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GES Attitude Observers - Part II: Single Vector Observations ⋆

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Abstract: This paper presents an attitude observer based on single vector observations with globally exponentially stable error dynamics. The proposed solution is computationally efficient and, as the observer does not evolve on the Special Orthogonal Group $SO(3)$, an explicit solution on $SO(3)$ is also provided, whose error is shown to converge to zero for all initial conditions. Singularities, unwinding phenomena, or topological limitations for achieving global asymptotic stabilization are absent and the distinct roles of the inertial and the corresponding body-fixed vector are also examined. Possible extensions are discussed and simulation results are shown that illustrate the achievable performance in the presence of low-grade sensor specifications.

Keywords: Attitude estimation; observers for nonlinear systems; nonlinear observer design; time-varying systems.

1. INTRODUCTION

Attitude estimation has been a hot topic of research in the past decades, as evidenced by the large number of publications on the subject, see e.g. Metni et al. (2006); Sanyal et al. (2008); Vasconcelos et al. (2010); Tayebi et al. (2007); Rehbinder and Ghosh (2003); Batista et al. (2009); Mahony et al. (2008); Thienel and Sanner (2003); and references therein. The reader is referred to Crassidis et al. (2007) for a survey on this topic. It turns out that only recently has attitude estimation been studied based on time-varying reference vectors and, in particular, single vector observations, see Kinsey and Whitcomb (2007), Lee et al. (2007), and Mahony et al. (2009). This paper presents a novel theoretical attitude estimation framework based on single vector observations. Applications include, e.g., attitude estimation of unmanned vehicles that depend on electromagnetic or acoustic feedback of direction vectors of known landmarks. For space applications, it is interesting e.g. in attitude estimation in trajectories with gravity gradient effects or even using magnetometers and sun sensor readings, as the corresponding inertial vectors are slowly time-varying.

The main contribution of this paper is the development of a novel attitude observer resorting to a single general vector observation that has globally exponentially stable (GES) error dynamics. It is based on the exact angular motion kinematics and it only requires a persistent excitation condition of the reference vector, building on well-established Lyapunov results and linear systems theory. Moreover, it is computationally efficient and it allows for norm changes of the vector observation. As in the companion paper Batista et al. (2011), the approach followed in this paper is to design an observer on $\mathbb{R}^{3\times 3}$ that yields estimates that converge asymptotically to $SO(3)$, therefore overcoming the topological limitations for achieving global asymptotic stabilization that are present on $SO(3)$. However, only a single vector is now available for feedback purposes, instead of multiple vectors, which motivates novel definitions and additional estimation structures. The vector may even be null for some time, as long as a persistent excitation condition is satisfied. Finally, an explicit solution on $SO(3)$ is also provided resorting to a projection operator, an approach that has been employed previously in the design of interpolation methods on $SE(3)$, see Belta and Kumar (2002). The error of the solution on $SO(3)$ is shown to converge to zero for all initial conditions.

The paper is organized as follows. The attitude kinematics and some definitions are given in Section 2, while the problem addressed in the paper is described in Section 3. The design and stability analysis of the attitude observer based on single vector observations is presented in Section 4. In addition, the role of the inertial and body-fixed vectors is discussed, as well as some refinements to the proposed solution. Simulation results that illustrate the achievable performance are presented in Section 5 and, finally, Section 6 summarizes the main contributions and conclusions of the paper.

1.1 Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as $\operatorname{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \times \mathbf{y}$ represents the cross product.

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2. PRELIMINARIES

This paper follows the attitude kinematic conventions described in Batista et al. (2011), which are standard. Let $\{I\}$ be an inertial reference frame, $\{B\}$ a body-fixed reference frame, and $\mathbf{R}(t) \in SO(3)$ the rotation matrix from $\{B\}$ to $\{I\}$. The attitude kinematics are given by

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left[\boldsymbol{\omega}(t)\right],$$

where $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is the angular velocity of $\{B\}$, expressed in $\{B\}$, and $\mathbf{S}(.)$ is the skew-symmetric matrix such that $\mathbf{S}(\boldsymbol{x}) \times \boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$. The angular velocity is assumed to be a continuous bounded signal, available for observer design purposes.

The following definitions are useful in the sequel.

Definition 1. A continuous norm-bounded vector $\mathbf{a}(t) \in \mathbb{R}^3$ is called *persistently non-constant* if

$$\underset{\substack{\alpha>0\\ \delta>0}}{\exists} \; \underset{t\geq t_0}{\forall} \; \underset{\substack{\mathbf{d}\in\mathbb{R}^3\\ \|\mathbf{d}\|=1}}{\forall} \int_t^{t+\delta} \|\mathbf{a}(\sigma) \times \mathbf{d}\| \, d\sigma \geq \alpha$$

and

$$\underset{\epsilon>0}{\exists} \; \underset{t\geq t_0}{\forall} \; \|\mathbf{a}(t)\| > \epsilon.$$

The following two definitions are similar to those introduced in Batista et al. (2011) but allow for piecewise continuous vectors.

