Single Range Navigation in the presence of Constant Unknown Drifts

Pedro Batista, Carlos Silvestre, and Paulo Oliveira

Abstract— **This paper addresses the problem of navigation/source localization by mobile agents based on the range to a single source. The agent is allowed to have a constant unknown drift relative to an inertial reference frame, which is common, for example, in ocean robotic vehicles in the presence of sea currents, and the source may also have a constant unknown velocity relative to the inertial frame. The contribution of the paper is twofold: i) necessary and sufficient conditions on the observability of the system are derived that are useful for the motion planning and control of the agent; ii) a linear model is developed that mimics the exact dynamics of the nonlinear range-based system, and a Kalman filter is synthesized to estimate the position of the source, as well as the difference between the agent and the source drift velocities. Simulation results that illustrate the performance of the proposed solution in the presence of realistic measurement noise are presented and discussed.**

I. INTRODUCTION

The problem of source localization has been subject of intensive research in recent years [1]. Roughly speaking, an agent has access to a set of measurements and aims to estimate the position of a source. The set of measurements depends on the environment in which the operation occurs and the mission scenario itself. If the source has a transponder the agent may have access to the distance to the source, designated as range in the sequel, but other kinds of information are possible, e.g., bearing to the source and even time differences of arrival when the agent has multiple receivers. Either way, this information is clearly not enough to estimate the position of the source, since the agent must also have some kind of self-awareness of its own movements, be it in terms of its position relative to an inertial reference frame, its velocity, acceleration, etc. When more than one agent communicate in order to determine the position of the source, in a cooperative manner, the setting differs completely and is not within the scope of this paper, which addresses the problem of source localization by a single mobile agent, based on range measurements, in the presence of constant unknown drifts. This last point is of critical importance as, in many applications, both the source and the agent may drift with constant unknown velocities. This is the case of marine applications, where autonomous underwater vehicles (AUVs) or autonomous surface crafts (ASCs) dwell with ocean currents, assumed constant for small periods of operation (or slowly time-varying). While in most robotic applications the velocity of the mobile platform relative to the fluid is available, the inertial velocity, the drift velocity and/or the position of the platform may be unavailable. Thus the importance of including constant unknown drifts in the problem. In the particular case of marine robotics, this problem can be seen as a navigation problem, where the vehicle aims to estimate its own position based on the range to a fixed source (typically a moored buoy or a base station) whose inertial position is known a priori or communicated using acoustic modems.

Previous work in the field can be found in [2], where the authors propose a localization algorithm based on the range to the source (more specifically its square) and the inertial position of the agent, which provides the necessary self-awareness of the agent motion. Global exponential stability (GES) is achieved under a persistent excitation condition and the analysis is extended to non-stationary sources, being shown that it is possible to achieve tracking up to some bounded error. A so-called Synthetic Long Baseline (SLBL) navigation algorithm for underwater vehicles is proposed in [3]. The vehicle is assumed to have access to range measurements to a single transponder, from time to time, and between sampling instants, a high performance dead-reckoning system is used to extrapolate the motion of the vehicle. A discrete-time Kalman filter is synthesized to a linearized model of the system to obtain the required estimates. In [4] the authors deal with the problem of underwater navigation in the presence of unknown currents based on range measurements to a single beacon. An observability analysis is presented based on the linearization of the nonlinear system which yields local results. Based on the linearized system dynamics, a Luenberger observer is introduced but in practice an extended Kalman filter (EKF) is implemented, with no warranties of global asymptotic stability. More recently, the same problem has been studied in [5] and [6], where EKFs have been extensively used to solve the navigation problem based on single beacon range measurements.

This paper addresses the problem of navigation/source localization based on range measurements to a single source in the presence of unknown constant drifts. The contribution is twofold: i) the observability of the nonlinear system is analyzed and necessary and sufficient conditions are derived that characterize this aspect of the system; ii) a filter design methodology is proposed, based on a linear model for the system, that captures the exact dynamics of the nonlinear system. Central to the observability analysis and the filter design is the derivation of a linear time-varying system (LTVS) that captures the dynamics of the nonlinear system, whose output is a nonlinear function of the state. The LTVS model is achieved through appropriate state augmentation,

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which is shown to mimic the nonlinear system. Applications of the proposed solution are many. In Section II-A particular emphasis is given to underwater navigation based on range measurements to a single beacon. The proposed framework allows to solve this problem, including the estimation of constant unknown ocean currents.

The paper is organized as follows. The system dynamics are introduced in Section II, where motivation examples are also provided. Section III refers to the observability analysis, while the filter design is proposed in Section IV. Simulation results are presented in Section V and Section VI summarizes the main conclusions of the paper.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as $\mathrm{diag}(\mathbf{A}_1,\ldots,\mathbf{A}_\mathrm{n})$. If $\mathbf{x}$ and $\mathbf{y}$ are two vectors of identical dimensions, $\mathbf{x}\times\mathbf{y}$ and $\mathbf{x}\cdot\mathbf{y}$ are the cross and inner products, respectively.

