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Decentralized state observers for range-based position and velocity estimation in acyclic formations with fixed topologies

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SUMMARY

This paper addresses the problem of decentralized position and velocity estimation in formations of autonomous vehicles. A limited number of vehicles in the formation have access to absolute position measurements, while the rest must rely on range measurements to neighboring agents, local sensor data, and limited communication capabilities to estimate their own position and velocity. The contribution is threefold: (i) a method for designing local state observers for each agent in the formation that rely only on locally available information is presented; (ii) the stability of the continuous-time linear time-varying Kalman filter subject to exponentially decaying perturbations in some variables is studied; and (iii) the stability of the error dynamics of the resulting decentralized state observer is analyzed for acyclic formations with fixed topologies, and it is shown that the error converges exponentially fast to the origin for all initial conditions. Simulation results are presented and discussed to validate the proposed solution, as well as assessing its performance under the influence of measurement noise. Copyright © 2015 John Wiley & Sons, Ltd.

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KEY WORDS: decentralized state observers; Kalman filtering; autonomous vehicles

1. INTRODUCTION

There are many applications where the use of formations composed by multiple autonomous vehicles working cooperatively is advantageous or, in some cases, crucial. To list only a few examples, formations of autonomous underwater vehicles (AUVs) can be used for applications such as minesweeping and oceanographic sampling over large areas, see, for example, [1] and [2], and close formation flight of Unmanned Aerial Vehicles (UAVs) allows for more efficient fuel usage, see [3] and [4]. As a result of that, the field of control and estimation in formations of autonomous agents has been the subject of extensive research in the last decade, see, for example, [5–10].

Conceptually, the easiest way to tackle control and estimation problems in formations of vehicles is to design a centralized solution, in which a central processing node receives all relevant information from the agents, performs all the computations, and spreads the results through the formation by communication. However, the implementation of a centralized solution poses problems in practical settings, as the heavy computational load in the central processing computer and the extensive communication that are required may translate into unacceptable delays and high communication

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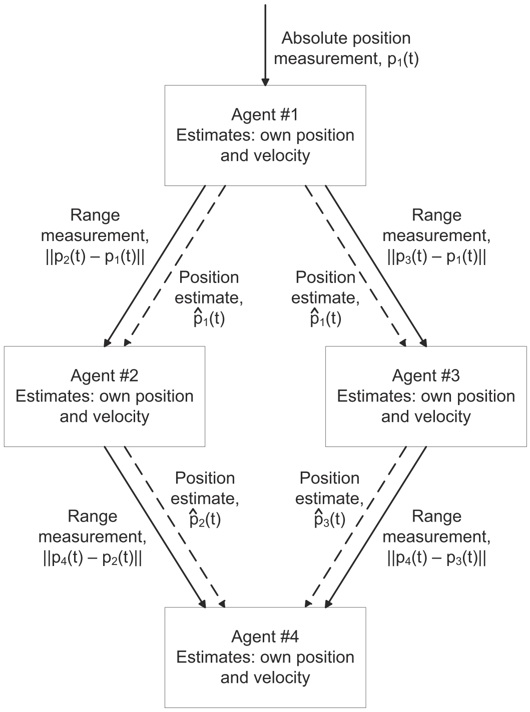

Figure 1. Simplified measurement and communication scheme for a sample four-agent formation.

loads in the formation. The alternative is to design a decentralized or distributed solution, in which each agent in the formation only concerns itself with a fraction of the computations, and relies only on locally available data (provided by a combination of sensor measurements and/or communication with other agents in the formation). Although a decentralized solution will fall behind a centralized one in terms of performance because of the fact that each local processing node must work with incomplete information, the much lower computational and communication load makes it very attractive for practical applications where communication, computational, and energy constraints are commonplace and often of paramount importance. For more on the pros and cons of decentralized solutions and their relevance in practical applications, see, for example, [11–13].

