(a) Error $\tilde{\mathbf{b}}_1(t)$



(b) Error $\tilde{\mathbf{b}}_2(t)$

Fig. 5. Initial convergence of the errors $\tilde{\mathbf{b}}_1(t)$ and $\tilde{\mathbf{b}}_2(t)$

where $0 \leq \tilde{\theta}_i(t) \leq \pi$ and $\tilde{\mathbf{d}}_i(t) \in \mathbb{R}^3$, $\left\|\tilde{\mathbf{d}}_i(t)\right\| = 1$, are the angles and axes that represent the rotation errors, the performance of the proposed filters is identified from the evolutions of $\tilde{\theta}_i(t)$, $i = 1, 2, 3$. The mean angle errors, computed for $t \geq 180$ s, are $0.0975°$ for θ_1, $0.0893°$ for θ_2, and $0.0868°$ for θ_3. These are indeed very good results and clearly evidence the filtering effect of the proposed solutions. In comparison, the mean raw rotation errors, computed based on the rotation matrices obtained directly from the direction measurements, are $1.3608°$ for θ_1, $1.5347°$ for θ_2, and $1.5545°$ for θ_3. Even worse, the raw angle errors reach values up to 6°, which compare to a maximum error of 0.25° with the proposed solution. Therefore, one can conclude that the proposed solutions effectively provided excellent filtered estimates of the relative rotation matrices between the 3 platforms. In steady-state, the rage gyro bias error remains, most of the time, below 0.01 $°/s$, which compares to the standard deviation of the noise of the rate angular velocity measurements, set to 0.01 $°/s$.

V. Conclusions

This paper addressed the problem of relative attitude estimation between formations of three platforms based on direction measurements, in addition to rate gyro readings. In short, while algebraic and optimization-based methods exist to obtain estimates of the relative attitudes based on the direction measurements, these are not filtered and are hence very prone to sensor noise. In this paper an observer was proposed that incorporates rate gyro readings, which drive the system dynamics, while the raw rotation matrix measurements are used to close the loop, while estimating at the same time the rate gyro biases. The stability of the proposed solution was analyzed and it was shown that, under some mild constraints, the error converges to zero for all initial conditions. Simulation results were presented, in the presence of sensor noise, that clearly show the filtering effect of the proposed solutions, effectively reducing the error by a factor of more than 10. Future work will include extensive Monte Carlo simulations prior to experimental evaluation.

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