II. Problem Statement

Consider the nonlinear system

$$\left\{\begin{array}{l}\dot{\mathbf{x}}_1(t)=\mathbf{x}_2(t)+\mathbf{u}(t)\\ \dot{\mathbf{x}}_2(t)=\mathbf{0}\\ y(t)=\|\mathbf{x}_1(t)\|^2\end{array}\right., \tag{1}$$

where $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^3$ are the system states, $\mathbf{u}\in\mathbb{R}^3$ is the system input, assumed to be a continuous function of time, and $y\in\mathbb{R}$ is the system output. The problem considered in the paper is the analysis of the observability of (1) and the design of a state observer to estimate the system states.

A. Motivation

Consider a point-mass agent moving in a scenario and suppose that the agent has access to

$$z(t)=f\left(r(t)\right),$$

where r is the range to a fixed source and $f:\mathbb{R}\rightarrow\mathbb{R}$ is an injective known function. The problem of source localization is that of estimating the position of the source from the knowledge of $z(t)$.

Evidently, the signal $z(t)$ does not suffice to estimate the position of the source without some knowledge about the motion of the agent itself. To complete the problem framework, let $\{I\}$ denote an inertial reference coordinate frame and $\{B\}$ a coordinate frame attached to the agent, denominated in the sequel as the body-fixed coordinate frame. The linear motion of the agent can be written as

$$\dot{\mathbf{p}}(t)=\mathbf{R}(t)\mathbf{v}(t),$$

where $\mathbf{p}\in\mathbb{R}^3$ denotes the inertial position of the agent, $\mathbf{v}\in\mathbb{R}^3$ is the velocity of the agent relative to $\{I\}$ and expressed in body-fixed coordinates, and $\mathbf{R}$ is the rotation matrix from $\{B\}$ to $\{I\}$, which satisfies

$$\dot{\mathbf{R}}(t)=\mathbf{R}(t)\mathbf{S}\left(\boldsymbol{\omega}(t)\right),$$

where $\boldsymbol{\omega}\in\mathbb{R}^3$ is the angular velocity of $\{B\}$, expressed in body-fixed coordinates, and $\mathbf{S}\left(\boldsymbol{\omega}\right)$ is the skew-symmetric matrix such that $\mathbf{S}\left(\boldsymbol{\omega}\right)\boldsymbol{x}$ is the cross product $\boldsymbol{\omega}\times\boldsymbol{x}$. Let s denote the inertial position of the source. Then, the range to the source is given by

$$r(t)=\|\mathbf{r}(t)\|,$$

where

$$\mathbf{r}(t):=\mathbf{R}^T(t)\left[\mathbf{s}-\mathbf{p}(t)\right] \tag{2}$$

is the location of the source relative to the agent, expressed in body-fixed coordinates, precisely the quantity that the agent aims to estimate. The time derivative of (2) is given by

$$\dot{\mathbf{r}}(t)=-\mathbf{v}(t)-\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{r}(t).$$

Thus, the problem of source localization could be stated as that of estimating $\mathbf{r}(t)$ from the knowledge of $z(t)$, $\mathbf{v}(t)$, $\mathbf{R}(t)$ and $\boldsymbol{\omega}(t)$. While the attitude and the angular velocity of the agent, $\mathbf{R}(t)$ and $\omega(t)$, respectively, are usually available, it is not always possible to measure the inertial velocity of the agent, take, e.g., the case of an autonomous underwater vehicle (AUV) moving far away from the seabed and in the presence of constant unknown ocean currents. Nevertheless, the relative velocity of the agent is usually available, a Doppler velocity log, e.g., would provide this quantity in a marine environment. Thus, it is assumed that the agent is moving in a fluid that has a constant unknown velocity, and that the velocity of the agent relative to the fluid is available from the sensor suite installed on-board. Let $\mathbf{v}_r$ and $\mathbf{v}_f$ denote the velocity of the agent relative to the fluid and the velocity of the fluid relative to $\{I\}$, respectively, both expressed in body-fixed coordinates. Then, it is possible to further write

$$\left\{\begin{array}{l}\dot{\mathbf{r}}(t)=-\mathbf{v}_r(t)-\mathbf{v}_f(t)-\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{r}(t)\\ \dot{\mathbf{v}}_f(t)=-\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{v}_f(t)\end{array}\right..$$

Now, let

$$\left[\begin{array}{c}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\end{array}\right]=\left[\begin{array}{cc}\mathbf{R}(t)&\mathbf{0}\\ \mathbf{0}&-\mathbf{R}(t)\end{array}\right]\left[\begin{array}{c}\mathbf{r}(t)\\ \mathbf{v}_f(t)\end{array}\right], \tag{3}$$

which is a Lyapunov coordinate transformation previously used by the authors [7]. It is straightforward to show that the dynamics of (3) satisfy (1), with $\mathbf{u}(t)=-\mathbf{R}(t)\mathbf{v}_r(t)$ and $y(t)=\|\mathbf{r}(t)\|^2=r^2(t)$.