The problem addressed in this paper is the design of a decentralized state observer to estimate linear motion quantities (position and linear velocity), in a formation of vehicles with fixed topology, moving in a fluid with constant drift velocity. In order to achieve a decentralized architecture, each agent in the formation must rely only on locally available measurements and limited communication with other agents in the vicinity to estimate its inertial position and velocity. In the scenario envisioned in this paper, some agents in the formation have access to measurements of their absolute position, while the rest rely on measurements of their distance, or range, to other agents in the vicinity, as well as the position estimates of those agents, received through communication. Figure 1 depicts a simplified scheme of the measurements and communication in a sample formation composed by 4 agents. Additionally, to provide awareness of its movement, it is assumed that each agent has access to a measurement of its velocity relative to the fluid, measurements of its attitude, and estimates of its depth or altitude. This type of scenario is relevant in underwater applications with formations of AUVs, as tasks such as oceanographic sampling require accurate inertial position estimates, and the attenuation of electromagnetic waves in water does not allow the use of the GPS. In such a scenario, one or more AUVs closer to the surface can obtain absolute position measurements using, for example, range readings to a single mobile source such as an autonomous surface craft (ASC) [14] or to several fixed beacons [15], while the rest can use acoustic transceivers to obtain range readings and communicate state estimates. Measurements of the velocity relative to the water can be obtained using a Doppler Velocity Log (DVL), an attitude and heading reference system (AHRS) can provide the attitude estimates, and the depth measurements can be obtained using a

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pressure sensor. Note that, while the AHRS and DVL provide measurements with a fast sampling rate, due to the limitations of communication and sensing in underwater environments, the range and absolute position measurements will typically be obtained at a rate one sample every few seconds. Thus, applying the results of this paper to a practical situation will require a continuous-discrete implementation such as the one detailed in [16]. Our approach to this decentralized estimation problem can be divided into three main parts:

1. The design of a local state observer for each agent, which features globally exponentially stable (GES) error dynamics when the position estimates received from neighboring agents are exact. This problem is tackled by obtaining, through state and output augmentation, a linear time-varying (LTV) system, which mimics exactly the nonlinear dynamics of a given agent in the formation. Two cases are considered, depending on the number of range sources available to the agent, and sufficient conditions for observability of the LTV system are derived for each case. This reduces the problem of designing the local state observers to a classical linear state observer design problem.
2. The stability analysis of the continuous-time LTV Kalman filter when some of the variables of the system are corrupted by exponentially decaying perturbations. This is relevant to problem at hand because, as it will be discussed ahead, the position estimates received through communication are used to compute estimates of the input of the augmented LTV system, as well as its output matrix.
3. The stability analysis of the decentralized state observer composed by the local Kalman filters of all the agents in the formation. It is shown that, when the directed graph associated with the formation is acyclic and under mild assumptions, the estimation error of the decentralized state observer converges exponentially to the origin for all initial conditions.

Preliminary results on this subject were presented in [17]. This paper extends the previous work by providing an improved theoretical basis, as well as more detailed simulation results. In particular, uniform complete observability results are presented to strengthen the observability analysis, and the properties of the perturbed LTV Kalman filter are studied in much greater detail to improve the main result on the stability of the decentralized state observer. In recent years, several research groups have studied the problem of cooperative range-based localization. One of the first approaches to the problem can be found in [18], which presents a necessary and sufficient condition for local observability and an extended Kalman filter (EKF)-based solution. In [19], another EKF-based solution is presented to estimate the relative motion between two AUVs, featuring an observability analysis, which results in weak local observability guarantees. Similarly, in [20], the problem of localization based on range measurements to a single fixed beacon is addressed with weak local observability results based on the linearization of the dynamics of the problem and an EKF solution. Other approaches include the one proposed in [21], in which successive measurements at different points of the trajectory are used to recover the present position, while [22] presents a centralized EKF solution backed by extensive experimental results. Finally, [23] studies the minimum number of distinct range measurements to compute both relative position and orientation in a 2D setting and proposes an algorithm to solve the problem, as well as a local observability result. To summarize, in most of the previous approaches to the problem, the observability analysis revolves around the concept of weak local observability introduced in [24], and the filtering solution is EKF-based. In comparison with the previous-referenced works, the novelty of the proposed cooperative localization solution resides in two main points: while other works such as [18] and [25] present similar conditions for observability, the framework that is employed in this paper allows for the derivation of observability conditions that hold globally, and the proposed state estimation solution features exponentially convergent error dynamics for any initial condition. As it is shown in the simulation results, the proposed solution achieves a similar performance to the EKF while offering these observability and stability guarantees, which are significant as it is well known that the EKF can quickly diverge because of poor choice of initial condition.

