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Globally Asymptotically Stable Filter for Navigation aided by Direction and Depth Measurements

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Abstract— **This paper presents a navigation solution for autonomous vehicles based on direction and depth measurements, in addition to relative velocity readings, with globally asymptotically stable (GAS) error dynamics. A constant unknown drift velocity disturbance is also assumed present during the operation of the agent, which is explicitly considered in the system dynamics. The observability of the system is studied resorting to linear time-varying system theory, in an exact way, in spite of the nonlinear nature of the original nonlinear system dynamics. Realistic simulation results are presented, including measurement noise, that illustrate the performance of the achieved solution. Comparison with the Extended Kalman Filter is also carried out, revealing that similar performances are achieved for the proposed approach.**

I. Introduction

While there exist many solutions in the literature, the topic of navigation of autonomous vehicles is still a very active field of research, with the pursuit of stability and performance guarantees and the employment of different sensors. Navigation Systems based on the Global Positioning System (GPS) proliferate but the GPS is not always available, e.g. in underwater environments. Long Baseline (LBL) acoustic positioning systems offer an alternative to the GPS in marine scenarios, however its deployment is time consuming and it is not always possible, e.g. in very large operation scenarios in open waters. More recently, navigation algorithms based on single range measurements have been proposed [1], [2], [3], [4]. This paper presents a novel navigation solution based on direction and depth measurements with globally asymptotically stable error dynamics.

The problem of localization of a mobile robot using bearing measurements was addressed in [5], where a nonlinear transformation of the measurement equation into a higher dimensional space is performed. This has allowed to obtain tight, possibly complex-shaped, bounding sets for the feasible states in a closed-form representation. Relevant work related to the problem of navigation based on direction measurements can be found in [6], where the problem of target localization is addressed, in 2-D, based on bearing measurements, in addition to the trajectory of the agent. The estimation error dynamics were shown to be globally exponentially stable (GES) under an appropriate persistent excitation condition and a circumnavigation control law was also proposed. Earlier work on the observability issues of target motion analysis based on angle readings, in 2-D, is available in [7], which was later extended to 3-D in [8]. The specific observability criteria thereby derived resort to complicated nonlinear differential equations and some tedious mathematics are needed for the solution, giving conditions that are necessary for system observability. Another related framework in the domain of target motion analysis (TMA) can be found in [9], where frequency measurements are also included. This topic was further studied in [10], where Cramer Rao analysis revealed the parametric dependencies of TMA with angle-only tracking and angle/frequency tracking, giving also an idea of the increase in estimation accuracy using the later. Recent work by the authors with single direction measurements was presented in [11], where appropriate persistent excitation conditions were derived, leading to estimation solutions with GAS error dynamics.

This paper addresses the problem of navigation based on depth and direction measurements to a single source in the presence of unknown constant drifts. The observability of the system is studied and a time-varying Kalman filter with GAS error dynamics is proposed, without any system linearization and yielding performances comparable to those of the Extended Kalman Filter but with GAS guarantees. Central to the design is the augmentation of the system state, which allows to consider linear time-varying (LTV) system dynamics. In comparison with [11], the present solution differs from the inclusion of depth measurements, which are readily available in most situations (depth cells for underwater vehicles, pressure sensors for aerial vehicles), and that allow to derive less demanding observability conditions.

The paper is organized as follows. Section II introduces the problem at hand and presents the system dynamics, while the proposed solution is derived in Section III. Simulation results, including the comparison with the Extended Kalman Filter, are discussed in Section IV. Section V presents some concluding remarks.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as $\operatorname{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_\mathrm{n})$ and the set of unit vectors on $\mathbb{R}^3$ is denoted by $S(2)$. The Dirac delta function is written as $\delta(t)$.

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II. PROBLEM STATEMENT

A. Motivation and framework

The EU project TRIDENT, whose consortium consists of 8 partners, aims to develop an Intervention Autonomous Underwater Vehicle (I-AUV) that works in cooperation with an Autonomous Surface Craft (ASC) in order to carry out surveying and intervention missions. Naturally, both vehicles must be properly geo-referenced, with appropriate navigation systems.

While the ASC is equipped with a navigation system based on a GPS and an Inertial Measurement Unit, a different approach must be pursued for the navigation system of the AUV. In this case, the ASC and the AUV work cooperatively and as such it is possible to assume that the ASC transmits its inertial position $\mathbf{s}(t)$ and inertial velocity $\mathbf{v}_s(t)$ to the AUV. In this framework, the goal of the AUV is to determine its own position in inertial coordinates $\mathbf{p}(t)$, as well as its drift velocity $\mathbf{v}_c(t)$ due to ocean currents, given the information provided by the ASC, the relative velocity readings $\mathbf{v}_r(t)$ of the AUV, provided by a Doppler Velocity Log (DVL), direction measurements $\mathbf{d}(t)$ of the ASC relative to the AUV, which may be obtained, e.g., with a passive Ultra-Short Baseline acoustic system, and the depth of the AUV, provided by a depth sensor.

B. System dynamics

Consider an autonomous vehicle (the agent) moving in a mission scenario where there is a source, which may be another vehicle or a fixed object, such that the agent is able to measure the direction to the source and its own vertical position, while it receives the position of the source in inertial coordinates. Roughly speaking, the problem addressed in this paper is that of estimating the position of the agent, in inertial coordinates, given direction and vertical position measurements between the source and the agent.