Interestingly enough, the dynamic system (1) also captures the case where the source is moving with constant unknown velocity relative to $\{I\}$. Indeed, let $\mathbf{v}_s$ denote the velocity of the source relative to $\{I\}$, expressed in body-fixed coordinates, and define $\mathbf{v}_{fs}(t):=\mathbf{v}_f(t)-\mathbf{v}_s(t)$. It is easy to see that

$$\left\{\begin{array}{l}\dot{\mathbf{r}}(t)=-\mathbf{v}_r(t)-\mathbf{v}_{fs}(t)-\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{r}(t)\\ \dot{\mathbf{v}}_{fs}(t)=-\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{v}_{fs}(t)\end{array}\right.,$$

which is readily transformed into the form of (1) using the coordinate transformation

$$\left[\begin{array}{c}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\end{array}\right]=\left[\begin{array}{cc}\mathbf{R}(t)&\mathbf{0}\\ \mathbf{0}&-\mathbf{R}(t)\end{array}\right]\left[\begin{array}{c}\mathbf{r}(t)\\ \mathbf{v}_{fs}(t)\end{array}\right].$$

A two dimensional view of this scenario is depicted in Fig. 1 for the sake of illustrativeness.

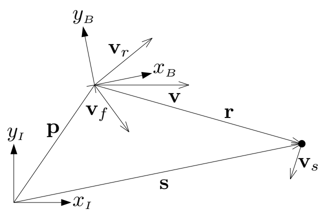

Fig. 1. Example of a source localization scenario in 2D

III. Observability Analysis

While the observability of linear systems is nowadays fairly well understood, the observability of nonlinear systems is still an open field of research, as evidenced (and in spite of) the large number of recent publication on the subject, see [8], [9], [10], [11], and the references therein. The analysis of the linearization of a nonlinear system does not yield definite results either. This section provides an analysis of the observability of (1) through state augmentation. With the proposed method, it will be possible to derive a linear system that captures the behavior of (1). Necessary and sufficient conditions on the observability of the linear system are derived and finally it is shown that these conditions also apply to the nonlinear system.

A. State Augmentation

To derive a linear system that mimics the dynamics of the nonlinear system (1), define three additional scalar state variables as

$$\begin{cases} x_3(t) := y(t) \\ x_4(t) := \mathbf{x}_1^T(t)\mathbf{x}_2(t) \\ x_5(t) := \|\mathbf{x}_2(t)\|^2 \end{cases}$$

and denote by

$$\mathbf{x}(t) = \left[\mathbf{x}_1^T(t)\, \mathbf{x}_2^T(t)\, x_3(t)\, x_4(t)\, x_5(t)\right]^T \in \mathbb{R}^n,\ n = 9,$$

the augmented state. It is easy to verify that the dynamics of the augmented system can be written as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) \end{cases}, \tag{4}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \\ 2\mathbf{u}^T(t) & \mathbf{0} & 0 & 2 & 0 \\ 0 & \mathbf{u}^T(t) & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{5}$$

and $\mathbf{C} = [\mathbf{0}\,\mathbf{0}\,1\,0\,0]$.

The dynamic system (4) can be regarded as a linear time-varying system, even though the system matrix $\mathbf{A}(t)$ depends explicitly on the system input, as evidenced by (5). Nevertheless, this just suggests that the observability of (4) may be connected with the evolution of the system input, which is not common and does not happen when this matrix does not depend on the system input. The observability analysis of (4) will follow using classical theory of linear systems applied to the pair $(\mathbf{A}(t), \mathbf{C})$. Notice that there is nothing in the system dynamics (4) imposing

$$\begin{cases} y(t) = \|\mathbf{x}_1(t)\|^2 \\ x_4(t) = \mathbf{x}_1^T(t)\mathbf{x}_2(t) \\ x_5(t) = \|\mathbf{x}_2(t)\|^2 \end{cases}. \tag{6}$$

Although these restrictions could be easily added including artificial outputs, e.g., $y_2(t) = x_4(t) - \mathbf{x}_1^T(t)\mathbf{x}_2(t) = 0\ \forall t$, this form is preferred since it allows to consider the system as linear. However, care must be taken when extrapolating conclusions from the observability of (4) to the observability of (1).

