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Relative Attitude Observers for Three-Platform Formations with Inertial Spread Observations

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Abstract— **The problem of relative attitude estimation for a formation of three mobile platforms is addressed in this paper. In previous work by the authors rate gyro bias has been considered for two of the three platforms. This paper proposes an extended framework that includes rate gyro bias estimation for all platforms. In order to do so, two additional vector observations of constant inertial vectors are assumed available, which may be distributed among the platforms. Cascade observers are then proposed, where the first stage acts as a bias observer and the second allows to filter attitude estimates directly on the special orthogonal group. The first is globally exponentially stable and the second is locally input-to-state stable with respect to the errors of the first, with the region of convergence for the initial attitude estimate better described as semi-global. Lastly, simulation results are presented that exemplify the performance of the proposed estimators in the presence of sensor noise.**

I. Introduction

Empowered by impressive, fast-paced technological and theoretical advances, a trend shift is occurring to the use of multiple agents as an alternative to individual robotic systems. Indeed, multi-agent solutions are now recognized as a superior answer to many problems, taking advantage of important unique features such as flexible distributed sensing and intervention abilities, complementary heterogeneity, robustness to individual failures through redundancy, scalability in time and space, smaller deployment times, reduced operational costs, improved performance, and dispensableness of cheap individuals in dangerous tasks. Operational formations of such robotic platforms can tackle a wider range of potential problems and a common requirement in most (if not all) scenarios is that each robotic agent needs to obtain estimates of its attitude and position, in absolute coordinates or relative to other vehicles of the formation, in order to perform the tasks at hand.

The topic of attitude estimation has attracted a lot of interest in recent years, in spite of the fact that is has been extensively studied for some decades now. The solution of the Wahba's problem [1] provides an algebraic estimate of the rotation between two coordinate frames given two sets of corresponding vectors expressed in each of the frames. However, it does not incorporate any filtering effect, which can be achieved by using information of other sensors such as rate gyros. The extended Kalman filter (EKF) has been proposed as an alternative to this algebraic solution for the problem of pose estimation, see e.g. [2] and [3]. More recently, several alternative solutions that avoid the drawbacks and limitations of the EKF have been proposed in [4], [5], [6], [7], [8], [9], and [10], formulated resorting to differential geometry tools and exploiting the manifold properties, namely $SO(3)$ and/or $SE(3)$. Previous work by the authors on the subject of attitude estimation includes [11], [12], and [13], where globally asymptotically or exponentially stable observers and filters have been proposed.

The problem of relative attitude determination for formations of three spacecrafts was addressed in [14] based on line of sight measurements. Deterministic solutions were proposed and the covariance of the error was analysed. Solutions for the same problem, in a stochastic setting, were also proposed by some of the authors of the previous paper in [15], resorting to an EKF, leading to a solution where quaternions were used to express the relative attitude. A related approach can also be found in [16], where multiple constraints were incorporated.

In [17] the authors addressed the problem of relative attitude estimation for a formation of three platforms based on direction and rate gyro measurements. In particular, each platform measures the direction to the other two in its own coordinate frame and two of the platforms also explicitly take into account the rate gyro bias, while a third is assumed to measure bias-free angular velocities. The solution that was proposed avoids the limitations and drawbacks of EKFs and it was shown that the estimation error converges to zero for all initial states under appropriate conditions on the observer gains. The greatest drawback of that approach is the fact that one of the platforms must have access to bias-free angular velocities. This paper proposes a novel solution, which includes additional measurements, that allows for the estimation of rate gyro bias on all three platforms.

The framework that is proposed to tackle this problem considers a small modification: two additional observations of constant inertial vectors are available. These vector observations may be spread over the formation, i.e., both vectors may be measured by a single platform or by two distinct

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platforms, each one measuring one of the vectors. The estimation solution is also different from the one proposed in [17]. Firstly, a cascade approach is considered, where the first block acts as a bias observer, while the second fuses all the measurements to obtain filtered estimates of the attitude. Moreover, while the solution proposed in [17] considers topological relaxations in order to achieve strong stability results, but providing estimates that are not directly obtained on the special orthogonal group $SO(3)$, the attitude observers herein proposed are built directly on $SO(3)$. Interestingly enough, the stability analysis reveals a different condition on the observer gains for convergence, which is best characterized as semi-global in this case.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. For $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ represent the inner and the cross product, respectively. The set of unit vectors in $\mathbb{R}^3$ is denoted by $S(2) := \left\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\right\}$ and the Special Orthogonal Group of order 3 is denoted by $SO(3) := \left\{\mathbf{X} \in \mathbb{R}^{3\times 3} : \mathbf{X}\mathbf{X}^T = \mathbf{X}^T\mathbf{X} = \mathbf{I} \wedge \det(\mathbf{X}) = 1\right\}$. The associated Lie algebra is the set of skew-symmetric matrices $\mathfrak{so}(3) := \left\{\mathbf{S} \in \mathbb{R}^{3\times 3} : \mathbf{S} = -\mathbf{S}^T\right\}$. For convenience, we define the operators $\mathbf{S}(.) : \mathbb{R}^3 \to \mathfrak{so}(3)$ as

$$\mathbf{S}(\mathbf{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

and $\mathbf{vex}(.) : \mathfrak{so}(3) \to \mathbb{R}^3$ as

$$\mathbf{vex}(\mathbf{S}(\mathbf{x})) = \mathbf{x}, \ \mathbf{x} \in \mathbb{R}^3.$$

II. Problem Statement

Consider three mobile platforms with 3 body-fixed frames associated to each one of them, as depicted in Fig. 1. Each platform measures the directions to the other two, expressed in its own coordinates. Additionally, all platforms have angular velocity measurements provided by rate gyros. Finally, two vector observations of constant inertial vectors are assumed available to the formation. Notice that these two vectors may be either measured by a single platform or two platforms may measure each one of the vectors. The latter situation is depicted in Fig. 1. The problem considered here is that of obtaining filtered estimates of the relative attitudes between the platforms and the rate gyro biases.