In order to properly set the problem framework, let $\mathbf{p}(t) \in \mathbb{R}^3$ denote the position of the autonomous vehicle (the agent), in inertial coordinates, moving in a scenario where there is a source whose position, in inertial coordinates, is denoted by $\mathbf{s}(t) \in \mathbb{R}^3$. Suppose that the source is moving with velocity $\mathbf{v}_s(t) \in \mathbb{R}^3$ relative to the inertial frame, i.e., $\dot{\mathbf{s}}(t) = \mathbf{v}_s(t)$, while the linear motion kinematics of the agent are given by

$$\begin{cases} \dot{\mathbf{p}}(t) = \mathbf{v}_c(t) + \mathbf{v}_r(t) \\ \dot{\mathbf{v}}_c(t) = \mathbf{0} \end{cases},$$

where $\mathbf{v}_c(t) \in \mathbb{R}^3$ is a constant unknown drift velocity of the agent and $\mathbf{v}_r(t) \in \mathbb{R}^3$ is a known input. Further consider that the agent measures the direction to the source

$$\mathbf{d}(t) = \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} \in S(2), \tag{1}$$

with $\mathbf{r}(t) := \mathbf{s}(t) - \mathbf{p}(t) \in \mathbb{R}^3$, and the vertical position (depth, in the case of autonomous underwater vehicles) $h(t) := [0\ 0\ 1]\,\mathbf{p}(t) \in \mathbb{R}$. Then, the system dynamics can be written as

$$\begin{cases} \dot{\mathbf{p}}(t) = \mathbf{v}_c(t) + \mathbf{v}_r(t) \\ \dot{\mathbf{v}}_c(t) = \mathbf{0} \\ \mathbf{d}(t) = \frac{\mathbf{s}(t)-\mathbf{p}(t)}{\|\mathbf{s}(t)-\mathbf{p}(t)\|} \\ h(t) := [0\ 0\ 1]\,\mathbf{p}(t) \end{cases} \tag{2}$$

and the problem considered here is that of estimating both the position of the agent $\mathbf{p}(t)$ and the drift velocity $\mathbf{v}_c(t)$, or, in other words, to design an estimator for the nonlinear system (2).

The following assumption is required in the sequel.

Assumption 1: The relative velocity is continuous and continuously differentiable. Moreover, both $\mathbf{v}_r(t)$ and $\dot{\mathbf{v}}_r(t)$ are norm-bounded.

This is a mild assumption for observability and observer design purposes as the actuation systems of agents or vehicles necessarily limit the available force and torque, which implies upper bounds on the velocities and accelerations. In this paper, in particular, it allows to consider that both $\dot{\mathbf{d}}(t)$ and $\ddot{\mathbf{d}}(t)$ are norm-bounded. The values of the bounds are not required as they are not explicitly used.

III. NAVIGATION FILTER DESIGN

This section presents a filter design methodology for the problem stated in Section II. First, an augmented linear time-varying system is introduced in Section III-A. Afterwards, in Section III-B, the observability of this system is analyzed and its relation with the original nonlinear system is established. Finally, the filter design is discussed in Section III-C.

A. System dynamics

In order to derive an augmented linear system for navigation based on direction and depth measurements, define the system states as

$$\begin{cases} \mathbf{x}_1(t) = \mathbf{p}(t) \\ \mathbf{x}_2(t) = \mathbf{v}_c(t) \\ x_3(t) = \|\mathbf{r}(t)\| \end{cases}.$$

From (1) it follows that $\mathbf{x}_1(t) + x_3(t)\mathbf{d}(t) = \mathbf{s}(t)$ for all t. Let $\mathbf{x}(t) = \left[\mathbf{x}_1^T(t)\,\mathbf{x}_2^T(t)\,x_3(t)\right]^T \in \mathbb{R}^7$. Then, the system dynamics can be written as the LTV system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}, \tag{3}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{d}^T(t) & 0 \end{bmatrix} \in \mathbb{R}^{7\times 7},$$

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{d}^T(t) & \mathbf{d}^T(t) \end{bmatrix} \in \mathbb{R}^{7\times 6},$$

$$\mathbf{C}(t) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{d}(t) \\ 0\ 0\ 1 & \mathbf{0} & 0 \end{bmatrix} \in \mathbb{R}^{4\times 7},$$

and

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{v}_r(t) \\ \mathbf{v}_s(t) \end{bmatrix} \in \mathbb{R}^6.$$

B. Observability analysis

The observability of the problem of navigation with relative velocity, depth, and direction measurements is studied in this section. The following proposition [Proposition 4.2, [12]] is useful in the sequel.