The rest of the paper is organized in the following way: Section 2 describes the problem addressed in this work in more detail and presents the dynamics of the agents in the formation, posed as nonlinear dynamic systems. Section 3 details the observability analysis of the aforementioned nonlinear

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systems, providing a method for designing local state observers for each agent in the formation, and Section 4 analyzes the stability of the Kalman filter when some of the variables are subject to exponentially decaying perturbations. In Section 5, the resulting decentralized observer is analyzed and conditions for its estimation error to converge globally exponentially to zero are derived, and in Section 6, the results of simulations that were carried out to assess the performance of the proposed decentralized estimation solution under measurement noise are presented. Finally, Section 7 summarizes the main conclusions of the paper.

1.1. Notation

Throughout the paper, the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. Whenever relevant, the dimensions of an $n \times n$ identity matrix are indicated as $\mathbf{I}_n$. A block diagonal matrix is represented as diag $(\mathbf{A}_1, \ldots, \mathbf{A}_\mathrm{n})$. The superscript x^i is used to identify the i-th component of a vector $\mathbf{x}$. For a time-dependent vector $\mathbf{x}(t)$, the notation $\mathbf{x}^{[1]}(t, t_0)$ and $\mathbf{x}^{[2]}(t, t_0)$ denotes the time integrals

$$\mathbf{x}^{[1]}(t, t_0) := \int_{t_0}^{t} \mathbf{x}(\sigma) d\sigma$$

and

$$\mathbf{x}^{[2]}(t, t_0) := \int_{t_0}^{t} \int_{t_0}^{\sigma_1} \mathbf{x}(\sigma_2) d\sigma_2 \sigma_1,$$

respectively. For a matrix $\mathbf{A}$, $\|\mathbf{A}\|$ denotes its induced 2-norm, and $\|\mathbf{A}\|_F$ denotes its Frobenius norm. The notation vec$(\mathbf{A})$ denotes the vectorizing operator, which returns a vector constructed by stacking the columns of the matrix $\mathbf{A}$.

2. PROBLEM STATEMENT

Consider a formation composed by N agents, evolving in 3D and under the influence of an unknown constant current. To discriminate between different agents, a distinct integer label between 1 and N is assigned to each one. It is assumed that each agent has access to either:

- An absolute position measurement, provided, for example, by GPS or a long baseline positioning system, see e.g. [15] and [26]; or
- Range measurements to one or more neighboring agents, as well as position estimates communicated by those agents.

Additionally, it is assumed that the sensor suite mounted on-board each agent has: a DVL, which provides the velocity of the agent relative to the fluid; a sensor, which provides accurate measurements of its depth/altitude (which is readily available in most situations: depth cells for underwater vehicles, pressure sensors for aerial vehicles); and an AHRS to measure its attitude.

The first of three problems considered here is the design of local state estimators for each agent to accurately estimate its position and velocity, relying only on locally available measurements and limited communication with neighboring agents. This process will result in a decentralized state observer, in which each agent only concerns itself with a fraction of the total information available in the formation to estimate its state. The second problem to address is the stability analysis of the local state estimators subject to errors in the received state estimates, which will evidently happen because they are provided by other local estimators. The third and final problem to address is that of using the previous results to establish conditions that guarantee the stability of the decentralized estimator.

2.1. Motion kinematics and measurements

Let $\{I\}$ denote an inertial reference coordinate frame and $\{B_i\}$ a coordinate frame attached to agent i, which is usually denoted as the body-fixed coordinated frame. The linear motion of agent i follows

$$\dot{\mathbf{p}}_i(t) = \mathbf{R}_i(t)\mathbf{v}_i(t),$$

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where $\mathbf{p}_i(t) \in \mathbb{R}^3$ is its inertial position, $\mathbf{v}_i(t) \in \mathbb{R}^3$ is its velocity relative to $\{I\}$, expressed in body-fixed coordinates, and $\mathbf{R}_i(t) \in SO(3)$ is the rotation matrix from $\{B_i\}$ to $\{I\}$. For the agents that have access to absolute position measurements, the design of a navigation system is well understood, and it is not elaborated here (for further details, see [15, 26–28] and references therein).