Definition 2. A set of N piecewise continuous norm-bounded vectors $\mathcal{A} = \left\{\mathbf{a}_i(t) \in \mathbb{R}^3, i = 1, \ldots, N\right\}$ is called *persistently non-collinear* if there exist at least two vectors $\mathbf{a}_l(t)$ and $\mathbf{a}_m(t)$, $l, m \in \{1, \ldots, N\}$, such that

$$\underset{\substack{\alpha>0\\ \delta>0}}{\exists} \; \underset{t\geq t_0}{\forall} \int_t^{t+\delta} \|\mathbf{a}_l(\sigma) \times \mathbf{a}_m(\sigma)\| \, d\sigma \geq \alpha.$$

Definition 3. A set of N piecewise continuous norm-bounded vectors $\mathcal{A} = \left\{\mathbf{a}_i(t) \in \mathbb{R}^3, i = 1, \ldots, N\right\}$ is called *persistently non-planar* if there exist at least three distinct vectors $\mathbf{a}_l(t)$, $\mathbf{a}_m(t)$, and $\mathbf{a}_n(t)$, $l, m, n \in \{1, \ldots, N\}$, such that

$$\underset{\substack{\alpha>0\\ \delta>0}}{\exists} \; \underset{t\geq t_0}{\forall} \; \underset{\substack{\mathbf{d}\in\mathbb{R}^3\\ \|\mathbf{d}\|=1}}{\forall} \int_t^{t+\delta} \|\mathbf{M}(\sigma)\, \mathbf{d}\| \, d\sigma \geq \alpha,$$

where

$$\mathbf{M}(t) := \begin{bmatrix} \mathbf{a}_l^T(t) \\ \mathbf{a}_m^T(t) \\ \mathbf{a}_n^T(t) \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

3. PROBLEM STATEMENT

Consider a continuous norm-bounded reference vector $\mathbf{r}_1(t) \in \mathbb{R}^3$, expressed in inertial coordinates, and the corresponding vector $\mathbf{v}_1(t) \in \mathbb{R}^3$, expressed in body-fixed coordinates, that satisfies

$$\mathbf{r}_1(t) = \mathbf{R}(t)\mathbf{v}_1(t). \tag{1}$$

Suppose that $\mathbf{r}_1(t)$ is persistently non-constant and that the angular velocity $\boldsymbol{\omega}(t)$ is continuous and bounded. The problem of attitude estimation considered in the paper is that of designing an observer for the rotation matrix $\mathbf{R}(t)$ with globally exponentially stable error dynamics based on $\mathbf{v}_1(t)$, $\mathbf{r}_1(t)$, and $\boldsymbol{\omega}(t)$.

4. OBSERVER DESIGN AND STABILITY ANALYSIS

4.1 Attitude observer

Before presenting the observer design, it is first shown how to construct a set of vectors in body-fixed coordinates, associated to a persistently non-planar set of reference vectors, based on a persistently non-constant reference vector and the corresponding vector in body-fixed coordinates. This is established in the following theorem.

Theorem 1. Consider a persistently non-constant reference vector $\mathbf{r}_1(t) \in \mathbb{R}^3$, with $\alpha > 0$ and $\delta > 0$ such that

$$\underset{t\geq t_0}{\forall} \; \underset{\substack{\mathbf{d}\in\mathbb{R}^3\\ \|\mathbf{d}\|=1}}{\forall} \int_t^{t+\delta} \|\mathbf{r}_1(\sigma) \times \mathbf{d}\| \, d\sigma \geq \alpha, \tag{2}$$

and the corresponding vector in body-fixed coordinates $\mathbf{v}_1(t) \in \mathbb{R}^3$ such that (1) holds. Suppose that the angular velocity $\boldsymbol{\omega}(t)$ is continuous and bounded. Define the set of reference vectors $\mathcal{R} := \{\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t)\}$, where

$$\mathbf{r}_2(t) := \mathbf{r}_1(t_i), \; t_i \leq t < t_{i+1}, \; i \in \mathbb{N}_0, \tag{3}$$

with $t_i := t_0 + i\delta$, $i \in \mathbb{N}_0$, and

$$\mathbf{r}_3(t) := \mathbf{r}_1(t) \times \mathbf{r}_2(t) \in \mathbb{R}^3. \tag{4}$$

Define also the set of vectors in body-fixed coordinates $\mathcal{V} := \{\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t)\}$, where $\mathbf{v}_2(t)$ is the piecewise continuous vector

$$\begin{cases} \mathbf{v}_2(t_i) := \mathbf{v}_1(t_i) \\ \dot{\mathbf{v}}_2(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}_2(t), \; t_i \leq t < t_{i+1} \end{cases}, \; i \in \mathbb{N}_0 \tag{5}$$

and

$$\mathbf{v}_3(t) := \mathbf{v}_1(t) \times \mathbf{v}_2(t) \in \mathbb{R}^3. \tag{6}$$

Then, i) the set of reference vectors $\mathcal{R}$ is compatible with the set of body-fixed vectors $\mathcal{V}$, i.e.,

$$\mathbf{r}_i(t) = \mathbf{R}(t)\mathbf{v}_i(t), \; i = 1, 2, 3; \tag{7}$$

and ii) the set of reference vectors $\mathcal{R}$ is persistently non-planar.