Proposition 1: Let $\mathbf{f}(t) : [t_0, t_f] \subset \mathbb{R} \rightarrow \mathbb{R}^n$ be a continuous and i-times continuously differentiable function on $\mathcal{I} := [t_0, t_f]$, $T := t_f - t_0 > 0$, and such that

$$\mathbf{f}(t_0) = \dot{\mathbf{f}}(t_0) = \ldots = \mathbf{f}^{(i-1)}(t_0) = \mathbf{0}.$$

Further assume that $\left\|\mathbf{f}^{(i+1)}(t)\right\| \leq C$ for all $t \in \mathcal{I}$. If there exist $\alpha > 0$ and $t_1 \in \mathcal{I}$ such that $\left\|\mathbf{f}^{(i)}(t_1)\right\| \geq \alpha$ then there exist $0 < \delta \leq T$ and $\beta > 0$ such that $\|\mathbf{f}(t_0 + \delta)\| \geq \beta$.

The following theorem characterizes the observability of the LTV system (3).

Theorem 1: Let $\mathcal{I} := [t_0, t_f]$. If the direction $\mathbf{d}(t)$ does not remain in the horizontal plane for all $t \in \mathcal{I}$ or, equivalently,

$$\underset{t_1 \in \mathcal{I}}{\exists} \quad d_3(t_1) \neq 0, \tag{4}$$

where $\mathbf{d}(t) = [d_1(t)\ d_2(t)\ d_3(t)]^T$, then the LTV system (3) is observable on $\mathcal{I}$.

Proof: The observability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C}(t))$ on $\mathcal{I}$ is defined as

$$\boldsymbol{\mathcal{W}}(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\phi}^T(\tau, t)\,\mathbf{C}^T(\tau)\mathbf{C}(\tau)\boldsymbol{\phi}(\tau, t)\,d\tau,$$

where $\boldsymbol{\phi}(t, t_0)$ denotes the transition matrix associated with $\mathbf{A}(t)$, which is given by

$$\boldsymbol{\phi}(t, t_0) = \begin{bmatrix} \mathbf{I} & (t - t_0)\,\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\int_{t_0}^{t} \mathbf{d}^T(\tau)\,d\tau & 1 \end{bmatrix} \in \mathbb{R}^{7\times 7}.$$

Let $\mathbf{c} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & c_3 \end{bmatrix} \in \mathbb{R}^7$, $\mathbf{c}_i \in \mathbb{R}^3$, $i = 1, 2$, $c_3 \in \mathbb{R}$, be a unit vector, i.e., $\|\mathbf{c}\| = 1$. Then, it is possible to write

$$\mathbf{c}^T \boldsymbol{\mathcal{W}}(t_0, t_f)\,\mathbf{c} = \int_{t_0}^{t_f} \|\mathbf{f}(\tau)\|^2\,d\tau$$

for all $\|\mathbf{c}\| = 1$, where

$$\mathbf{f}(\tau) = \begin{bmatrix} \mathbf{f}_1(\tau) \\ f_2(\tau) \end{bmatrix} \in \mathbb{R}^4, \ \tau \in \mathcal{I},$$

with

$$\begin{aligned} \mathbf{f}_1(\tau) \quad &= \quad \mathbf{c}_1 + \left[(\tau - t_0)\,\mathbf{I} - \mathbf{d}(\tau)\int_{t_0}^{\tau} \mathbf{d}^T(\sigma)d\sigma\right]\mathbf{c}_2 \\ &\quad\ + c_3\mathbf{d}(\tau) \in \mathbb{R}^3 \end{aligned}$$

and

$$f_2(\tau) = c_{13} + (\tau - t_0)\,c_{23} \in \mathbb{R},$$

where $\mathbf{c}_1 = [c_{11}\ c_{12}\ c_{13}]^T$ and $\mathbf{c}_2 = [c_{21}\ c_{22}\ c_{23}]^T$. The first derivatives of $\mathbf{f}_1(\tau)$ and $f_2(\tau)$ are given by

$$\begin{aligned} \frac{d}{d\tau}\mathbf{f}_1(\tau) \quad &= \quad \left[\mathbf{I} - \mathbf{d}(\tau)\mathbf{d}^T(\tau) - \dot{\mathbf{d}}(\tau)\int_{t_0}^{\tau} \mathbf{d}^T(\sigma)d\sigma\right]\mathbf{c}_2 \\ &\quad\ + c_3\dot{\mathbf{d}}(\tau) \end{aligned}$$

and

$$\frac{d}{d\tau} f_2(\tau) = c_{23}.$$

To show that (4) is a sufficient condition, suppose that the LTV system is not observable and (4) is verified. Then, if the system is not observable, there exists a unit vector $\mathbf{c}$ such that