In order to properly set the problem framework, let $\{\mathcal{I}\}$ denote an inertial reference frame and $\{\mathcal{B}_i\}$ denote the body-fixed frame associated with the i-th platform, $i = 0, 1, 2$. The rotation matrix $\mathbf{R}_i^j(t) \in SO(3)$ represents the rotation from $\{\mathcal{B}_i\}$ to $\{\mathcal{B}_j\}$, $i, j = 0, 1, 2$, whereas $\mathbf{R}_i^I(t) \in SO(3)$ denotes the rotation matrix from $\{\mathcal{B}_i\}$ to $\{\mathcal{I}\}$, $i = 0, 1, 2$. The angular velocity of $\{\mathcal{B}_i\}$ with respect to $\{\mathcal{I}\}$ and expressed in $\{\mathcal{B}_i\}$ is denoted by $\boldsymbol{\omega}_i(t)$, $i = 0, 1, 2$, and hence

$$\dot{\mathbf{R}}_i^I(t) = \mathbf{R}_i^I(t)\mathbf{S}(\boldsymbol{\omega}_i(t)), \ i = 0, 1, 2. \tag{1}$$

As previously mentioned, each platform measures, in its own reference frame, the directions to the other two. Let

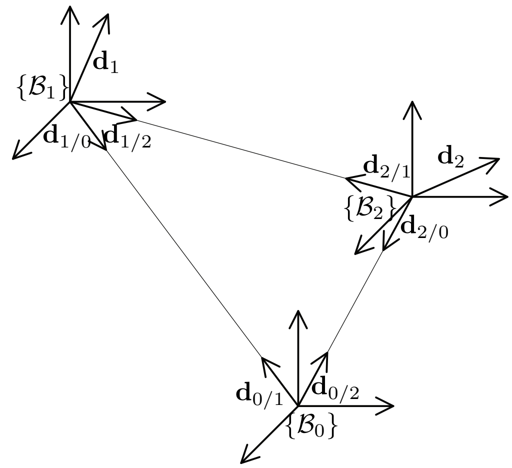

Fig. 1. Formation of three platforms: body-fixed frames and direction measurements

$\mathbf{d}_{i/j}(t) \in S(2)$ denote the direction from the i-th platform to the j-th platform, expressed in $\{\mathcal{B}_i\}$, i.e.,

$$\mathbf{d}_{i/j}(t) = \left[\mathbf{R}_i^I(t)\right]^T \frac{\mathbf{p}_j(t) - \mathbf{p}_i(t)}{\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\|}, \ i, j = 0, 1, 2,$$

where $\mathbf{p}_i(t)$ is the inertial position of the i-th platform. Two additional vector measurements of inertial constant vectors are also available. Without loss of generality, it is here assumed that one of these measurements is done by platform 1 in its own coordinate frame and platform 2 measures the second vector in its own coordinate frame, i.e.,

$$\mathbf{d}_i(t) = \left[\mathbf{R}_i^I(t)\right]^T {}^I\mathbf{d}_i, \ i = 1, 2, \tag{2}$$

where ${}^I\mathbf{d}_i$ and $\mathbf{d}_i(t)$, $i = 1, 2$, are the inertial and vector measurements, respectively. The case where both measurements are measured by the same platform is discussed in detail when required. Finally, let the rate gyro measurements be given by

$$\boldsymbol{\omega}_{im}(t) = \boldsymbol{\omega}_i(t) + \mathbf{b}_i(t), \ i = 0, 1, 2,$$

where $\mathbf{b}_i(t)$ corresponds to the rate gyro bias, which is assumed constant, i.e.

$$\dot{\mathbf{b}}_i(t) = \mathbf{0}, \ i = 0, 1, 2.$$

The following assumptions are considered throughout the paper.

Assumption 1: All angular velocities are bounded.

Assumption 2: The positions of the three platforms define a plane and the inertial vectors ${}^I\mathbf{d}_i$, $i = 1, 2$, are non-collinear.

The first is a common technical assumption that is evidently verified for all systems in practice, as one cannot have arbitrarily large angular velocities [9]. The second ensures that it is possible to determine the relative attitudes between the platforms with the set of measurements that is here considered, as well as the rate gyro bias of all platforms.

The problem addressed in the paper is that of designing observers with stability and convergence guarantees for the relative rotation matrices $\mathbf{R}_1^0(t)$, $\mathbf{R}_2^0(t)$, and $\mathbf{R}_1^2(t)$, as well as for the rate gyro biases $\mathbf{b}_0(t)$, $\mathbf{b}_1(t)$, and $\mathbf{b}_2(t)$, based on the rate gyro measurements $\boldsymbol{\omega}_i(t)$, $i = 0, 1, 2$, the relative direction measurements $\mathbf{d}_{i/j}(t)$, $i, j = 0, 1, 2$, $i \neq j$, and $\mathbf{d}_i(t)$, $i = 1, 2$. Notice that the actual inertial vectors ${}^I\mathbf{d}_i$ are not required.