Proof. First, notice that, by assumption, (7) is verified for $i = 1$. From (5) it is straightforward to show that

$$\mathbf{v}_2(t) = \mathbf{R}^T(t)\mathbf{R}(t_i)\,\mathbf{v}_2(t_i), \; t_i \leq t < t_{i+1}, \; i \in \mathbb{N}_0. \tag{8}$$

By definition,

$$\mathbf{v}_2(t_i) = \mathbf{v}_1(t_i), \; i \in \mathbb{N}_0. \tag{9}$$

Substituting (9) in (8) and using (1) gives

$$\begin{aligned}\mathbf{v}_2(t) &= \mathbf{R}^T(t)\mathbf{R}(t_i)\,\mathbf{v}_1(t_i) \\ &= \mathbf{R}^T(t)\mathbf{R}(t_i)\,\mathbf{R}^T(t_i)\,\mathbf{r}_1(t_i) \\ &= \mathbf{R}^T(t)\mathbf{r}_1(t_i), \; t_i \leq t < t_{i+1}, \; i \in \mathbb{N}_0. \end{aligned} \tag{10}$$

Now, substituting (3) in (10) immediately gives that (7) is verified for $i = 2$. As (7) is verified for $i = 1$ and $i = 2$, it is trivial to show, from (4) and (6), that it is also verified for $i = 3$,

$$\mathbf{v}_3(t) = \left[\mathbf{R}^T(t)\mathbf{r}_1(t)\right] \times \left[\mathbf{R}^T(t)\mathbf{r}_2(t)\right] = \mathbf{R}^T(t)\mathbf{r}_3(t),$$

which concludes the first part of the proof. Next, it is shown that the set of vectors $\{\mathbf{r}_1(t), \mathbf{r}_2(t)\}$ is persistently non-collinear. By assumption, there exist positive constants α and δ such that (2) holds. Let $\delta' := 2\delta$. Then,

$$\int_{t}^{t+\delta'} \|\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)\| \, d\sigma = \int_{t_i}^{t_{i+1}} \|\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)\| \, d\sigma$$
$$+ \int_{t}^{t_i} \|\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)\| \, d\sigma + \int_{t_{i+1}}^{t+\delta'} \|\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)\| \, d\sigma \quad (11)$$

for all $t \geq t_0$, where i corresponds to the smallest integer such that $t_i \geq t$. Since all terms are positive, it follows from (11) that

$$\int_{t}^{t+\delta'} \|\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)\| \, d\sigma \geq \int_{t_i}^{t_{i+1}} \|\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)\| \, d\sigma \quad (12)$$

for all $t \geq t_0$. Now, notice that, by definition, $\mathbf{r}_2(t)$ is constant on $[t_i, t_{i+1}[$. Therefore, it is possible to write, from (12),

$$\int_{t}^{t+\delta'} \|\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)\| \, d\sigma \geq \int_{t_i}^{t_i+\delta} \|\mathbf{r}_1(\sigma)\times\mathbf{r}_2(t_i)\| \, d\sigma$$
$$\geq \|\mathbf{r}_2(t_i)\| \int_{t_i}^{t_i+\delta} \left\| \mathbf{r}_1(\sigma) \times \frac{\mathbf{r}_2(t_i)}{\|\mathbf{r}_2(t_i)\|} \right\| d\sigma \quad (13)$$

for all $t \geq t_0$. Now, applying (2) to (13) gives

$$\int_{t}^{t+\delta'} \|\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)\| \, d\sigma \geq \alpha \|\mathbf{r}_2(t_i)\| \quad (14)$$

for all $t \geq t_0$. Substituting (3) in (14), and from the fact that, by assumption, $\mathbf{r}_1(t)$ is persistently non-constant and therefore bounded from below (and above), it follows that the set of reference vectors $\{\mathbf{r}_1(t), \mathbf{r}_2(t)\}$ is persistently non-collinear. Now, notice that the set of vectors $\mathcal{R}$ corresponds to the set of vectors $\{\mathbf{r}_1(t), \mathbf{r}_2(t)\}$, which is persistently non-collinear, augmented with the vector $\mathbf{r}_1(t)\times\mathbf{r}_2(t)$. Therefore, it follows that the augmented set $\mathcal{R}$ is persistently non-planar, see (Batista et al., 2011, Proposition 1), which concludes the second part of the proof. Notice that, although in (Batista et al., 2011, Proposition 1) the vectors are assumed continuous, the proof remains unchanged for piecewise continuous vectors. □