$$\mathbf{c}^T \boldsymbol{\mathcal{W}}(t_0, t)\,\mathbf{c} = 0 \tag{5}$$

for all $t \in \mathcal{I}$. This implies that $\mathbf{f}(\tau) = \mathbf{0}$ and $\frac{d}{d\tau}\mathbf{f}(\tau) = 0$ for all $\tau \in \mathcal{I}$. In particular, for $\tau = t_0$, it follows that $f_2(t_0) = 0 \Leftrightarrow c_{13} = 0$ and $\frac{d}{d\tau} f_2(t_0) = 0 \Leftrightarrow c_{23} = 0$. With $c_{13} = 0$ and $c_{23} = 0$ on $\mathbf{f}_1(\tau)$ allows to conclude that $c_3 d_3(\tau) = 0$ for all $\tau \in \mathcal{I}$. But then, as there exists $t_1 \in \mathcal{I}$ such that $d_3(t_1) \neq 0$, it must be $c_3 = 0$. Now, with $c_3 = 0$ on $\mathbf{f}_1(\tau)$ for $\tau = t_0$ yields $\mathbf{f}_1(t_0) = \mathbf{0} \Leftrightarrow \mathbf{c}_1 = \mathbf{0}$. Now, with $\mathbf{c}_1 = \mathbf{0}$, $c_{23} = 0$, and $c_3 = 0$, it follows that $\frac{d}{d\tau}\mathbf{f}_1(t_0) = \mathbf{0}$ implies either: i) $\mathbf{c}_2 = \mathbf{0}$; or ii) $\mathbf{c}_2 = \pm\mathbf{d}(t_0)$. The first case corresponds to $\mathbf{c} = \mathbf{0}$, which contradicts the hypothesis of existence of a unit vector such that (5) holds for all $t \in \mathcal{I}$. As such, it must be $\mathbf{c}_1 = \mathbf{0}$, $c_{23} = 0$, $c_3 = 0$, $\mathbf{c}_2 = \pm\mathbf{d}(t_0)$. Next it is shown that this cannot hold either. Indeed, as (4) is assumed to be true and it must be $d_3(t_0) = 0$ (as $\mathbf{c}_2 = \mathbf{d}(t_0)$ and $c_{23} = 0$), there must exist $t_i \in \mathcal{I}$ such that

$$\begin{aligned} &\tfrac{d}{d\tau}\mathbf{f}_1(t_i) = \mathbf{0} \\ \Leftrightarrow\ &\pm\mathbf{d}(t_0) \mp \left[\mathbf{d}(t_i)\mathbf{d}^T(t_0)\right]\mathbf{d}(t_i) \mp \\ &\left[\textstyle\int_{t_0}^{t_i} \mathbf{d}^T(\sigma)\mathbf{d}(t_0)\,d\sigma\right]\dot{\mathbf{d}}(t_i) = \mathbf{0}, \end{aligned} \tag{6}$$

where $\dot{\mathbf{d}}(t_i)$ cannot be expressed as a linear combination of $\mathbf{d}(t_0)$ and $\mathbf{d}(t_1)$. Hence (6) cannot hold and there is no unit vector $\mathbf{c}$ such that (5) holds for all $t \in \mathcal{I}$, which means that the system (3) is observable, which concludes the proof. ■

Remark 1: Notice that Theorem 1 provides only a sufficient condition, which is mild for the application in mind: indeed, as long as the direction does not remain horizontal, the system is observable. When the direction is always horizontal, the system may still be observable if a persistent excitation condition is met, in line with what happens for the case of single directions but not so demanding, see [11], but with $d_3(t) = 0$ for all t. This will be addressed in future work.

Before proceeding, it is important to remark that there is nothing in (3) imposing the nonlinear restriction $x_3(t) = \|\mathbf{r}(t)\|$ and as such the observability of the LTV system (3) does not immediately entail the observability of the original nonlinear system (2). This issue is addressed in the following theorem.

Theorem 2: Under the conditions of Theorem 1, the initial condition of the LTV system (3) corresponds to the initial condition of the original nonlinear system (2), i.e.,

$$\begin{cases} \mathbf{x}_1(t_0) = \mathbf{p}(t_0) \\ \mathbf{x}_2(t_0) = \mathbf{v}_c(t_0) \\ x_3(t_0) = \|\mathbf{r}(t_0)\| \end{cases}. \tag{7}$$

Proof: Under the terms of Theorem 1, the initial condition of the LTV system (3) is uniquely determined by the corresponding system output and input. The proof follows by showing that (7) explains the system output. The output of the LTV system (3) is given by $\mathbf{y}(t) = \begin{bmatrix} \mathbf{y}_1(t) & y_2(t) \end{bmatrix}$, with

$$\begin{aligned}\mathbf{y}_1(t) &= \mathbf{x}_1(t_0) + (t-t_0)\,\mathbf{x}_2(t_0) + \int_{t_0}^{t}\mathbf{v}_r(\tau)d\tau + x_3\,(t_0)\,\mathbf{d}(t)\\ &\quad + \int_{t_0}^{t}\left[\mathbf{v}_s\,(\tau) - \mathbf{v}_r(\tau) - \mathbf{x}_2\,(t_0)\right]^T \mathbf{d}(\tau)d\tau\mathbf{d}(t)\\ &= \mathbf{s}(t) \end{aligned} \tag{8}$$

and

$$\begin{aligned} y_2(t) &= [0\;0\;1]\left[\mathbf{x}_1\,(t_0) + (t-t_0)\,\mathbf{x}_2\,(t_0) + \int_{t_0}^{t}\mathbf{v}_r\,(\tau)\,d\tau\right]\\ &= [0\;0\;1]\,\mathbf{p}\,(t) \end{aligned} \tag{9}$$

for all $t \in \mathcal{I}$, $\mathcal{I} = [t_0, t_f]$. Substituting (7) in (8) gives

$$\begin{aligned}\mathbf{y}_1(t) &= \mathbf{p}(t_0) + (t-t_0)\,\mathbf{v}_c(t_0) + \int_{t_0}^{t}\mathbf{v}_r(\tau)d\tau + \|\mathbf{r}\,(t_0)\|\,\mathbf{d}(t)\\ &\quad + \int_{t_0}^{t}\left[\mathbf{v}_s\,(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c\,(t_0)\right]^T \mathbf{d}(\tau)d\tau\mathbf{d}(t) \end{aligned} \tag{10}$$