III. Observer design

The first step in the design proposed in this paper is to recognize that, given the available measurements in the framework proposed in Section II, it is possible to determine the relative attitude between all platforms, as briefly detailed in Section III-A. However, this estimate is solely algebraic and does not incorporate and filtering effect. Next, a bias observer is proposed that makes use of the relative attitudes computed directly from the direction measurements, as well as the vector observations of inertial measurements. This will be detailed in Section III-B. Making use of the bias estimates, as well as all other information, an observer for the relative attitude of the platforms is proposed in Section III-C and its stability analyzed considering the cascade structure of the proposed solution.

A. Deterministic solution for the relative attitude

The problem of deterministic determination of the relative attitude for three platforms can be solved resorting to the algorithm proposed in [18], where the relative attitude between two platforms is computed based on measurements from each platform to the other, as well as measurements of both platforms to a third common object. In the scenario envisioned in Section II, when determining the relative attitude between two of the platforms, the thirds acts as the common object to which both platforms also measure the direction. As this has been previously studied in detail, only a brief summary is here presented.

Consider Fig. 2, where two reference frames are depicted, $\{\mathcal{V}\}$ and $\{\mathcal{W}\}$, and a third object is also included, visible from both reference frames. Denote by $\mathbf{R}_V^W \in SO(3)$ the rotation matrix from $\{\mathcal{V}\}$ to $\{\mathcal{W}\}$. In the figure, $\mathbf{w}_1$ corresponds to the direction from $\{\mathcal{W}\}$ to $\{\mathcal{V}\}$, expressed in $\{\mathcal{W}\}$, whereas $\mathbf{v}_1$ corresponds to the same direction but expressed in $\{\mathcal{V}\}$, i.e., $\mathbf{w}_1 = \mathbf{R}_V^W \mathbf{v}_1$. The direction $\mathbf{w}_2$, from $\{\mathcal{W}\}$ to the third object, is expressed in $\{\mathcal{W}\}$, whereas the direction $\mathbf{v}_2$, from $\{\mathcal{V}\}$ to the third object, is expressed in $\{\mathcal{V}\}$. Then, the rotation matrix from $\{\mathcal{V}\}$ to $\{\mathcal{W}\}$ can be

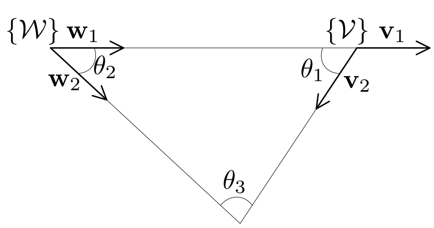

Fig. 2. Geometry for deterministic determination of the relative attitude between two platforms

obtained from $\mathbf{R}_V^W = \mathbf{R}_2\mathbf{R}_1$, where

$$\mathbf{R}_1 := \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{1 + \mathbf{v}_1^T\mathbf{w}_1} - \mathbf{I}$$

and

$$\mathbf{R}_2 := \cos(\gamma)\,\mathbf{I} + [1 + \cos(\gamma)]\,\mathbf{w}_1\mathbf{w}_1^T - \sin(\gamma)\,\mathbf{S}(\mathbf{w}_1),$$

with

$$\gamma := \operatorname{atan2}\left(\mathbf{w}_2\mathbf{S}(\mathbf{w}_1)\mathbf{R}_1\mathbf{v}_2, \mathbf{w}_2\mathbf{S}(\mathbf{w}_1)^2\mathbf{R}_1\mathbf{v}_2\right) + \pi,$$

where atan2 denotes to the four-quadrant inverse of the tangent function, see [18] for further details. Notice that this solution is computationally inexpensive and, more importantly, it has no ambiguities as long as the third object defines, together with the origins of $\{\mathcal{V}\}$ and $\{\mathcal{W}\}$, a plane.

The relative attitude between the platforms, expressed by the rotation matrices $\mathbf{R}_1^0(t)$, $\mathbf{R}_2^0(t)$, and $\mathbf{R}_1^2(t)$, can be readily obtained using this deterministic solution based on the direction measurements $\mathbf{d}_{i/j}(t)$, $i, j = 0, 1, 2$, $i \neq j$. As an example, to obtain $\mathbf{R}_1^0$ one considers $\{\mathcal{V}\} = \{\mathcal{B}_1\}$, $\{\mathcal{W}\} = \{\mathcal{B}_0\}$, $\mathbf{v}_1 = -\mathbf{d}_{1/0}$, $\mathbf{v}_2 = \mathbf{d}_{1/2}$, $\mathbf{w}_1 = \mathbf{d}_{0/1}$, and $\mathbf{w}_2 = \mathbf{d}_{0/2}$. The other two relative rotation matrices are computed in a similar way and hence the details are omitted.

B. Bias observer

The design of rate gyro bias observers for all three platforms is presented in this section. In short, vector observations in body-fixed coordinates of inertial constant inertial vectors are shown to exist. Afterwards, a rate gyro bias observer previously developed by the authors is applied, yielding globally exponentially stable error dynamics.