On the other hand, for the agents that only have access to range measurements, the conditions for observability, as well as the design of stable state observers, are considerably more complex and are henceforth the main focus of this section and the next. The DVL provides a measurement of the velocity relative to the fluid in body-fixed coordinates, so it makes sense to divide $\mathbf{v}_i(t)$ into $\mathbf{v}_{ri}(t) \in \mathbb{R}^3$ (velocity of the agent relative to the fluid, in body-fixed coordinates) and $\mathbf{v}_f(t) \in \mathbb{R}^3$ (velocity of the fluid relative to $\{I\}$, in inertial coordinates), yielding

$$\dot{\mathbf{p}}_i(t) = \mathbf{R}_i(t)\mathbf{v}_{ri}(t) + \mathbf{v}_f(t).$$

A range measurement from agent i to agent j will be denoted as

$$r_{ij}(t) = \|\mathbf{p}_j(t) - \mathbf{p}_i(t)\| \in \mathbb{R},$$

and the depth measurement of agent i is $z_i(t) = p_i^3(t) \in \mathbb{R}$.

2.2. Agent dynamics

To proceed from the point of view of some agent i, we define the following nonlinear system by joining the linear motion kinematics and the available measurements:

$$\begin{cases} \dot{\mathbf{p}}_i(t) = \mathbf{R}_i(t)\mathbf{v}_{ri}(t) + \mathbf{v}_f(t) \\ \dot{\mathbf{v}}_f(t) = 0 \\ z_i(t) = p_i^3(t) \\ r_{ij}(t) = \|\mathbf{p}_j(t) - \mathbf{p}_i(t)\|, \quad j \in D_i \end{cases},$$

where D_i is the set of other agents to whom agent i measures its range. Figure 2 details the quantities involved in the local state estimation problem for agent i. Define

$$\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} := \begin{bmatrix} \mathbf{p}_i(t) \\ \mathbf{v}_f(t) \end{bmatrix} \in \mathbb{R}^6,$$

and let $\mathbf{u}(t) := \mathbf{R}_i(t)\mathbf{v}_{ri}(t) \in \mathbb{R}^3$. Note that, although $\mathbf{u}(t)$ is equivalent to an inertial velocity measurement in the absence of noise, the fact that it is constructed from different measurements is reflected later in the simulations by adding sensor noise to both the Euler angles used to construct $\mathbf{R}(t)$ and the body-fixed velocity measurements. Suppose that agent i has access to measurements of its range to M other agents in its vicinity and, for the sake of simplicity, take indexes from 1 to M to

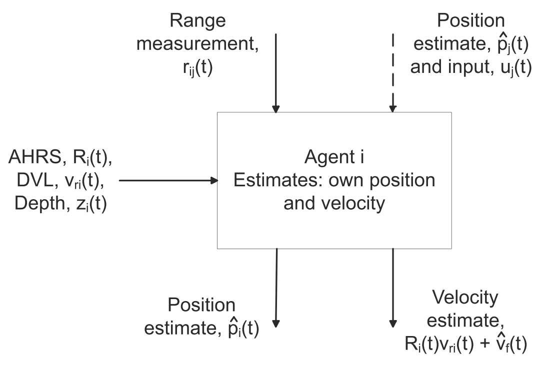

Figure 2. For agent i measuring its range to agent j, available data and quantities estimated.

identify each of those agents. Additionally, suppose that they transmit both $\mathbf{u}_j(t) := \mathbf{R}_j(t)\mathbf{v}_{rj}(t) \in \mathbb{R}^3$ and an estimate of $\mathbf{p}_j(t)$ to agent i. Then, the system takes the form

$$\begin{cases} \dot{\mathbf{x}}_1(t) = \mathbf{x}_2(t) + \mathbf{u}(t) \\ \dot{\mathbf{x}}_2(t) = \mathbf{0} \\ z(t) = x_1^3(t) \\ r_j(t) = \|\mathbf{p}_j(t) - \mathbf{x}_1(t)\|, \quad j = 1, 2, \dots, M \end{cases}. \tag{1}$$

The time derivative of $r_j(t)$, which will be useful in the next section, is given by

$$\dot{r}_j(t) = \frac{\left[\mathbf{p}_j(t) - \mathbf{x}_1(t)\right] \cdot \left[\mathbf{u}_j(t) - \mathbf{u}(t)\right]}{r_j(t)}. \tag{2}$$

Remark 1
While the assumption that agents receive the inputs of other agents in the vicinity might seem strong, it can realistically be achieved in many practical settings. For example, in underwater applications, one way to obtain reliable range measurements between vehicles is by using acoustic transceivers and computing the distance from the round trip time. This effectively creates a communication channel between the agents, and the input can be transmitted along with the signal used to compute the range.