Next, it is shown that (10) is equal to $\mathbf{s}(t)$ for all $t \in \mathcal{I}$. Substituting $t = t_0$ in (10) yields

$$\mathbf{y}_1\,(t_0) = \mathbf{p}\,(t_0) + \|\mathbf{r}\,(t_0)\|\,\mathbf{d}\,(t_0) = \mathbf{p}\,(t_0) + \mathbf{r}\,(t_0) = \mathbf{s}\,(t_0)\,.$$

The time derivative of (10) is given by

$$\begin{aligned}\dot{\mathbf{y}}_1(t) &= \|\mathbf{r}\,(t_0)\|\,\dot{\mathbf{d}}(t)\\ &\quad + \int_{t_0}^{t}\left[\mathbf{v}_s\,(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c\,(t_0)\right]^T \mathbf{d}(\tau)d\tau\dot{\mathbf{d}}(t)\\ &\quad + \mathbf{v}_r\,(t) + \mathbf{v}_c\,(t_0)\\ &\quad + \left[\mathbf{v}_s\,(t) - \mathbf{v}_r(t) - \mathbf{v}_c\,(t_0)\right]^T \mathbf{d}(t)\mathbf{d}(t). \end{aligned} \tag{11}$$

As $\mathbf{v}_c(t)$ is constant, it is possible to rewrite (11) as

$$\begin{aligned}\dot{\mathbf{y}}_1(t) &= \|\mathbf{r}\,(t_0)\|\,\dot{\mathbf{d}}(t)\\ &\quad + \int_{t_0}^{t}\left[\mathbf{v}_s\,(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c\,(\tau)\right]^T \mathbf{d}(\tau)d\tau\dot{\mathbf{d}}(t)\\ &\quad + \mathbf{v}_r\,(t) + \mathbf{v}_c\,(t)\\ &\quad + \left[\mathbf{v}_s\,(t) - \mathbf{v}_r(t) - \mathbf{v}_c\,(t)\right]^T \mathbf{d}(t)\mathbf{d}(t). \end{aligned} \tag{12}$$

The derivative of $\|\mathbf{r}(t)\|$ is given by

$$\frac{d}{dt}\,\|\mathbf{r}(t)\| = \left[\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t)\right]^T \mathbf{d}(t),$$

which allows to write

$$\|\mathbf{r}(t)\| = \|\mathbf{r}\,(t_0)\| + \int_{t_0}^{t}\left[\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(\tau)\right]^T\mathbf{d}(\tau)d\tau. \tag{13}$$

On the other hand, the time derivative of (1) is given by

$$\begin{aligned}\dot{\mathbf{d}}(t) &= \frac{\mathbf{v}_s\,(t) - \mathbf{v}_r(t) - \mathbf{v}_c\,(t)}{\|\mathbf{r}(t)\|}\\ &\quad - \frac{\left[\mathbf{v}_s\,(t) - \mathbf{v}_r(t) - \mathbf{v}_c\,(t)\right]^T\mathbf{d}(t)}{\|\mathbf{r}(t)\|}\mathbf{d}(t). \end{aligned} \tag{14}$$

Substituting (13) and (14) in (12) gives $\dot{\mathbf{y}}_1(t) = \mathbf{v}_s(t)$. As it is shown that, with the initial condition (7), $\mathbf{y}_1\,(t_0) = \mathbf{s}\,(t_0)$ and $\dot{\mathbf{y}}_1(t) = \mathbf{v}_s(t) = \dot{\mathbf{s}}(t)$, then it must be $\mathbf{y}_1(t) = \mathbf{s}(t)$ for all $t \in \mathcal{I}$, which means that the initial condition (7) explains the system output $\mathbf{y}_1(t)$. Next, it is shown that the initial condition (7) also explains $y_2(t)$. To that purpose, substitute (7) in (9), which gives

$$y_2(t) = [0\;0\;1]\left[\mathbf{p}\,(t_0) + (t-t_0)\,\mathbf{v}_c\,(t_0) + \int_{t_0}^{t}\mathbf{v}_r\,(\tau)\,d\tau\right]. \tag{15}$$

Substituting $t = t_0$ in (15) gives $y_2\,(t_0) = [0\;0\;1]\,\mathbf{p}\,(t_0)$. The time derivative of (15) is given by

$$\dot{y}_2(t) = [0\;0\;1]\left[\mathbf{v}_r(t) + \mathbf{v}_c\,(t_0)\right]. \tag{16}$$

As $\mathbf{v}_c(t)$ is constant, it is possible to rewrite (16) as

$$\dot{y}_2(t) = [0\;0\;1]\left[\mathbf{v}_r(t) + \mathbf{v}_c\,(t)\right] = [0\;0\;1]\,\dot{\mathbf{p}}(t).$$

Thus, it was shown that, with (7), it is $y_2\,(t_0) = [0\;0\;1]\,\mathbf{p}\,(t_0)$ and $\dot{y}_2(t) = [0\;0\;1]\,\dot{\mathbf{p}}(t)$, which implies that $y_2(t) = [0\;0\;1]\,\mathbf{p}(t)$. This concludes the proof, as it was shown that (7) explains the output. As the initial condition is uniquely determined under the conditions of the theorem, it must be (7). ∎

In order to design observers (or filtering) solutions with GAS error dynamics, stronger forms of observability are convenient. The following theorem addresses this issue.