Consider first the scenario depicted in Fig. 1. In this situation, platform 1 measures, in its own coordinate frame, $\mathbf{d}_1(t)$, which corresponds to an inertial vector ${}^I\mathbf{d}_1$. On the other hand, platform 2 measures $\mathbf{d}_2(t)$, which corresponds to an inertial vector ${}^I\mathbf{d}_2$. In order to determine the rate gyro bias of platform 0, notice that with the relative attitude matrices computed in the previous section one may express both vectors in $\{\mathcal{B}_0\}$ by computing

$$ {}^0\mathbf{d}_1(t) = \mathbf{R}_1^0(t)\mathbf{d}_1(t) \tag{3}$$

and

$$ {}^0\mathbf{d}_2(t) = \mathbf{R}_2^0(t)\mathbf{d}_2(t), \tag{4}$$

where ${}^0\mathbf{d}_1(t)$ and ${}^0\mathbf{d}_2(t)$ correspond to $\mathbf{d}_1(t)$ and $\mathbf{d}_2(t)$ expressed in $\{\mathcal{B}_0\}$, respectively. Now, notice that ${}^0\mathbf{d}_1(t)$ and ${}^0\mathbf{d}_2(t)$ correspond to vector observations, available for feedback purposes, expressed in $\{\mathcal{B}_0\}$, of constant inertial vectors. Indeed, substituting (2) in (3) and (4) gives

$$ {}^0\mathbf{d}_1(t) = \mathbf{R}_1^0(t)\left[\mathbf{R}_1^I(t)\right]^T {}^I\mathbf{d}_1 = \left[\mathbf{R}_0^I(t)\right]^T {}^I\mathbf{d}_1$$

and

$$ {}^0\mathbf{d}_2(t) = \mathbf{R}_2^0(t)\left[\mathbf{R}_2^I(t)\right]^T {}^I\mathbf{d}_2 = \left[\mathbf{R}_0^I(t)\right]^T {}^I\mathbf{d}_2,$$

respectively. The same reasoning can be done for the other two platforms, only in that case only one of the vector measurements needs to be rotated. As an example, for platform 1, one has, in addition to (2) for i = 1,

$$ {}^1\mathbf{d}_2(t) = \mathbf{R}_2^1(t)\mathbf{d}_2(t) = \mathbf{R}_2^1(t)\left[\mathbf{R}_2^I(t)\right]^T {}^I\mathbf{d}_2 = \left[\mathbf{R}_1^I(t)\right]^T {}^I\mathbf{d}_2.$$

In the scenario depicted in Fig. 1 the two vector observations $\mathbf{d}_i(t)$, $i=1,2$, are spread over the formation. Evidently, the reasoning just described also applies if both inertial vectors are measured by a single platform, instead of being measured by two different platforms. Hence, in any case, one may obtain two vector observations expressed in each of the body-frames of constant inertial vectors.

Once two vector observations of constant inertial vectors are available, a number of rate gyro bias observers is available in the literature to estimate the bias of the rate gyros installed in each platform. Here, the observer proposed in [12] by the authors is considered. For the sake of completeness, it is here summarized for the case of platform 0. The others follow in a similar fashion.

The observer has essentially a Luenberger-like structure and is given by

$$\begin{cases} {}^0\dot{\hat{\mathbf{d}}}_1(t) = -\mathbf{S}\left(\boldsymbol{\omega}_{m0}(t)\right){}^0\hat{\mathbf{d}}_1(t) - \mathbf{S}\left({}^0\mathbf{d}_1(t)\right)\hat{\mathbf{b}}_0(t) + a_1{}^0\tilde{\mathbf{d}}_1(t) \\ {}^0\dot{\hat{\mathbf{d}}}_2(t) = -\mathbf{S}\left(\boldsymbol{\omega}_{m0}(t)\right){}^0\hat{\mathbf{d}}_2(t) - \mathbf{S}\left({}^0\mathbf{d}_2(t)\right)\hat{\mathbf{b}}_0(t) + a_2{}^0\tilde{\mathbf{d}}_2(t), \\ \dot{\hat{\mathbf{b}}}_0(t) = b_1\mathbf{S}\left({}^0\mathbf{d}_1(t)\right){}^0\tilde{\mathbf{d}}_1(t) + b_2\mathbf{S}\left({}^0\mathbf{d}_2(t)\right){}^0\tilde{\mathbf{d}}_2(t) \end{cases}$$

where ${}^0\hat{\mathbf{d}}_1(t)$, ${}^0\hat{\mathbf{d}}_2(t)$, and $\hat{\mathbf{b}}_0(t)$ are the estimates of ${}^0\mathbf{d}_1(t)$, ${}^0\mathbf{d}_2(t)$, and $\mathbf{b}_0(t)$, respectively, $a_i > 0$ and $b_i > 0$, $i=1$, 2, are positive observer gains, and ${}^0\tilde{\mathbf{d}}_1(t) := {}^0\mathbf{d}_1(t) - {}^0\hat{\mathbf{d}}_1(t)$, ${}^0\tilde{\mathbf{d}}_2(t) := {}^0\mathbf{d}_2(t) - {}^0\hat{\mathbf{d}}_2(t)$, $\tilde{\mathbf{b}}_0(t) := \mathbf{b}_0(t) - \hat{\mathbf{b}}_0(t)$ are the observer errors. Notice that the first two are available for feedback purposes. Under Assumptions 1 and 2, the origin of the rate gyro bias observer error dynamics is a globally exponentially stable equilibrium point [12, Theorem 1].

Remark 1: Notice that, for the purpose of the design of the bias observers, the inertial vectors need not to be known, one only needs the vector observations in body-fixed coordinates. They are not needed later to estimate the relative attitude either. In fact, they would only be needed if one wanted to estimate, in addition to the relative attitudes, the attitude relative to the inertial frame.