The first problem addressed in the paper is the analysis of the observability of the nonlinear system (1). As the observability analysis is constructive, the estimator design follows naturally. Two cases are considered, depending on the number of range sources available to the agent. In the first, the agent has access to only one range source. In the other, it has access to $M \geqslant 3$ range sources. The consideration of two distinct cases is pertinent because of the different observability conditions that can be expected for each case: when the agent has three or more range sources, and given the availability of depth measurements, the position can be recovered through trilateration as long as the range sources are not positioned in a straight line. On the other hand, with only 1 range source, trilateration is impossible and the relative movement between the two agents must be rich enough to allow for some sort of trilateration over time. The intermediate case, two range sources, is not elaborated on here, but the observability condition can be expected to be a hybrid of the two other cases: while trilateration is still not possible, there will be less restrictions on the motion of the agents.

For the second case, $M \geqslant 3$, consider the set $\mathcal{P}_i$ of time-varying vectors in $\mathbb{R}^2$ composed by the projections on the horizontal plane (any plane orthogonal to the depth/altitude axis) of the positions of the agents corresponding to the range measurements available to agent i. The following is assumed:

Assumption 1
There exists a $t_{nc} \in \left[t_0, t_f\right]$ for which at least three of the vectors in $\mathcal{P}_i$ are non-collinear.

Remark 2
In (1), $\mathbf{x}(t)$ is used to denote the state of agent i instead of $\mathbf{x}_i(t)$ or some other form of indexation, as it simplifies the notation in the rest of the paper considerably. As the observability analysis is carried out from the point of view of agent i, $\mathbf{x}_1$ and $\mathbf{x}_2$ always refer to the state of the relevant agent, i, while $\mathbf{p}_j$ always denotes the position of another agent, associated with the range measurement r_j.

3. OBSERVABILITY ANALYSIS

This section details the observability analysis of the nonlinear system (1). The method applied here consists in obtaining, through state augmentation, a LTV system that mimics exactly the dynamics of (1), allowing to carry out the observability analysis in a linear system framework.

The section is organized as follows. First, new state variables are defined to obtain an LTV representation of the nonlinear dynamics of (1). For the case with three or more range sources, output augmentation is also performed: new outputs are defined by combining available measurements,

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which are used later in the section to derive the observability results. Then, the LTV systems are studied resorting to classical linear system theory, and sufficient conditions for observability are obtained for both cases. Following this, the augmented LTV systems are compared with the nonlinear system (1) to show that the observability conditions are in fact applicable to the original nonlinear system. Finally, sufficient conditions are derived for uniform complete observability of the augmented LTV systems. While these conditions are more restrictive than the ones obtained previously in the section, uniform complete observability guarantees globally exponential stability of the error dynamics of a Kalman filter for the corresponding LTV system. Previous work by the authors proposed navigation solutions for single-AUV scenarios using a similar methodology for the observability analysis process, see for example, [15] and [14]. The results here differ due to differences in modeling, and the observability conditions are required for the results of subsequent sections and for the implementation of the solution in the simulations and potential practical applications. The proofs for the results detailed in this section can be found in Appendix A.

3.1. State augmentation

Define an additional state variable, $\mathbf{x}_3 \in \mathbb{R}^M$, with each component equal to one of the range measurements, and let $\Delta\mathbf{u}_j(t) := \mathbf{u}_j(t) - \mathbf{u}(t) \in \mathbb{R}^3$. Then, attending to (2), the dynamics of the augmented system can be described by the following LTV system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_L(t)\mathbf{x}(t) + \mathbf{B}_L(t)\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{p}_1(t)\\ \vdots \\ \mathbf{p}_M(t)\end{bmatrix}, \\ y(t) = \mathbf{C}_A\mathbf{x}(t) \end{cases} \tag{3}$$

where

$$\mathbf{A}_L(t) := \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ -\frac{\Delta\mathbf{u}_1^{\mathrm{T}}(t)}{r_1(t)} & \mathbf{0} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\Delta\mathbf{u}_M^{\mathrm{T}}(t)}{r_M(t)} & \mathbf{0} & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{(6+M)\times(6+M)},$$

$$\mathbf{B}_L(t) := \begin{bmatrix} \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \frac{\Delta\mathbf{u}_1^T(t)}{r_1(t)} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \frac{\Delta\mathbf{u}_M^T(t)}{r_M(t)} \end{bmatrix} \in \mathbb{R}^{(6+M)\times 3(1+M)},$$

$$\text{and } \mathbf{C}_A := \begin{bmatrix} \begin{bmatrix}0 & 0 & 1\end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \in \mathbb{R}^{(1+M)\times(6+M)}.$$