Following the notation introduced in Batista et al. (2011), consider a column representation of the rotation matrix $\mathbf{R}(t)$ given by $\boldsymbol{x}(t) = \begin{bmatrix} \mathbf{z}_1^T(t) & \mathbf{z}_2^T(t) & \mathbf{z}_3^T(t) \end{bmatrix}^T \in \mathbb{R}^9$, where

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{z}_1^T(t) \\ \mathbf{z}_2^T(t) \\ \mathbf{z}_3^T(t) \end{bmatrix}, \; \mathbf{z}_i(t) \in \mathbb{R}^3, \; i = 1, \ldots, 3.$$

Then, the dynamics of $\boldsymbol{x}(t)$ can be written as

$$\dot{\boldsymbol{x}}(t) = -\mathbf{S}_3 [\boldsymbol{\omega}(t)] \, \boldsymbol{x}(t),$$

where $\mathbf{S}_3(\mathbf{x}) := \operatorname{diag}(\mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{x})) \in \mathbb{R}^{9\times 9}$, $\mathbf{x} \in \mathbb{R}^3$. Let $\mathbf{r}_1(t)$ be a persistently non-constant vector, associated to a body-fixed vector $\mathbf{v}_1(t)$, which satisfies (1). Define the sets of vectors $\mathcal{V}$ and $\mathcal{R}$ as in Proposition 1 and let $\mathbf{v}(t) := \begin{bmatrix} \mathbf{v}_1^T(t) & \mathbf{v}_2^T(t) & \mathbf{v}_3^T(t) \end{bmatrix} \in \mathbb{R}^9$. It is straightforward to show, using (7), that

$$\mathbf{v}(t) = \mathbf{C}(t)\boldsymbol{x}(t),$$

where

$$\mathbf{C}(t) := \begin{bmatrix} r_{11}(t)\mathbf{I} & r_{12}(t)\mathbf{I} & r_{13}(t)\mathbf{I} \\ r_{21}(t)\mathbf{I} & r_{22}(t)\mathbf{I} & r_{23}(t)\mathbf{I} \\ r_{31}(t)\mathbf{I} & r_{32}(t)\mathbf{I} & r_{33}(t)\mathbf{I} \end{bmatrix} \in \mathbb{R}^{3\times 3},$$

with $\mathbf{r}_i(t) = \begin{bmatrix} r_{i1}(t) & r_{i2}(t) & r_{i3}(t) \end{bmatrix}^T \in \mathbb{R}^3$, $i = 1, 2, 3$.

Consider the attitude observer given by

$$\dot{\hat{\boldsymbol{x}}}(t) = -\mathbf{S}_3(\boldsymbol{\omega}(t))\,\hat{\boldsymbol{x}}(t) + \mathbf{C}^T(t)\mathbf{Q}\left[\mathbf{v}(t) - \mathbf{C}(t)\hat{\boldsymbol{x}}(t)\right], \quad (15)$$

where $\mathbf{Q} = \mathbf{Q}^T \in \mathbb{R}^{9\times 9}$ is a positive definite matrix, and define the error variable $\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$. Then, the observer error dynamics are given by

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{\mathcal{A}}(t)\tilde{\boldsymbol{x}}(t), \quad (16)$$

where $\boldsymbol{\mathcal{A}}(t) := -\left[\mathbf{S}_3[\boldsymbol{\omega}(t)] + \mathbf{C}^T(t)\mathbf{Q}\mathbf{C}(t)\right]$ is a piecewise continuous matrix.

The following theorem is the main result of the paper.

Theorem 2. Suppose that the vector $\mathbf{r}_1(t)$ is persistently non-constant and consider the attitude observer (15), where $\mathbf{Q} \succ \mathbf{0}$ is a design parameter and the sets of vectors $\mathcal{R}$ and $\mathcal{V}$ were chosen according to Theorem 1. Then, the origin of the observer error dynamics (16) is a globally exponentially stable equilibrium point.

Proof. The observer error dynamics (16) are identical to those of the observer proposed in (Batista et al., 2011, Theorem 1) for a persistently non-planar set of reference vectors. The only difference steams from the fact that $\boldsymbol{\mathcal{A}}(t)$ is piecewise continuous here instead of continuous, which results from the fact that the vectors employed in the observer design in this case are piecewise continuous instead of continuous. However, the proof of (Batista et al., 2011, Theorem 1) is still valid in this case. The proof is briefly completed here for the sake of completeness. Let $V(t) := \frac{1}{2}\|\tilde{\boldsymbol{x}}(t)\|^2$ be a Lyapunov candidate function. As $\mathbf{S}_3(.)$ is skew-symmetric, the time derivative of $V(t)$ is simply given by $\dot{V}(t) = -\tilde{\boldsymbol{x}}^T(t)\mathbf{C}^T(t)\mathbf{Q}\mathbf{C}(t)\tilde{\boldsymbol{x}}(t)$, which can be written as $\dot{V}(t) = -\tilde{\boldsymbol{x}}^T(t)\boldsymbol{\mathcal{C}}^T(t)\boldsymbol{\mathcal{C}}(t)\tilde{\boldsymbol{x}}(t)$, where $\boldsymbol{\mathcal{C}}(t) := \mathbf{Q}^{\frac{1}{2}}\mathbf{C}(t)$. Clearly, $\frac{1}{2}\|\tilde{\boldsymbol{x}}(t)\|^2 \leq V(t) \leq \frac{1}{2}\|\tilde{\boldsymbol{x}}(t)\|^2$ and $\dot{V}(t) \leq 0$. If, in addition,