C. Relative Attitude Observers

This section presents the design of observers for the relative attitude between the platforms. These observers build on the relative attitude algebraic estimates obtained as described in Section III-A and the rate gyro bias as provided by the observers detailed in Section III-B, in addition to the rate gyro measurements. First, the kinematics of the relative attitudes are briefly derived.

The rotation matrix $\mathbf{R}_i^j(t)$ can be written as

$$\mathbf{R}_i^j(t) = \left[\mathbf{R}_j^I(t)\right]^T \mathbf{R}_i^I(t), \tag{5}$$

i, $j = 0$, 1, 2. Taking the time derivative of (5) and using (1) and (5), as well as the fact that $\mathbf{S}\left(.\right)$ is a skew-symmetric matrix, gives

$$\dot{\mathbf{R}}_i^j(t) \quad = \quad -\mathbf{S}\left(\boldsymbol{\omega}_j(t)\right)\mathbf{R}_i^j(t) + \mathbf{R}_i^j(t)\mathbf{S}\left(\boldsymbol{\omega}_i(t)\right). \tag{6}$$

Then, using the property

$$\mathbf{R}\mathbf{S}\left(\mathbf{x}\right)\mathbf{R}^T = \mathbf{S}\left(\mathbf{R}\mathbf{x}\right) \tag{7}$$

for all $\mathbf{R} \in SO(3)$, $\mathbf{x} \in \mathbb{R}^3$, allows to write the relative attitude kinematics (6) as

$$\dot{\mathbf{R}}_i^j(t) = \mathbf{R}_i^j(t)\mathbf{S}\left(\boldsymbol{\omega}_i(t) - \left[\mathbf{R}_i^j(t)\right]^T\boldsymbol{\omega}_j(t)\right). \tag{8}$$

Next, an observer for $\mathbf{R}_1^0(t)$ is proposed and its stability is analyzed. The design builds essentially on two premises: i) the bias estimates obtained in the previous section are used to cancel out the bias of rate gyro measurements, resulting in a cascade structure; and ii) the observer structure is such that the estimates belong, by construction, to $SO(3)$, provided that is also true for the initial estimate. The analysis builds on common methods used in attitude estimation, resorting to quaternions to express the estimation error, with two key differences: i) an additional nonlinear term appears due to the fact that the observer is for relative attitude estimates and hence the angular velocity of the other platform also affects the relative attitude kinematics; and ii) due to the cascade structure of the observer, the effect of the errors of the bias estimates must be considered in the stability analysis.

Let $\hat{\mathbf{R}}_1^0(t) \in SO(3)$ denote the observer estimate of $\mathbf{R}_1^0(t)$ and define the error variable

$$\tilde{\mathbf{R}}(t) := \mathbf{R}_1^0(t)\left[\hat{\mathbf{R}}_1^0(t)\right]^T \in SO(3). \tag{9}$$

Consider the attitude observer given by

$$\dot{\hat{\mathbf{R}}}_1^0(t) = \hat{\mathbf{R}}_1^0(t)\mathbf{S}\left(\boldsymbol{\omega}_{obs}(t)\right), \tag{10}$$

with

$$\begin{aligned} \boldsymbol{\omega}_{obs}(t) \;=\; & \boldsymbol{\omega}_{1m}(t) - \hat{\mathbf{b}}_1(t) - \left[\hat{\mathbf{R}}_1^0(t)\right]^T\left[\boldsymbol{\omega}_{0m}(t) - \hat{\mathbf{b}}_0(t)\right] \\ & + \left[\mathbf{R}_1^0(t)\right]^T\mathbf{K}(t)\mathbf{vex}\left(\frac{\tilde{\mathbf{R}}(t) - \tilde{\mathbf{R}}^T(t)}{2}\right), \end{aligned}$$

where $\mathbf{K}(t) \succ \mathbf{0}$ is a possibly time varying positive definite observer gain and $\hat{\mathbf{b}}_0(t)$ and $\hat{\mathbf{b}}_1(t)$ are bias estimates obtained from the observers proposed in the previous section. Notice that, to compute the feedback term, the algebraic solution for $\mathbf{R}_1^0(t)$ described in Section III-A can be used to compute the feedback error $\tilde{\mathbf{R}}(t)$. Then, the observer error dynamics can be written as

$$\dot{\tilde{\mathbf{R}}}(t) = -\mathbf{S}\left(\boldsymbol{\omega}_f(t) + \tilde{\boldsymbol{\omega}}_f(t)\right)\tilde{\mathbf{R}}(t), \tag{11}$$

with

$$\boldsymbol{\omega}_f(t) = \mathbf{K}(t)\mathbf{vex}\left(\frac{\tilde{\mathbf{R}}(t) - \tilde{\mathbf{R}}^T(t)}{2}\right) + \left[\mathbf{I} - \tilde{\mathbf{R}}(t)\right]\boldsymbol{\omega}_0(t) \tag{12}$$

and

$$\tilde{\boldsymbol{\omega}}_f(t) = \mathbf{R}_1^0(t)\tilde{\mathbf{b}}_1(t) - \tilde{\mathbf{R}}(t)\tilde{\mathbf{b}}_0(t).$$