The transition matrix associated with $\mathbf{A}_L(t)$, which is instrumental to the observability analysis of (3), is given by

$$\mathbf{\Phi}_L(t,t_0) = \begin{bmatrix} \mathbf{I} & (t-t_0)\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ -\int_{t_0}^{t}\frac{\Delta\mathbf{u}_1^T(\sigma)}{r_1(\sigma)}\,d\sigma & -\int_{t_0}^{t}\frac{(\sigma-t_0)\Delta\mathbf{u}_1^T(\sigma)}{r_1(\sigma)}\,d\sigma & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\int_{t_0}^{t}\frac{\Delta\mathbf{u}_M^T(\sigma)}{r_M(\sigma)}\,d\sigma & -\int_{t_0}^{t}\frac{(\sigma-t_0)\Delta\mathbf{u}_M^T(\sigma)}{r_M(\sigma)}\,d\sigma & 0 & \dots & 1 \end{bmatrix}.$$

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3.2. Output augmentation

Regarding specifically the second case, that is, when there are $M \geqslant 3$ range sources, note that

$$r_j(t) - r_k(t) + \frac{2\left[\mathbf{p}_j(t) - \mathbf{p}_k(t)\right] \cdot \mathbf{x}_1(t)}{r_j(t) + r_k(t)} = \frac{\left\|\mathbf{p}_j(t)\right\|^2 - \|\mathbf{p}_k(t)\|^2}{r_j(t) + r_k(t)},$$

for all $j, k = 1, 2, \ldots, M,\ j \neq k$. The right side is assumed to be known, so the left side can be included as an additional output. In fact, including all distinct combinations provide $(M-1)M/2$ new outputs to the system, leading to the following LTV system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_L(t)\mathbf{x}(t) + \mathbf{B}_L(t) \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{p}_1(t) \\ \vdots \\ \mathbf{p}_M(t) \end{bmatrix} \\ y(t) = \mathbf{C}_B(t)\mathbf{x}(t) \end{cases}, \tag{4}$$

where

$$\mathbf{C}_B(t) := \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{C}_1(t) & \mathbf{0} & \mathbf{C}_2 \end{bmatrix} \in \mathbb{R}^{[1+(M+1)M/2] \times (6+M)},$$

in which

$$\mathbf{C}_1(t) := \begin{bmatrix} \frac{2(\mathbf{p}_1(t) - \mathbf{p}_2(t))^T}{r_1(t)+r_2(t)} \\ \frac{2(\mathbf{p}_1(t) - \mathbf{p}_3(t))^T}{r_1(t)+r_3(t)} \\ \vdots \\ \frac{2(\mathbf{p}_{M-1}(t) - \mathbf{p}_M(t))^T}{y_{M-1}(t)+y_M(t)} \end{bmatrix} \in \mathbb{R}^{[(M-1)M/2] \times 3}$$

and

$$\mathbf{C}_2 := \begin{bmatrix} 1 & -1 & 0 & 0 \ldots \ldots & 0 \\ 1 & 0 & -1 & 0 \ldots \ldots & 0 \\ & & \vdots & & \\ 0 & \ldots \ldots & 0 & 0 \quad 1 & -1 \end{bmatrix} \in \mathbb{R}^{[(M-1)M/2] \times M}.$$

Remark 3
Note that, even though $\mathbf{A}_L(t)$, $\mathbf{B}_L(t)$, and $\mathbf{C}_B(t)$ depend on the system's input and output, systems (3) and (4) can still be regarded as LTV systems for analysis purposes, as all involved quantities are known [14, Lemma 1].

3.3. Observability of the linear time-varying system

The following result provides a sufficient condition for the observability of the LTV system (3) when there is only $M = 1$ range source.

Lemma 1
Suppose that the functions of the set

$$\mathcal{F} := \left\{\Delta u_1^1(t), \Delta u_1^2(t), (t-t_0)\Delta u_1^1(t), (t-t_0)\Delta u_1^2(t)\right\} \tag{5}$$

are linearly independent in $\left[t_0, t_f\right],\ t_0 < t_f$. Then, the LTV system (3) with one range source is observable on $\left[t_0, t_f\right]$.