$$V(t+\delta) - V(t) < -\alpha V(t) \quad (17)$$

for all $t \geq t_0$ and some $\delta > 0$ and $\alpha > 0$, then the origin $\tilde{\boldsymbol{x}} = \mathbf{0}$ is a globally exponentially stable equilibrium point, see (Khalil, 2001, Theorem 8.5). The observer error dynamics correspond to a linear time-varying system and, even though the system matrix is piecewise continuous, the transition matrix is well defined. Let $\boldsymbol{\phi}(t, t_0)$ be the transition matrix associated with $\boldsymbol{\mathcal{A}}(t)$ from t_0 to t. Then, it is possible to write

$$V(t+\delta) - V(t) = \int_t^{t+\delta} \dot{V}(\sigma) d\sigma$$
$$= -\int_t^{t+\delta} \tilde{\boldsymbol{x}}^T(\sigma)\boldsymbol{\mathcal{C}}^T(\sigma)\boldsymbol{\mathcal{C}}(\sigma)\tilde{\boldsymbol{x}}(\sigma) d\sigma$$
$$= -\int_t^{t+\delta} \tilde{\boldsymbol{x}}^T(t)\boldsymbol{\phi}^T(\sigma, t)\boldsymbol{\mathcal{C}}^T(\sigma)\boldsymbol{\mathcal{C}}(\sigma)\boldsymbol{\phi}(\sigma, t)\,\tilde{\boldsymbol{x}}(t) d\sigma$$
$$= -\tilde{\boldsymbol{x}}^T(t) \int_t^{t+\delta} \boldsymbol{\phi}^T(\sigma, t)\boldsymbol{\mathcal{C}}^T(\sigma)\boldsymbol{\mathcal{C}}(\sigma)\boldsymbol{\phi}(\sigma, t)\, d\sigma \tilde{\boldsymbol{x}}(t)$$
$$= -\tilde{\boldsymbol{x}}^T(t)\boldsymbol{\mathcal{W}}(t, t+\delta)\,\tilde{\boldsymbol{x}}(t),$$

where $\boldsymbol{\mathcal{W}}(t_0, t_f)$ is the observability Gramian associated with the pair $(\boldsymbol{\mathcal{A}}(t), \boldsymbol{\mathcal{C}}(t))$ on $[t_0, t_f]$, which is also well-defined, even though the corresponding system matrices are piecewise continuous instead of continuous. Evidently, if this pair is uniformly completely observable, it follows that (17) is verified for some positive constants α and δ. But that is easily shown following the same arguments of (Batista et al., 2011, Theorem 1) as the set of reference vectors is also persistently non-planar, thus concluding the proof. □

4.2 Solution on $SO(3)$

The attitude observer previously proposed yields estimates of the rotation matrix $\mathbf{R}(t)$ given by

$$\hat{\mathbf{R}}(t) = \begin{bmatrix} \hat{\mathbf{z}}_1^T(t) \\ \hat{\mathbf{z}}_2^T(t) \\ \hat{\mathbf{z}}_3^T(t) \end{bmatrix},\ \hat{\mathbf{z}}_i(t) \in \mathbb{R}^3,\ i = 1, \ldots, 3,$$

where $\hat{\boldsymbol{x}}(t) = \begin{bmatrix} \hat{\mathbf{z}}_1^T(t)\ \hat{\mathbf{z}}_2^T(t)\ \hat{\mathbf{z}}_3^T(t) \end{bmatrix} \in \mathbb{R}^9$. However, the estimate of the rotation matrix, $\hat{\mathbf{R}}(t)$, is not necessarily a rotation matrix as there is nothing in the observer structure imposing the restriction $\hat{\mathbf{R}}(t) \in SO(3)$. In fact, if this restriction is imposed, it is actually impossible to achieve global asymptotic stabilization due to topological limitations, see Bhat and Bernstein (2000). Nevertheless, the estimation error of the proposed observer converges globally exponentially fast to zero and therefore the corresponding rotation matrix restrictions are verified asymptotically. When the observer error is sufficiently small, one orthogonalization cycle suffices, as given by

$$\hat{\mathbf{R}}_o(t) = \frac{1}{2}\left(\hat{\mathbf{R}}(t) + \left[\hat{\mathbf{R}}^T(t)\right]^{-1}\right),$$

to obtain an estimate sufficiently close to an element of $SO(3)$, see Bar-Itzhack and Meyer (1976). In spite of the fact that the orthogonalization cycle is an extremely efficient method to obtain an estimate of the rotation matrix that is very close to $SO(3)$, it may happen that an explicit solution on SO(3) is required. This is established in the following theorem, see also Batista et al. (2011).