Theorem 3: If

$$\underset{\substack{\alpha>0\\ \delta>0}}{\exists}\;\underset{t\geq t_0}{\forall}\;\underset{t_i\in[t,t+\delta]}{\exists}\;|d_3\,(t_i)| \geq \alpha \tag{17}$$

then the LTV system (3) is uniformly completely observable.

Proof: The LTV system (3) is uniformly completely observable if there exist positive constants α_1, α_2, and δ such that, for all $t \geq t_0$, it is true that

$$\alpha_1\mathbf{I} \preceq \boldsymbol{\mathcal{W}}\,(t, t+\delta) \preceq \alpha_2\mathbf{I},$$

which is equivalent to say that, for all unit vectors $\mathbf{c}$,

$$\alpha_1 \leq \mathbf{c}^T\boldsymbol{\mathcal{W}}\,(t, t+\delta)\,\mathbf{c} \leq \alpha_2. \tag{18}$$

As in the proof of Theorem 1, it is possible to write

$$\mathbf{c}^T\boldsymbol{\mathcal{W}}\,(t, t+\delta)\,\mathbf{c} = \int_{t}^{t+\delta}\|\mathbf{f}\,(\tau, t)\|^2\,d\tau$$

for all $\|\mathbf{c}\| = 1$, where

$$\mathbf{f}\,(\tau, t) = \begin{bmatrix} \mathbf{f}_1\,(\tau, t)\\ f_2\,(\tau, t) \end{bmatrix} \in \mathbb{R}^4,$$

for all $\tau \in [t, t+\delta]$, with

$$\begin{aligned}\mathbf{f}_1\,(\tau, t) &= \mathbf{c}_1 + \left[(\tau - t)\,\mathbf{I} - \mathbf{d}(\tau)\int_{t}^{\tau}\mathbf{d}^T(\sigma)d\sigma\right]\mathbf{c}_2\\ &\quad + c_3\mathbf{d}(\tau) \in \mathbb{R}^3 \end{aligned}$$

and

$$f_2\,(\tau, t) = c_{13} + (\tau - t)\,c_{23} \in \mathbb{R},$$

where $\mathbf{c}_1 = [c_{11}\;c_{12}\;c_{13}]^T$ and $\mathbf{c}_2 = [c_{21}\;c_{22}\;c_{23}]^T$. The right side of (18) is always verified as $\mathbf{f}\,(\tau, t)$ is a continuous bounded function for all $\tau \in [t, t+\delta]$, with $t \geq t_0$. Next, it is shown that, if (17) holds, then the left side of (18) is also verified. Fix δ and α such that (17) is true. Suppose first that $c_{13} \neq 0$. Then, $|f_2(t)| = |c_{13}| > 0$ for all $t \geq t_0$ and, using Proposition 1, there exists $\beta_1 > 0$ such that, for all $t \geq t_0$, it is true that $\mathbf{c}^T \boldsymbol{\mathcal{W}}(t, t+\delta)\,\mathbf{c} \geq \beta_1$. Suppose now that $c_{23} \neq 0$. Then,

$$\left| \frac{d}{d\tau} f_2(\tau, t) \right|_{\tau=t} \Bigg| = |c_{23}| > 0$$

for all $t \geq t_0$. As such, using Proposition 1 twice, it follows that there exists $\beta_2 > 0$ such that, for all $t \geq t_0$, it is true that $\mathbf{c}^T \boldsymbol{\mathcal{W}}(t, t+\delta)\,\mathbf{c} \geq \beta_2$. Suppose now that $c_{13} = 0$, $c_{23} = 0$, and $c_3 \neq 0$. Then, using (17), it follows that, for all $t \geq t_0$, it is possible to choose $t_i \in [t, t+\delta]$ such that $\|\mathbf{f}_1(t_i, t)\| \geq |\alpha c_3|$. But that implies, using Proposition 1 again, that there exists $\beta_3 > 0$ such that, for all $t \geq t_0$, $\mathbf{c}^T \boldsymbol{\mathcal{W}}(t, t+\delta)\,\mathbf{c} \geq \beta_3$. Suppose now that $c_{13} = 0$, $c_{23} = 0$, $c_3 = 0$ and $\mathbf{c}_1 \neq \mathbf{0}$. Then, it follows that $\|\mathbf{f}_1(t, t)\| \geq \|\mathbf{c}_1\| > 0$, which implies that, using Proposition 1, there exists $\beta_4 > 0$ such that, for all $t \geq t_0$, $\mathbf{c}^T \boldsymbol{\mathcal{W}}(t, t+\delta)\,\mathbf{c} \geq \beta_4$. Suppose finally that $\mathbf{c}_1 = \mathbf{0}$, $c_{23} = 0$, $c_3 = 0$ and $\mathbf{c}_2 \neq \mathbf{0}$. For $\tau = t$, it follows that