B. Observability of the Linear System

Before providing necessary and sufficient conditions for the observability of (4), it is convenient to compute the observability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C})$, and, to do that, the transition matrix associated with $\mathbf{A}(t)$. Let

$$\mathbf{u}^{[1]}(t, t_0) := \int_{t_0}^{t} \mathbf{u}(\sigma)d\sigma,$$

where $(.)^{[i]}$ represents the i^{th} integral of the quantity. Then, it is straightforward to show that the transition matrix associated with $\mathbf{A}(t)$ can be written as

$$\boldsymbol{\phi}(t, t_0) = \begin{bmatrix} \mathbf{I} & (t-t_0)\,\mathbf{I} & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{I} & 0 & 0 & 0 \\ 2\left[\mathbf{u}^{[1]}(t,t_0)\right]^T & 2\,(t-t_0)\left[\mathbf{u}^{[1]}(t,t_0)\right]^T & 1 & 2\,(t-t_0) & (t-t_0)^2 \\ \mathbf{0} & \left[\mathbf{u}^{[1]}(t,t_0)\right]^T & 0 & 1 & t-t_0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

The observability Gramian for the pair $(\mathbf{A}(t), \mathbf{C})$ is given by

$$\boldsymbol{\mathcal{W}}(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\phi}^T(t, t_0)\, \mathbf{C}^T\mathbf{C}\boldsymbol{\phi}(t, t_0)\, dt$$

and, after a few steps, it is possible to further write

$$\boldsymbol{\mathcal{W}}(t_0, t_f) = \begin{bmatrix} \boldsymbol{\mathcal{W}}_A(t_0, t_f) & \boldsymbol{\mathcal{W}}_B(t_0, t_f) \\ \left[\boldsymbol{\mathcal{W}}_B(t_0, t_f)\right]^T & \boldsymbol{\mathcal{W}}_C(t_0, t_f) \end{bmatrix},$$

where

$$\boldsymbol{\mathcal{W}}_A(t_0, t_f) = \int_{t_0}^{t_f} \begin{bmatrix} 2\mathbf{u}^{[1]}(t,t_0) \\ 2\,(t-t_0)\,\mathbf{u}^{[1]}(t,t_0) \end{bmatrix} \begin{bmatrix} 2\mathbf{u}^{[1]}(t,t_0) \\ 2\,(t-t_0)\,\mathbf{u}^{[1]}(t,t_0) \end{bmatrix}^T dt,$$

$$\boldsymbol{\mathcal{W}}_B(t_0, t_f) = \int_{t_0}^{t_f} \begin{bmatrix} 2\mathbf{u}^{[1]}(t,t_0) \\ 2\,(t-t_0)\,\mathbf{u}^{[1]}(t,t_0) \end{bmatrix} \begin{bmatrix} 1 \\ 2\,(t-t_0) \\ (t-t_0)^2 \end{bmatrix}^T dt,$$

and

$$\boldsymbol{\mathcal{W}}_C(t_0, t_f) = \begin{bmatrix} t_f - t_0 & (t_f-t_0)^2 & \frac{(t_f-t_0)^3}{3} \\ (t_f-t_0)^2 & \frac{4(t_f-t_0)^3}{3} & \frac{(t_f-t_0)^4}{2} \\ \frac{(t_f-t_0)^3}{3} & \frac{(t_f-t_0)^4}{2} & \frac{(t_f-t_0)^5}{5} \end{bmatrix}.$$

The following result presents a necessary condition for the system (4) to be observable.

Theorem 1: If the linear time-varying system (4) is observable on $[t_0, t_f]$, $t_0 < t_f$, then

$$\underset{\substack{\mathbf{u}_0 \in \mathbb{R}^3 \\ \|\mathbf{u}_0\| = 1 \\ c \in \mathbb{R}}}{\nexists} \quad \underset{t \in [t_0, t_f]}{\forall} \quad : \quad \mathbf{u}_0^T \mathbf{u}(t) = c. \tag{8}$$

Proof: Suppose that (8) does not hold. Then,

$$\underset{\substack{\mathbf{u}_0 \in \mathbb{R}^3 \\ \|\mathbf{u}_0\| = 1 \\ c \in \mathbb{R}}}{\exists} \quad \underset{t \in [t_0, t_f]}{\forall} \quad : \quad \mathbf{u}_0^T \mathbf{u}(t) = c. \tag{9}$$

Now, notice that

$$\mathbf{u}_0^T \mathbf{u}^{[1]}(t, t_0) = \mathbf{u}_0^T \int_{t_0}^{t} \mathbf{u}(\sigma) d\sigma = \int_{t_0}^{t} \mathbf{u}_0^T \mathbf{u}(\sigma) d\sigma$$

and, using (9),

$$\underset{t \in [t_0, t_f]}{\forall} \quad \mathbf{u}_0^T \mathbf{u}^{[1]}(t, t_0) = c\,(t - t_0). \tag{10}$$