Theorem 3. Consider the estimate $\hat{\mathbf{R}}(t)$ obtained from the attitude observer (15) under the conditions of Theorem 2, with GES error dynamics. Further suppose that the initial estimate satisfies $\hat{\mathbf{R}}(t_0) \in SO(3)$ and define a new attitude estimate $\hat{\mathbf{R}}_f(t)$ of the rotation matrix $\mathbf{R}(t)$ as

$$\begin{cases} \hat{\mathbf{R}}_f(t) = \arg \min\limits_{\mathbf{X}(t) \in SO(3)} \left\|\mathbf{X}(t) - \hat{\mathbf{R}}(t)\right\|, \left\|\hat{\mathbf{R}}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I}\right\| \leq \epsilon \\ \dot{\hat{\mathbf{R}}}_f(t) = \hat{\mathbf{R}}_f(t)\mathbf{S}\left(\boldsymbol{\omega}(t)\right), \left\|\hat{\mathbf{R}}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I}\right\| > \epsilon \end{cases},$$

where $\epsilon > 0$. Then,

(1) $\hat{\mathbf{R}}_f(t) \in SO(3)$;
(2) there exists t_s such that $\left\|\hat{\mathbf{R}}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I}\right\| \leq \epsilon$ for all $t \geq t_s$ and therefore $\hat{\mathbf{R}}_f(t)$ corresponds to the projection on $SO(3)$ of $\hat{\mathbf{R}}(t)$ for all $t \geq t_s$; and
(3) the error $\left\|\mathbf{R}(t) - \hat{\mathbf{R}}_f(t)\right\|$ is bounded and

$$\lim_{t \to \infty} \left\|\mathbf{R}(t) - \hat{\mathbf{R}}_f(t)\right\| = 0.$$

Proof. See Batista et al. (2011).

Remark 1. A projection on $SO(3)$ of $\hat{\mathbf{R}}(t)$ is well known and it is readily obtained from the Singular Value Decomposition (SVD) of $\hat{\mathbf{R}}(t)$, see e.g. Belta and Kumar (2002).

Remark 2. Notice that Theorem 3 does not violate any of the results presented in Bhat and Bernstein (2000) for global asymptotic stabilization on $SO(3)$ as the solution provided by Theorem 3 is not guaranteed to be continuous for $t \leq t_s$.

4.3 Roles of the reference and the body-fixed vectors

This section clarifies the roles of the reference vector in inertial coordinates and the corresponding vector in body-fixed coordinates. In this paper the conditions for achieving GES were given in terms of persistent excitation hypothesis on the reference vector and no conditions were given for the corresponding vector in body-fixed coordinates. This is not by accident, as it is the reference vector that influences the observability of the system and not the body-fixed vector.

First, it is shown that, even though a reference vector may be persistently non-constant, which allows for the application of Theorem 2, the corresponding vector in body-fixed coordinates may not be persistently non-constant. Let $\mathbf{r}_1(t) \in \mathbb{R}^3$ be a persistently non-constant reference vector, in inertial coordinates, and let $\mathbf{v}_1(t)$ be the corresponding vector in body-fixed coordinates, which satisfies (1). Let the dynamics of the reference vector be given by

$$\dot{\mathbf{r}}_1(t) = -\mathbf{S}\left[\boldsymbol{\omega}_r(t)\right]\mathbf{r}_1(t) + u_r(t)\mathbf{r}_1(t),$$

where $u_r(t) \in \mathbb{R}$ is a continuous bounded function. Taking the time derivative of $\mathbf{v}_1(t)$ gives

$$\begin{aligned} \dot{\mathbf{v}}_1(t) = & -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}_1(t) - \mathbf{R}^T(t)\mathbf{S}\left[\boldsymbol{\omega}_r(t)\right]\mathbf{r}_1(t) \\ & + u_r(t)\mathbf{R}^T(t)\mathbf{r}_1(t). \end{aligned} \tag{18}$$

Using (1) and the fact that $\mathbf{R}(t)$ is a rotation matrix, together with cross product properties, allows to rewrite (18) as

$$\dot{\mathbf{v}}_1(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t) + \mathbf{R}^T(t)\boldsymbol{\omega}_r(t)\right]\mathbf{v}_1(t) + u_r(t)\mathbf{v}_1(t). \tag{19}$$

Let $\boldsymbol{\omega}(t) = -\mathbf{R}^T(t)\boldsymbol{\omega}_r(t)$. Then, (19) reads as $\dot{\mathbf{v}}_1(t) = u_r(t)\mathbf{v}_1(t)$, which means that $\mathbf{v}_1(t)$ has constant direction and therefore it is not persistently non-constant. This shows that it is possible to estimate the attitude even if the vector in body-fixed coordinates is not persistently non-constant.