$$\left. \frac{d}{d\tau} \mathbf{f}_1(\tau, t) \right|_{\tau=t} = \left[\mathbf{I} - \mathbf{d}(t)\mathbf{d}^T(t)\right] \mathbf{c}_2$$

for all t. The only non-null vector such that

$$\left. \frac{d}{d\tau} \mathbf{f}_1(\tau, t) \right|_{\tau=t} = \mathbf{0}$$

is $\mathbf{c}_2 = \pm \mathbf{d}(t)$. Now, recall that $d_3(t_0) \neq 0$. Then, as (17) holds, there exists $\beta_5 > 0$ such that it is possible to choose, for all $t \geq t_0$, $t_j \in [t, t+\delta]$ such that

$$\begin{aligned} \left\| \tfrac{d}{d\tau}\, \mathbf{f}_1(\tau, t)|_{\tau=t_j} \right\| &= \Big\| \mathbf{d}(t) - \left[\mathbf{d}(t_j)\mathbf{d}^T(t)\right] \mathbf{d}(t_j) - \\ &\left[\int_t^{t_j} \mathbf{d}^T(\sigma)\mathbf{d}(t)\, d\sigma \right] \dot{\mathbf{d}}(t_j) \Big\| \geq \beta_5, \end{aligned}$$

as $\dot{\mathbf{d}}(t_j)$ cannot be expressed as a linear combination of $\mathbf{d}(t)$ and $\mathbf{d}(t_j)$. As such, using Proposition 1 twice, it follows that there exists $\beta_6 > 0$ such that, for all $t \geq t_0$, $\mathbf{c}^T \boldsymbol{\mathcal{W}}(t, t+\delta)\,\mathbf{c} \geq \beta_6$. But this concludes the proof, as it was shown that (18) holds for all unit vectors $\mathbf{c}$. ■

C. Kalman filter

Section III-A introduced a LTV system for navigation based on direction and depth measurements and its observability was characterized in Section III-B. In particular, it was shown that the LTV system (3) is uniformly completely observable if (17) is satisfied. A natural estimation solution thus consists in the Kalman filter, with guarantee of globally asymptotically stable error dynamics. Including additive system disturbances and sensor noise, the system dynamics read as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{n}(t) \end{cases},$$

where $\mathbf{w}(t) \in \mathbb{R}^7$ is zero-mean white Gaussian noise, with $\mathrm{E}\left[\mathbf{w}(\mathrm{t})\,\mathbf{w}^{\mathrm{T}}(\mathrm{t}-\tau)\right] = \boldsymbol{\Xi}\delta(\tau)$, $\boldsymbol{\Xi} \succ \mathbf{0}$, $\mathbf{n}(t) \in \mathbb{R}^3$ is zero-mean white Gaussian noise, with $\mathrm{E}\left[\mathbf{n}(\mathrm{t})\,\mathbf{n}^{\mathrm{T}}(\mathrm{t}-\tau)\right] = \boldsymbol{\Theta}\delta(\tau)$, $\boldsymbol{\Theta} \succ \mathbf{0}$, and $\mathrm{E}\left[\mathbf{w}(\mathrm{t})\,\mathbf{n}^{\mathrm{T}}(\mathrm{t}-\tau)\right] = \mathbf{0}$. It is important to stress, however, that it is not possible to conclude that this is an optimal solution, as the actual system disturbances and sensor noise may not be additive. Nevertheless, the nominal filter error dynamics are globally asymptotically stable if the LTV system is uniformly completely observable [13]. The design of the time-varying Kalman filter is well known and therefore it is omitted.

IV. Simulation results

In order to assess the performance of the proposed solution, simulations were carried out, which are detailed in this section. The initial position of the source, which can be, for instance, an Autonomous Surface Craft, is $\mathbf{s}_0 = [4\ 0\ 0]^T$ (m), whereas the initial position of the agent, which can be an Autonomous Underwater Vehicle, is $\mathbf{p}_0 = [0\ 0\ 10]^T$ (m). The drift velocity of the source was set to $\mathbf{v}_s(t) = [1\ 0.5\ 0]^T$ (m/s), while the drift velocity of the agent was set to $\mathbf{v}_c(t) = [0\ 0.5\ 0]^T$ (m/s). The relative velocity of the agent is $\mathbf{v}_r(t) = [1\ 0\ 0]^T$ (m/s). The resulting trajectories of the source and the agent are those depicted in Fig. 1. Clearly, the observability conditions derived in the paper are met as the direction is not horizontal, which allows to employ the solution proposed in the paper. In the context of the EU project TRIDENT, this scenario configures a typical leader-following integrated guidance, navigation, and control approach, where the ASC acts as a fast communication relay between the I-AUV and the end user while providing, simultaneously, accurate positioning data for the I-AUV. A leader following task ensures that the ASC remains close to the vertical of the I-AUV, in the most accurate cone of the acoustic equipment coverage.