The linear system (4) is observable on $[t_0, t_f]$ if and only if the observability Gramian $\boldsymbol{\mathcal{W}}(t_0, t_f)$ is positive definite. Since $\boldsymbol{\mathcal{W}}_C(t_0, t_f)$ is positive definite, it follows that $\boldsymbol{\mathcal{W}}(t_0, t_f)$ is positive definite if and only if its Schur complement

$$\begin{aligned}\boldsymbol{\mathcal{S}}_{\boldsymbol{\mathcal{W}}}(t_0, t_f) &:= \boldsymbol{\mathcal{W}}_A(t_0, t_f) \\ &- \boldsymbol{\mathcal{W}}_B(t_0, t_f)\, \boldsymbol{\mathcal{W}}_C^{-1}(t_0, t_f) \left[\boldsymbol{\mathcal{W}}_B(t_0, t_f)\right]^T\end{aligned} \tag{11}$$

is positive definite. However, if (10) is true, then it is easy to show that

$$\left[\mathbf{u}_0^T \mathbf{u}_0^T\right] \boldsymbol{\mathcal{S}}_{\boldsymbol{\mathcal{W}}}(t_0, t_f) \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_0 \end{bmatrix} = 0, \tag{12}$$

which implies that the observability Gramian is not positive definite and, therefore, (4) is not observable, which completes the proof. ∎

The following theorem provides a sufficient condition for the observability of (4).

Theorem 2: Suppose that the set of functions

$$\mathcal{F} = \left\{1, (t - t_0), (t - t_0)^2, \int_{t_0}^{t} \mathbf{u}^T(\sigma) d\sigma, (t - t_0) \int_{t_0}^{t} \mathbf{u}^T(\sigma) d\sigma\right\} \tag{13}$$

is linearly independent on $[t_0, t_f]$, $t_0 < t_f$. Then, the linear time-varying system (4) is observable on $[t_0, t_f]$.

Proof: If there exist n distinct instants of time $t_i \in [t_i, t_f]$, $i = 1, \ldots, n$ such that the matrix

$$\mathbf{O}_n := \begin{bmatrix} \mathbf{C}\boldsymbol{\phi}(t_1, t_0) \\ \mathbf{C}\boldsymbol{\phi}(t_2, t_0) \\ \vdots \\ \mathbf{C}\boldsymbol{\phi}(t_n, t_0) \end{bmatrix} \tag{14}$$

has full rank, then the system is clearly observable, since

$$\mathbf{O}_n \mathbf{x}(t_0) = \begin{bmatrix} y(t_1) - \int_{t_0}^{t_1} \mathbf{C}\boldsymbol{\phi}(t_1, \sigma)\, \mathbf{B}\mathbf{u}(\sigma)\, d\sigma \\ y(t_2) - \int_{t_0}^{t_2} \mathbf{C}\boldsymbol{\phi}(t_2, \sigma)\, \mathbf{B}\mathbf{u}(\sigma)\, d\sigma \\ \vdots \\ y(t_n) - \int_{t_0}^{t_n} \mathbf{C}\boldsymbol{\phi}(t_n, \sigma)\, \mathbf{B}\mathbf{u}(\sigma)\, d\sigma \end{bmatrix}.$$

Expanding (14) yields

$$\mathbf{O}_n := \begin{bmatrix} 2\left[\mathbf{u}^{[1]}(t_1, t_0)\right]^T & 2(t_1 - t_0)\left[\mathbf{u}^{[1]}(t_1, t_0)\right]^T & 1 & 2(t_1 - t_0) & (t_1 - t_0)^2 \\ 2\left[\mathbf{u}^{[1]}(t_2, t_0)\right]^T & 2(t_2 - t_0)\left[\mathbf{u}^{[1]}(t_2, t_0)\right]^T & 1 & 2(t_2 - t_0) & (t_2 - t_0)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\left[\mathbf{u}^{[1]}(t_n, t_0)\right]^T & 2(t_n - t_0)\left[\mathbf{u}^{[1]}(t_n, t_0)\right]^T & 1 & 2(t_n - t_0) & (t_n - t_0)^2 \end{bmatrix}.$$

Clearly, if the set of functions $\mathcal{F}$ is linearly independent, there exist t_i, $i = 1, \ldots, n$, such that $\mathrm{rank}(\mathbf{O}_n) = n$, which concludes the proof. ∎

C. Observability of the Nonlinear System

Before presenting the main results of this section, it is convenient to introduce some definitions regarding the observability of nonlinear systems [12], [13].

Consider the nonlinear system

$$\begin{cases} \dot{\mathbf{x}}^n(t) = \mathbf{f}(t, \mathbf{x}^n(t), \mathbf{u}^n(t)) \\ \mathbf{y}^n(t) = \mathbf{g}(t, \mathbf{x}^n(t), \mathbf{u}^n(t)) \end{cases}, \tag{15}$$

where $\mathbf{x}^n$ is the system state, $\mathbf{u}^n$ is the system input, and is $\mathbf{y}^n$ the system output.