On the other hand, when the reference vector is not persistently exciting, in the limit situation it has constant direction. It is well known that, with just one constant direction in inertial coordinates, it is impossible to recover the attitude matrix. Nevertheless, the corresponding vector, in body-fixed coordinates, may be persistently exciting, e.g., if $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_v$, where $\boldsymbol{\omega}_v$ is orthogonal to $\mathbf{v}_1(t_0)$. This shows that it may be impossible to estimate the attitude even though the vector observation, in body-fixed coordinates, is persistently non-constant.

The previous discussion clarifies why the inertial vectors are denoted as reference vectors in this paper. Even though it is possible to express alternative conditions such that the observer error dynamics are GES, the observability of the attitude is fundamentally connected to the evolution of the reference vector.

4.4 Further discussion

This section provides an extended discussion on the set of attitude estimation solutions presented in this paper and the companion paper Batista et al. (2011).

Lower bound on the norm of the reference vector In the definition of a persistently non-constant vector, a lower bound is set on the norm of the vector. It turns out that it is possible to generalize the observer design proposed in Section 4.1 to discard this restriction. Essentially, this assumption allows to write (13), where $\|\mathbf{r}_2(t_i)\| = \|\mathbf{r}_1(t_i)\|$ is lower bounded. If this bound was not assumed, it would be possible to have a persistently non-constant vector $\mathbf{r}_1(t)$ in Theorem 1 such that $\mathbf{r}_2(t) = \mathbf{r}_3(t) = \mathbf{0}$ for all t, and therefore the set of vectors $\mathcal{R}$ would not be persistently non-planar. However, an alternate time-varying scheme for the definition of a set of reference vectors is possible where δ is allowed to increase for the update of the reference vector $\mathbf{r}_2(t)$ such that $\mathbf{r}_2(t)$ is not a null vector for all t.

Direction vs. norm of the reference vector Although it is not explicitly shown here due to the lack of space, it is rather straightforward to conclude that the persistent excitation conditions expressed in the paper are essentially related to the direction of the reference vectors, and the only condition that the norm has to satisfy is not to be convergent to zero, which would lead to the loss of excitation.

Observer design for non persistently non-collinear sets of reference vectors In Batista et al. (2011) attitude observers were derived for multiple time-varying vectors observations assuming at least persistently non-collinear sets of reference vectors. It is now evident that, for a set of multiple vectors, it is possible to apply the observer design presented in this paper even if the set of vectors is not persistently non-collinear, as long as there exists at least one persistently non-constant reference vector. This broadens the applicability of the observers proposed in this paper and in Batista et al. (2011).

5. SIMULATION RESULTS

In order to evaluate the performance of the proposed solutions, simulations were carried out in a realistic setup environment, considering a single persistently non-constant reference vector, which results in a persistently non-coplanar set of three reference vectors, as given by Theorem 1, where it was set $\delta = 10$ s. The evolution of the reference and body-fixed vectors is depicted in Fig. 1 and the initial attitude is $\mathbf{R}(0) = \mathbf{I}$. Sensor noise was considered for the angular velocity readings and the body-fixed vector observation. In particular, additive, zero-mean, white Gaussian noise was considered, with standard deviation of $1\,^{\circ}/s$ for the angular velocity and 0.01 for the body-fixed vector observation. Notice that the specification for the angular velocity corresponds to a very low-grade sensor, while for the body-fixed vectors it corresponds to a standard deviation of 0.5% of the range. It should be emphasized that, when the norms of the body-fixed vectors are low, e.g. on the time interval $[20, 40]$ s, the noise of the vector observations is relatively very large. Therefore, the present specifications correspond to a realistic low-cost sensor suite. Although a different reference vector could have been chosen, notice that $\mathbf{r}_1(t)$ does not satisfy the lower bound on the norm. However, as discussed in Section 4.4, that does not affect the results and this vector was chosen on purpose to illustrate this situation. The filter parameter was chosen as $\mathbf{Q} = 0.1\mathbf{I}$, while the initial rotation matrix estimate was chosen as $\hat{\mathbf{R}}(0) = \operatorname{diag}(-1, -1, 1)$.