When the attitude estimate is identical to the true attitude, the error rotation matrix $\tilde{\mathbf{R}}$ corresponds to the identity. To show the convergence results it is somehow simpler to introduce the equivalent attitude error in terms of quaternions [19]. Let $\left((\tilde{s}(t), \tilde{\mathbf{r}}(t)\right)$ denote the unit quaternion corresponding to the rotation error $\tilde{\mathbf{R}}(t)$, where $\tilde{s}(t)$ and $\tilde{\mathbf{r}}(t)$ are the so-called scalar and vector parts. Then,

$$\tilde{\mathbf{R}}(t) = \mathbf{I} + 2\tilde{s}(t)\mathbf{S}\left(\tilde{\mathbf{r}}(t)\right) + 2\left[\mathbf{S}\left(\tilde{\mathbf{r}}(t)\right)\right]^2 \tag{13}$$

and the quaternion dynamics are given by

$$\begin{cases} \dot{\tilde{s}}(t) = \frac{1}{2}\left[\boldsymbol{\omega}_f(t) + \tilde{\boldsymbol{\omega}}_f(t)\right] \cdot \tilde{\mathbf{r}}(t) \\ \dot{\tilde{\mathbf{r}}}(t) = -\frac{1}{2}\left[\tilde{s}(t)\mathbf{I} - \mathbf{S}\left(\tilde{\mathbf{r}}(t)\right)\right]\left[\boldsymbol{\omega}_f(t) + \tilde{\boldsymbol{\omega}}_f(t)\right] \end{cases}.$$

In the form of a quaternion, the attitude estimation error is zero when $\tilde{\mathbf{r}}$ is also zero.

The following theorem is the main result of the paper.

Theorem 1: Suppose that Assumptions 1 and 2 hold and consider the observer given by (10), where $\mathbf{K}(t)$ is a positive definite matrix whose minimum eigenvalue satisfies

$$\lambda_{min}\left(\mathbf{K}(t)\right) \geq \lambda > 0. \qquad (14)$$

Define the parameterized set

$$\mathcal{R}(\epsilon) := \left\{\tilde{\mathbf{R}}\left(\tilde{s}, \tilde{\mathbf{r}}\right) \in SO(3) : |\tilde{\mathbf{r}}|^2 \leq 1 - \epsilon\right\},$$

fix $0 < \epsilon < 1$, $0 < \theta < 1$, and define $r_u := \epsilon\lambda\theta\sqrt{1-\epsilon}$. Then, the error dynamics (11) are locally input-to-state stable with respect to $\tilde{\mathbf{u}}(t) := \left[\tilde{\mathbf{b}}_0^T(t)\ \tilde{\mathbf{b}}_1^T(t)\right]^T$, with the domain of attraction characterized by $\tilde{\mathbf{R}}\left(t_0\right) \in \mathcal{R}(\epsilon)$ and $\|\tilde{\mathbf{u}}(t)\| < r_u$. The proof is omitted due to space limitations.

Remark 2: Notice that, in the definition of the region of convergence $\mathcal{R}(\epsilon)$ in Theorem 1, ϵ can be chosen as an arbitrarily small positive constant, which is equivalent to say that the initial error rotation angle must be smaller than π by an arbitrarily small positive margin. Therefore, the stability result is best characterized as semi-global in what concerns the initial attitude error. On the other hand, the error dynamics of the rate gyro bias observers are globally exponentially stable. Hence, there exists a certain time instant t_c after which $\tilde{\mathbf{u}}(t)$ is within the region of convergence expressed in Theorem 1, i.e., $\|\tilde{\mathbf{u}}(t)\| \leq r_u$ for $t \geq t_c$.

The structure of the dynamics of $\mathbf{R}_2^0(t)$ is identical to that of $\mathbf{R}_1^0(t)$, hence a similar observer can be employed. The same could be done for $\mathbf{R}_1^2(t)$ or, alternatively, an estimate of $\mathbf{R}_1^2(t)$ could be readily given by

$$\hat{\mathbf{R}}_1^2(t) = \left[\hat{\mathbf{R}}_2^0(t)\right]^T \hat{\mathbf{R}}_1^0(t). \qquad (15)$$

IV. Simulation Results

Numerical simulations are presented in this section. The platforms are assumed, without loss of generality, to be positioned in the plane $z = 0$, in the inertial frame. The evolution of the positions of the three platforms, in the xOy plane, is depicted in Fig. 3, where the starting positions are marked with a cross. In short, the position of platform 0 is constant, while platforms 1 and 2 move along straight paths with constant velocity. The initial inertial attitude of all platforms was set to the identity matrix, i.e., $\mathbf{R}_0^I(0) = \mathbf{R}_1^I(0) = \mathbf{R}_2^I(0) = \mathbf{I}$, and the angular velocities of the platforms with respect to the inertial frame follow

$$\boldsymbol{\omega}_0(t) = \frac{\pi}{180}\begin{bmatrix} 15\sin\left(\frac{2\pi}{60}t\right) \\ \sin\left(\frac{2\pi}{180}t\right) \\ -5\sin\left(\frac{2\pi}{300}t\right) \end{bmatrix} (\mathrm{rad/s}),$$

$$\boldsymbol{\omega}_1(t) = \frac{\pi}{180}\begin{bmatrix} 2\sin\left(\frac{2\pi}{60}t\right) \\ 10\sin\left(\frac{2\pi}{180}t\right) \\ -5\sin\left(\frac{2\pi}{300}t\right) \end{bmatrix} (\mathrm{rad/s}),$$