Definition 1: Two states $\mathbf{x}_1^n$, $\mathbf{x}_2^n$ are said to be indistinguishable (denoted $\mathbf{x}_1^n \mathcal{I} \mathbf{x}_2^n$) for (15) if for every admissible input function $\mathbf{u}^n$, the output function

$$t \mapsto \mathbf{y}^n(t, t_0, \mathbf{x}_1^n, \mathbf{u}^n)$$

of the system with initial condition $\mathbf{x}^n(t_0) = \mathbf{x}_1^n$ and the output function

$$t \mapsto \mathbf{y}^n(t, t_0, \mathbf{x}_2^n, \mathbf{u}^n)$$

of the system with initial condition $\mathbf{x}^n(t_0) = \mathbf{x}_2^n$ are identical on their common domain of definition.

Definition 2: The system (15) is called observable if

$$\mathbf{x}_1^n \mathcal{I} \mathbf{x}_2^n \Rightarrow \mathbf{x}_1^n = \mathbf{x}_2^n.$$

The following theorem provides a necessary condition on the observability of the nonlinear system (1).

Theorem 3: If the nonlinear system (1) is observable on $[t_0, t_f]$, $t_0 < t_f$, then (8) holds.

Proof: Suppose that (8) is not true. Then, it is possible to write (9), i.e.,

$$\underset{\substack{\mathbf{u}_0 \in \mathbb{R}^3 \\ \|\mathbf{u}_0\| = 1 \\ c \in \mathbb{R}}}{\exists} \quad \underset{t \in [t_0, t_f]}{\forall} \quad : \quad \mathbf{u}_0^T \mathbf{u}(t) = c \tag{16}$$

and

$$\underset{t \in [t_0, t_f]}{\forall} \quad \mathbf{u}_0^T \mathbf{u}^{[1]}(t, t_0) = c\,(t - t_0).$$

The output of (1) is given by

$$y(t) = \left\| \mathbf{x}_1(t_0) + (t - t_0)\, \mathbf{x}_2(t_0) + \int_{t_0}^{t} \mathbf{u}(\sigma)\, d\sigma \right\|^2.$$

Let $y^a(t)$ denote the output of the system with initial condition

$$\begin{cases} \mathbf{x}_1^a(t_0) = \mathbf{u}_0 \\ \mathbf{x}_2^a(t_0) = -c\mathbf{u}_0 \end{cases} \tag{17}$$

and $y^b(t)$ denote the output of the system with initial condition

$$\begin{cases} \mathbf{x}_1^b(t_0) = -\mathbf{u}_0 \\ \mathbf{x}_2^b(t_0) = -c\mathbf{u}_0 \end{cases}. \quad (18)$$

It is straightforward to show that

$$\underset{t\in[t_0,t_f]}{\forall} \quad : \quad y^a(t) = y^b(t).$$

Thus, if (8) does not hold, there exist, at least, two states that are indistinguishable, and thus the system is not observable, which concludes the proof. ∎

The following theorem provides a sufficient condition for the observability of the nonlinear system (1), as well as a practical result that can be used in the design of state observers for that system.

Theorem 4: Suppose that the set of functions (13) is linearly independent on $[t_0, t_f]$, $t_0 < t_f$. Then, the nonlinear system (1) is observable on $[t_0, t_f]$. Moreover, a state observer with globally asymptotically stable error dynamics for the LTVS (4) is also a state observer for the nonlinear system (1), with globally asymptotically stable error dynamics.

Proof: Let $\left[\mathbf{x}_1^T(t_0)\ \mathbf{x}_2^T(t_0)\right]^T$ be the initial state of the nonlinear system (1). Then, the output is given by

$$\begin{aligned} y(t) &= \|\mathbf{x}_1(t)\|^2 \\ &= \left\|\mathbf{x}_1(t_0) + (t-t_0)\mathbf{x}_2(t_0) + \int_{t_0}^t \mathbf{u}(\sigma)\,d\sigma\right\|^2 \\ &= \|\mathbf{x}_1(t_0)\|^2 + \|\mathbf{x}_2(t_0)\|^2 (t-t_0)^2 + \left\|\int_{t_0}^t \mathbf{u}(\sigma)\,d\sigma\right\|^2 \\ &\quad +2\mathbf{x}_1^T(t_0)\mathbf{x}_2(t_0)(t-t_0) + 2\mathbf{x}_1^T(t_0)\int_{t_0}^t \mathbf{u}(\sigma)\,d\sigma \\ &\quad +2\mathbf{x}_2^T(t_0)(t-t_0)\int_{t_0}^t \mathbf{u}(\sigma)\,d\sigma. \end{aligned} \quad (19)$$