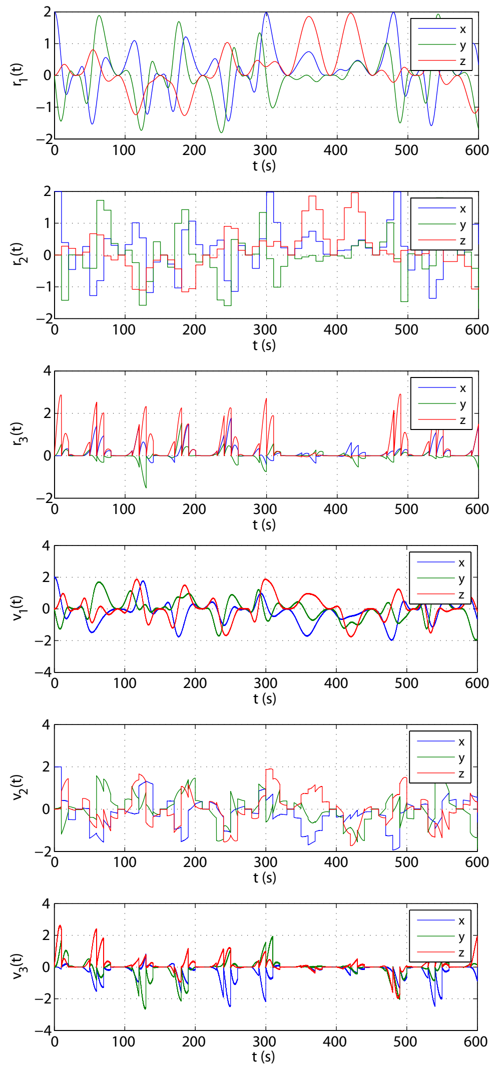

Fig. 1. Evolution of the reference and body-fixed vectors

The evolution of the Lyapunov function $V(t)$ is depicted in Fig. 2, and it is clear that the observer enters steady-state in less than 60 s, while the evolution of the error remains confined to a tight interval. The evolution of

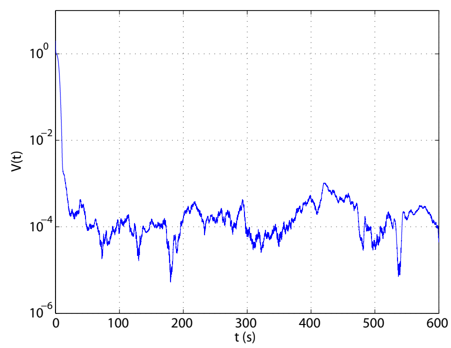

Fig. 2. Evolution of the Lyapunov function $V(t) = \frac{1}{2}\|\tilde{\boldsymbol{x}}(t)\|^2$

the orthogonality error, expressed as $\left\|\hat{\mathbf{R}}(t)\hat{\mathbf{R}}^T(t) - \mathbf{I}\right\|$, is shown in Fig. 3. The initial large error after the orthogonalization cycle appears due to the fact that, during the initial transients, the rotation estimate comes close to singularity. Nevertheless, the observer quickly enters the steady-state zone, and the effect of the orthogonalization cycle is visible, which translates in orthogonality errors around 10^{-4} in steady-state with one orthogonalization cycle and 10^{-8} with two orthogonalization cycles. Nevertheless, the solution provided in Theorem 3, instead of the use of orthogonalization steps, completely eliminates the orthogonality error, providing estimates on $SO(3)$. Although it is not show in the paper due to the lack of

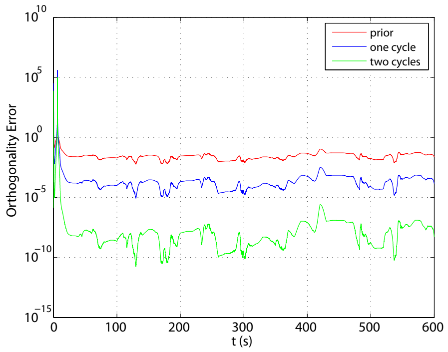

Fig. 3. Evolution of the orthogonality error, prior and after the orthogonalization steps

space, using an angle-axis representation for the rotation error defined as $\tilde{\mathbf{R}}_a(t) = \mathbf{R}^T(t)\hat{\mathbf{R}}(t)$, the angle error remains confined to a tight interval, in spite of the low-grade specifications of the sensors. The mean error is 0.68° which, considering the sensor suite specifications and the very limitted information, consists in a very good and promising result.

6. CONCLUSIONS

This paper presented an attitude observer based on single vectors with globally exponentially stable (GES) error dynamics. The proposed solution is computationally efficient and it does not have singularities, unwinding phenomena, or topological limitations for achieving global asymptotic stabilization. Central to the observer design is a persistent excitation condition defined for the reference vector, in inertial coordinates, that allows for the definition of augmented sets, both in inertial and body-fixed coordinates and available for observer design purposes. As the proposed observer does not evolve on $SO(3)$ but the estimates converge asymptotically to $SO(3)$, an additional solution is provided that yields attitude estimates on $SO(3)$, whose error is shown to converge to zero for all initial conditions. The distinct roles of the vector in inertial and body-fixed coordinates were examined and it was shown that the observability of the system is fundamentally connected to the evolution of the reference vector. Additional properties and extensions were also discussed. Finally, simulation results were provided that evidenced the performance of the proposed solution even in the presence of a sensor suite with very poor specifications.

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