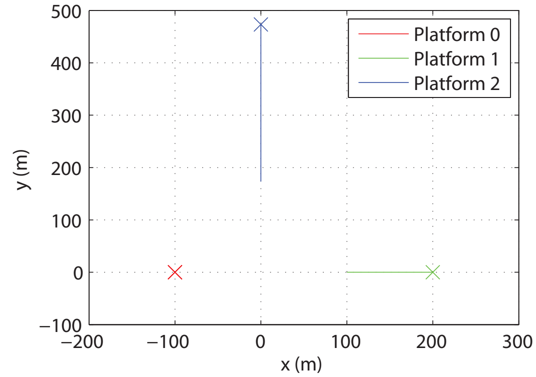

Fig. 3. Trajectories of the formation of three platforms

and

$$\boldsymbol{\omega}_2(t) = \frac{\pi}{180}\begin{bmatrix} 5\sin\left(\frac{2\pi}{60}t\right) \\ 10\sin\left(\frac{2\pi}{180}t\right) \\ -2\sin\left(\frac{2\pi}{300}t\right) \end{bmatrix} (\mathrm{rad/s}).$$

The inertial vectors were set to ${}^I\mathbf{d}_1 = [100]^T$ and ${}^I\mathbf{d}_2 = [010]^T$.

The conditions of the simulations here presented are similar to those of [17]. In particular, the measurements of the directions are assumed to be corrupted by zero-mean white Gaussian noise, with standard deviation of 0.01 on each axis. Afterwards, the directions are normalized again so that they correspond to unit vectors. The rate gyro measurements are also corrupted by additive, zero-mean, white Gaussian noise, with standard deviation of 0.1 $^\circ/s$ on each axis. The rate gyro biases were set to $\mathbf{b}_0 = \begin{bmatrix} -0.1 & 0.3 & 0.7 \end{bmatrix}$ $(^\circ/s)$, $\mathbf{b}_1 = \begin{bmatrix} -0.5 & 1 & 2 \end{bmatrix}$ $(^\circ/s)$, and $\mathbf{b}_2 = \begin{bmatrix} -1 & 0.5 & -2 \end{bmatrix}$ $(^\circ/s)$. A sampling frequency of 100 Hz was considered and the fourth-order Runge-Kutta method was employed in the simulations.

As in most nonlinear estimation problems, the observer gains were tuned empirically. In particular, for the bias observers, it was chosen $a_1 = a_2 = 1$ and $b_1 = b_2 = 0.1$, whereas the attitude observer gain was set to the identity, i.e., $\mathbf{K}(t) = \mathbf{I}$. The initial estimates were set to zero for the bias observers and with an attitude angle error close to 180 degrees for the initial attitude estimates.

Expressing the rotation error $\tilde{\mathbf{R}}(t) = \mathbf{R}(t)\hat{\mathbf{R}}^T(t)$ with the Euler angle-axis representation

$$\begin{aligned} \tilde{\mathbf{R}}(t) \;\; = \;\; & \mathbf{I}\cos\left(\tilde{\theta}(t)\right) + \left[1 - \cos\left(\tilde{\theta}(t)\right)\right]\tilde{\mathbf{d}}(t)\tilde{\mathbf{d}}^T(t) \\ & -\mathbf{S}\left(\tilde{\mathbf{d}}(t)\right)\sin\left(\tilde{\theta}(t)\right), \end{aligned}$$

where $0 \leq \tilde{\theta}(t) \leq \pi$ and $\tilde{\mathbf{d}}(t) \in \mathbb{R}^3$, $\left\|\tilde{\mathbf{d}}(t)\right\| = 1$, are the angle and axis that represent the rotation errors, the evolution of the attitude error can be inferred from the evolution of θ.

The initial convergence of the three relative attitude angle errors is depicted in Fig. 4, where the estimate for $\mathbf{R}_1^2(t)$ was obtained resorting to (15). In steady-state, the mean angle error is below 0.13°. The plots are not shown due to the lack of space. In comparison, the mean raw rotation errors, computed based on the rotation matrices obtained directly

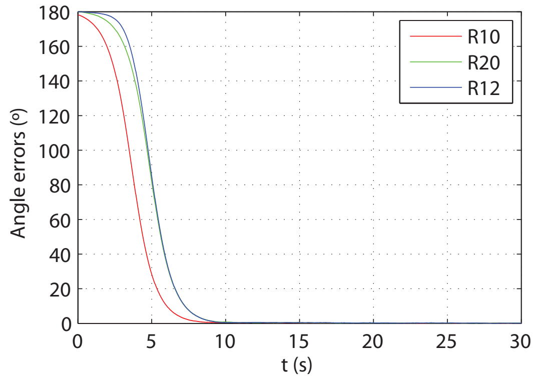

Fig. 4. Evolution of the estimation angle errors

from the direction measurements, are roughly 1.5°. Even worse, the raw angle errors reach values up to 6°, which compare to a maximum error of 0.4° with the proposed solution. Therefore, one can conclude that the proposed solutions effectively provided excellent filtered estimates of the relative rotation matrices between the 3 platforms.