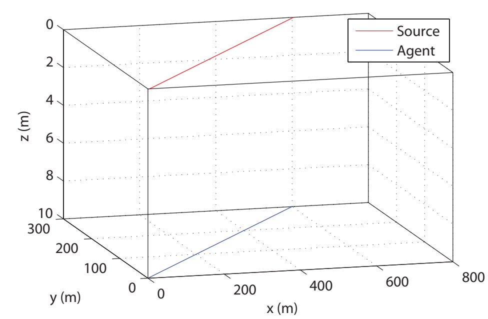

Fig. 1. Trajectories described by the agent and the source

Noise was considered for the direction, depth, and the relative velocity of the agent $\mathbf{v}_r(t)$. In particular, additive zero mean white Gaussian noise was considered for $\mathbf{v}_r(t)$, with standard deviation of 0.01 m/s, while the direction readings were assumed perturbed by rotations about random vectors of an angle modeled by zero-mean white Gaussian noise, with standard deviation of 1°. The depth measurements are assumed to be corrupted by additive zero mean white Gaussian noise, with standard deviation of 1 m. The source position and velocity data is also assumed to be corrupted by additive zero mean white Gaussian noise, with standard deviations of 1 m and 0.01 m/s, respectively. The Kalman filter parameters were set to $\boldsymbol{\Xi} = \mathrm{diag}\left(10^{-2}\mathbf{I}, 10^{-5}\mathbf{I}, 10^{-2}\right)$ and $\boldsymbol{\Theta} = \mathbf{I}$. The initial estimates were all set to zero.

The initial convergence of the filter errors is depicted in Fig. 2. As it is possible to see, the initial transients due to the mismatch of the initial conditions quickly fade out. Faster convergence rates would be possible by different choices of the Kalman filter parameters. For performance evaluation

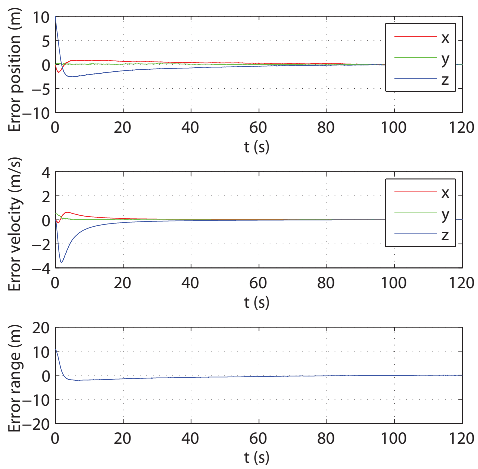

Fig. 2. Evolution of the estimation errors

purposes, an EKF was also devised, with the gains tuned for better accuracy. An example of the initial convergence with the EKF can be found in Fig. 3.

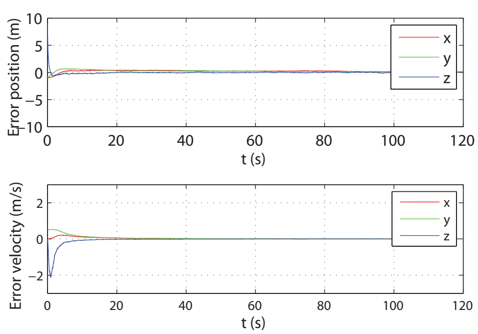

Fig. 3. Evolution of the estimation errors (EKF)

In order to better evaluate the performance of the proposed solution, the Monte Carlo method was applied. In particular, 1000 simulations were carried out with different, randomly generated noise signals. The mean and standard deviation of the errors were computed for each simulation and averaged over the set of simulations. The results are depicted in Table I. As it is possible to observe, the standard deviation of the errors is very low, adequate for the sensor suite that was considered. The results for the EKF are also shown and it is possible to conclude that similar performances are achieved, however the proposed solution guarantees GAS error dynamics, in contrast with the EKF.

TABLE I

STANDARD DEVIATION OF THE STEADY-STATE ESTIMATION ERRORS OF THE PROPOSED SOLUTION AND THE EKF, AVERAGED OVER 1000 RUNS OF THE SIMULATION

	Proposed solution	EKF
$\sigma_{\tilde{x}_{11}}$ (m)	5.97×10^{-3}	6.88×10^{-3}
$\sigma_{\tilde{x}_{12}}$ (m)	5.16×10^{-3}	6.14×10^{-3}
$\sigma_{\tilde{x}_{13}}$ (m)	5.12×10^{-3}	4.89×10^{-3}
$\sigma_{\tilde{x}_{21}}$ (m/s)	4.72×10^{-4}	4.66×10^{-4}
$\sigma_{\tilde{x}_{22}}$ (m/s)	4.71×10^{-4}	4.72×10^{-4}
$\sigma_{\tilde{x}_{23}}$ (m/s)	5.06×10^{-4}	4.58×10^{-4}
$\sigma_{\tilde{x}_{3}}$ (m)	8.30×10^{-3}	not explicitly estimated

V. CONCLUSIONS

This paper presented a novel time-varying Kalman filter with globally asymptotically stable error dynamics for the problem of navigation based on depth and direction measurements to a single source. The observability of the system was characterized, which allowed to conclude about the asymptotic stability of the Kalman filter. Simulations results were presented that illustrate the good performance achieved by the proposed solution, which was also compared with the EKF, achieving similar performance but with global asymptotic stability guarantees.

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