Since the set of functions (13) is assumed linearly independent on $[t_0, t_f]$ it follows, from Theorem 2, that the LTVS (4) is observable on $[t_0, t_f]$. Thus, given $y(t)$ for $t \in [t_0, t_f]$, the initial state of (4) is determined uniquely. Let $\left[\mathbf{z}_1^T(t_0)\ \mathbf{z}_2^T(t_0)\ z_3(t_0)\ z_4(t_0)\ z_5(t_0)\right]^T$ be the initial state of the linear system (4). Then, the output satisfies

$$\begin{aligned} y(t) &= 2\mathbf{z}_1^T(t_0)\int_{t_0}^t \mathbf{u}(\sigma)\,d\sigma + 2\mathbf{z}_2^T(t_0)(t-t_0)\int_{t_0}^t \mathbf{u}(\sigma)\,d\sigma \\ &\quad +z_3(t_0) + 2z_4(t_0)(t-t_0) + z_5(t_0)(t-t_0)^2 \\ &\quad +\left\|\int_{t_0}^t \mathbf{u}(\sigma)\,d\sigma\right\|^2. \end{aligned} \quad (20)$$

Notice that $y(t_0) = z_3(t_0) = \|\mathbf{x}_1(t_0)\|^2$. From the comparison between (19) and (20) it follows that

$$\begin{aligned} &2\left[\mathbf{x}_1^T(t_0)\mathbf{x}_2(t_0) - z_4(t_0)\right](t-t_0) \\ &+\left[\|\mathbf{x}_2(t_0)\|^2 - z_5(t_0)\right](t-t_0)^2 \\ &+2\left[\mathbf{x}_1^T(t_0) - \mathbf{z}_1^T(t_0)\right]\textstyle\int_{t_0}^t \mathbf{u}(\sigma)\,d\sigma \\ &2\left[\mathbf{x}_2^T(t_0) - \mathbf{z}_2^T(t_0)\right](t-t_0)\textstyle\int_{t_0}^t \mathbf{u}(\sigma)\,d\sigma = 0 \end{aligned} \quad (21)$$

for all $t \in [t_0, t_f]$. Since the set of functions $\mathcal{F}$ is assumed linearly independent, (21) implies that

$$\begin{cases} \mathbf{x}_1(t_0) = \mathbf{z}_1(t_0) \\ \mathbf{x}_2(t_0) = \mathbf{z}_2(t_0) \\ \mathbf{x}_1^T(t_0)\mathbf{x}_2(t_0) = z_4(t_0) \\ \|\mathbf{x}_2(t_0)\|^2 = z_5(t_0) \end{cases}.$$

This concludes the proof, as both the initial state of the nonlinear system (1) is uniquely determined and the initial state of the linear system (4) matches the initial state of the nonlinear system. ∎

Remark: Notice that, although the LTVS (4) mimics the original nonlinear system (1) under the conditions of Theorem 4, the algebraic restrictions (6) were not explicitly imposed.

IV. Filter Design

Given that (1) has been transformed into a linear system that captures the exact dynamics and its observability has been fairly well characterized, it is immediate to design a globally exponentially stable state observer [14]. In practice, it is common to have noisy measurements. Moreover, it is the range that is usually measured, not its square, which is a drawback of the previous solution since measurement noise may be greatly amplified for large distances with the square operation. These drawbacks are addressed in this section, where a Kalman filter is proposed for a different augmented system that relies on the range instead of its square.

Define a new augmented system state as

$$\boldsymbol{\chi}(t) = \left[\mathbf{x}_1^T(t)\,\mathbf{x}_2^T(t)\,\chi_3(t)\,x_4(t)\,x_5(t)\right]^T \in \mathbb{R}^n,\ n = 9,$$

where $\chi_3(t) := \|\mathbf{x}_1(t)\| = \sqrt{y(t)}$, and a new system output as $z(t) = \|\mathbf{x}_1(t)\|$. The new augmented system dynamics are given by

$$\begin{cases} \dot{\boldsymbol{\chi}}(t) = \boldsymbol{A}(t)\boldsymbol{\chi}(t) + \mathbf{B}\mathbf{u}(t) \\ z(t) = \mathbf{C}\boldsymbol{\chi}(t) \end{cases}, \quad (22)$$

where

$$\boldsymbol{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \\ \frac{1}{z(t)}\mathbf{u}^T(t) & \mathbf{0} & 0 & \frac{1}{z(t)} & 0 \\ \mathbf{0} & \mathbf{u}^T(t) & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \end{bmatrix},$$

$\mathbf{B}$ and $\mathbf{C}$ are as previously defined, and it is assumed that $z(t) > 0$ for all time. Notice that, further assuming that the output and its derivative are bounded for all time, both augmented states are related through a Lyapunov transformation, which preserves observability properties [15]. The advantage of the new representation (22) is that the output is the range, instead of its square. The design of a Kalman filter for (22) is now straightforward to obtain, as well as conditions that guarantee its observability [16], [17].