The initial evolution of the rate gyro bias estimation error for platform 0 is shown in Fig. 5. The evolution for the other two platforms is similar and hence it is ommitted. In steady-

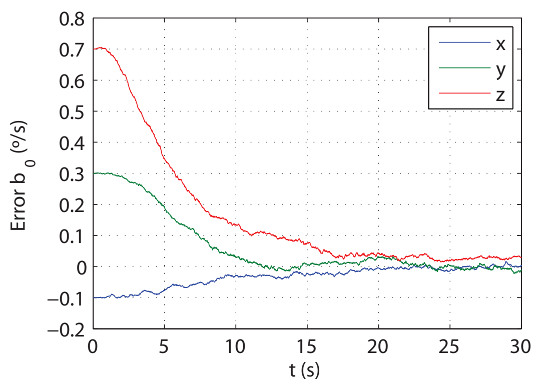

Fig. 5. Initial convergence of the errors $\tilde{\mathbf{b}}_0(t)$

state, the standard deviation of the rate gyro bias stays below $0.01\ ^\circ/s$. Its evolution is not shown due to the lack of space. In all cases, it is possible to conclude that the errors converge to a neighborhood of zero very fast. In the absence of noise, the errors converge to zero, as expected.

V. Conclusions

In previous work by the authors a solution for relative attitude estimation of formations of three platforms had been proposed considering direction measurements between the platforms and rate gyro bias in two of the platforms. However, a third platform required bias-free angular velocity measurements. This paper proposes an extended framework, with two additional vector observations of constant inertial vectors, which allows to estimate, in addition to the relative attitudes, the rate gyro biases of all three platforms. The observer has a cascade structure. The bias observers, which build on previous work by the authors, exhibit globally exponentially stable error dynamics. The error dynamics of the attitude observer, which exhibits a typical observer structure on $SO(3)$ is shown to be locally input-to-state stable with the bias estimation error as input. However, in what concerns the initial attitude error, the convergence is best characterized as semi-global. Finally, simulation results were presented, in the presence of sensor noise that clearly show the filtering effect of the proposed solutions.

References

[1] G. Wahba, "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review*, vol. 7, no. 3, p. 409, Jul. 1965.

[2] E. Lefferts, F. Markley, and M. Shuster, "Kalman filtering for spacecraft attitude estimation," *Journal of Guidance, Control, and Dynamics*, vol. 5, no. 5, pp. 417–429, Sep. 1982.

[3] Y. Bar-Shalom, X. Rong Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*. Wiley-Interscience, 2001.

[4] H. Grip, T. Fossen, T. Johansen, and A. Saberi, "Attitude Estimation Using Biased Gyro and Vector Measurements with Time-Varying Reference Vectors," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1332–1338, May 2012.

[5] A. Barrau and S. Bonnabel, "Intrinsic filtering on Lie groups with applications to attitude estimation," *IEEE Transactions on Automatic Control*, vol. 60, no. 2, pp. 436–449, Feb. 2015.

[6] S. Berkane, A. Abdessameud, and A. Tayebi, "Global Hybrid Attitude Estimation on the Special Orthogonal Group SO(3)," in *Proceedings of the 2016 American Control Conference*, Boston, USA, Jul. 2016.

[7] M. Izadi and A. Sanyal, "Rigid body attitude estimation based on the Lagranged'Alembert principle," *Automatica*, vol. 50, no. 10, pp. 2570–2577, Oct. 2014.

[8] T.-H. Wu, E. Kaufman, and T. Lee, "Globally Asymptotically Stable Attitude Observer on SO(3)," in *Proceedings of the 54th IEEE Conference on Decision and Control*, Osaka, Japan, Dec. 2015, pp. 2164–2168.

[9] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear Complementary Filters on the Special Orthogonal Group," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1218, Jun. 2008.

[10] A. Sanyal, M. Izadi, and E. Butcher, "Determination of relative motion of a space object from simultaneous measurements of range and range rate," in *Proceedings of the 2014 American Control Conference*, Portland, USA, Jun. 2014, pp. 1607–1612.

[11] P. Batista, C. Silvestre, and P. Oliveira, "Sensor-based Globally Asymptotically Stable Filters for Attitude Estimation: Analysis, Design, and Performance Evaluation," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2095–2100, Aug. 2012.

[12] P. Batista, C. Silvestre, and P. Oliveira, "Globally Exponentially Stable Cascade Observers for Attitude Estimation," *Control Engineering Practice*, vol. 20, no. 2, pp. 148–155, Feb. 2012.

[13] P. Batista, C. Silvestre, and P. Oliveira, "A GES Attitude Observer with Single Vector Observations," *Automatica*, vol. 48, no. 2, pp. 388–395, Feb. 2012.

[14] M. Andrle, J. Crassidis, R. Linares, Y. Cheng, and B. Hyun, "Deterministic relative attitude determination of three-vehicle formations," *Journal of Guidance, Control, and Dynamics*, vol. 32, no. 4, pp. 1077–1088, Jul.-Aug. 2009.

[15] K. Son-Goo, J. Crassidis, Y. Cheng, and A. Fosburyx, "Kalman Filtering for Relative Spacecraft Attitude and Position Estimation," *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 1, pp. 133–143, Jan.-Feb. 2007.

[16] R. Linares, J. Crassidis, and Y. Cheng, *Filtering solution to relative attitude determination problem using multiple constraints*, 2012, vol. 143, pp. 235–254.

[17] P. Batista, C. Silvestre, and P. Oliveira, "Globally Convergent Relative Attitude Observers for Three-Platform Formations," in *Proceedings of the 54th IEEE Conference on Decision and Control*, Osaka, Japan, Dec. 2015, pp. 2146–2151.

[18] R. Linares and J. Crassidis, "Constrained Relative Attitude Determination for Two-Vehicle Formations," *Journal of Guidance, Control, and Dynamics*, vol. 34, no. 2, pp. 545–553, Apr.-Mar. 2011.

[19] T. Fossen, *Handbook of Marine Craft Hydrodynamics and Motion Control*. Wiley, 2011.