V. Simulations

This section provides simulation results that demonstrate the performance of the Kalman filter solution proposed in Section IV.

As defined in Section II-A, let $\mathbf{p}(t)$ and $\mathbf{s}(t)$ denote the inertial positions of the agent and the source, respectively. In the simulation the agent starts at $\mathbf{p}(0) = [20\ 20\ 20]^T$ (m) and the source at $\mathbf{s}(0) = [0\ 50\ 0]^T$ (m). The source is assumed to be drifting with constant velocity $\mathbf{v}_s(t) = [0.5\ -0.5\ 0.5]^T$ (m/s) and the relative velocity of the agent, expressed in inertial coordinates, is

$$\mathbf{v}_r(t) = \begin{bmatrix} 0.5 - \cos\left(\frac{2\pi}{100}t\right) \\ -0.5 + 2\sin\left(\frac{2\pi}{200}t\right) \\ 0.5 - 0.5\cos\left(\frac{2\pi}{40}t\right) \end{bmatrix} \text{(m/s)}.$$

The position of the source relative to the agent, expressed in inertial coordinates, is $\mathbf{x}_1(t) = \mathbf{s}(t) - \mathbf{p}(t)$, and Fig. 2 depicts its evolution. Both the range and the agent velocity measurements are perturbed by zero-mean white Gaussian noises, with standard deviations of $0.2\,\mathrm{m}$ and $0.01\,\mathrm{m/s}$, which given the scale of the problem are reasonable. To tune the Kalman filter the state disturbance covariance matrix was chosen as $\mathbf{Q} = 0.01\mathrm{diag}\,(1, 1, 1, 0.001, 0.001, 0.001, 1, 1, 0.001)$ and the output noise variance as $R = 1$. The evolution of the error variables is depicted in Fig. 3. The initial large transients appear due to the mismatch of the initial conditions. In order to better evaluate the performance of the proposed solution, the error variables are depicted in greater detail in Fig. 4. Even in the presence of realistic measurement noise, the error variables remain confined to tight intervals.

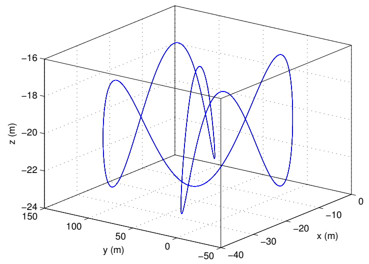

Fig. 2. Trajectory of $\mathbf{x}_1(t)$

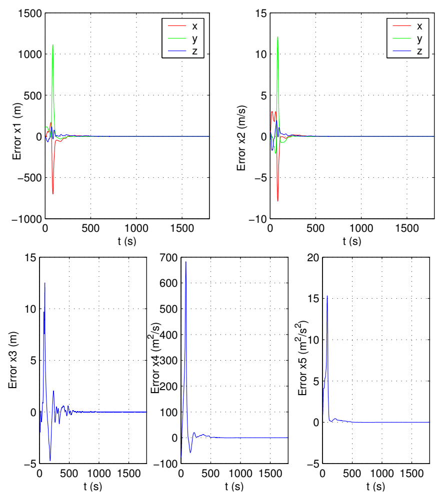

Fig. 3. Time evolution of the estimation error variables

VI. Conclusions

The problem of navigation/source localization by mobile agents based on range measurements, considering constant unknown drift velocities, was addressed in this paper. The observability of the nonlinear system was analyzed and necessary and sufficient conditions were derived. The results are closely related to the motion of the agent, which allows for the implementation of appropriate control strategies that render the system observable. To solve the estimation problem a Kalman filter was proposed based on a linear model that captures the exact dynamics of the system. Simulation results were presented that illustrate the attainable performance in the presence of realistic measurement noise.

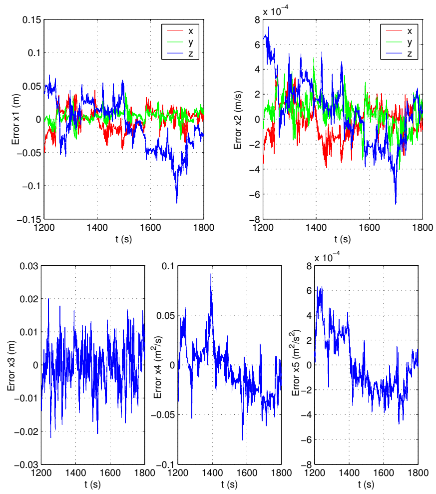

Fig. 4. Detailed evolution of the estimation error variables

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