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The following result provides a sufficient condition for the observability of the LTV system (4) when there are $M \geqslant 3$ range sources.

Lemma 2
Suppose that Assumption 1 holds. Then, the LTV system (4) with $M \geqslant 3$ range sources is observable on $\left[t_0, t_f\right],\ t_0 < t_f$.

Remark 4
While the physical meaning of the sufficient condition for observability in Lemma 2 is simple to infer (with $M \geqslant 3$ range sources non-collinearity is required for trilateration), the sufficient condition for observability detailed in Lemma 1 is, at first glance, more obscure. However, noting that $\Delta\mathbf{u}_1(t)$ is the relative velocity between the two vehicles, linearly independence of the functions of (5) guarantees enough relative motion between the vehicles to perform some sort of trilateration over time and isolate the effect of the constant current. In practical terms, to guarantee observability, an agent with only one range source should avoid being immobile or moving in a straight line relative to its source and instead vary its relative position in the horizontal plane. Note that this requirement of relative motion to guarantee observability is similar to the local observability conditions in other works on range-based localization such as [18–20] and [25], although the results presented herein are not local.

3.4. *Observability of the nonlinear system*

Comparing the nonlinear system (1) with the augmented LTV systems (3) and (4), one might be tempted to infer that observability of the corresponding LTV system entails observability of the nonlinear system, because the dynamics of the latter are essentially a subsystem of the dynamics of the former. However, this reasoning is incorrect as this equivalence is only verified if the algebraic restrictions

$$x_3^j(t) = \|\mathbf{p}_j(t) - \mathbf{x}_1(t)\|,\ j = 1, 2, \ldots, M$$

hold for all $t \in \left[t_0, t_f\right]$. Thus, equivalence between the nonlinear system and its augmented counterpart remains to be proved.

The following result presents a sufficient condition for the observability of the nonlinear system (1) when there is only $M = 1$ range source available, as well as a method for designing state observers for that system whose error converges exponentially fast to zero for all initial conditions.

Theorem 1
If the set of functions (5) is linearly independent, then the nonlinear system (1) with one range source is observable on $\left[t_0, t_f\right],\ t_0 < t_f$, in the sense that, given the outputs $z(t)$ and $r_1(t)$ and the input $\mathbf{u}(t)$, the initial state $\mathbf{x}(t_0)$ is uniquely defined. Moreover, a state observer for the LTV system (3) with GES error dynamics is also a state observer for the nonlinear system (1), whose error converges exponentially fast to zero for all initial conditions.

The following result presents a sufficient condition for the observability of the nonlinear system (1) when there are $M \geqslant 3$ range sources.

Theorem 2
If Assumption 1 holds, the nonlinear system (1) with $M \geqslant 3$ range sources is observable on $\left[t_0, t_f\right],\ t_0 < t_f$, in the sense that, given the outputs $z(t)$ and $r_j(t),\ j = 1, 2, \ldots, M$ and the input $\mathbf{u}(t)$, the initial state $\mathbf{x}(t_0)$ is uniquely defined. Moreover, a state observer for the LTV system (4) with GES error dynamics is also a state observer for the nonlinear system (1), whose error converges exponentially fast to zero for all initial conditions.

Remark 5
For the case where the agent has access to $M = 2$ range sources, it is easy to see that if the sufficient condition for observability of Theorem 1 holds for at least one of the two agents corresponding to

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the range measurements, then the nonlinear system (1) with two range sources is observable. Future work will focus on deriving a less restrictive sufficient condition for observability for that case.

3.5. Uniform complete observability

Following the previous results, it is straightforward to design Kalman filters for the LTV systems (3) or (4), depending on the number of available range sources, to estimate the state of the nonlinear system (1). However, to guarantee that the Kalman filter has GES error dynamics, uniform complete observability (UCO) is required. The focus of this subsection is the derivation of sufficient conditions for UCO of the augmented LTV systems (3) and (4). To do so, an additional assumption is needed.

Assumption 2
The quantities $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{u}(t)$ associated with each agent in the formation are bounded for all $t \in [t_0, t_f]$. Furthermore, there exist positive scalar constants y_m and y_M such that the range measurements obey $y_m \leqslant r_i(t) \leqslant y_M$ for all $t \in [t_0, t_f]$.

Note that, in practical terms, this assumption is very mild, as the state $\mathbf{x}(t)$ will be bounded by the specifications of the mission scenario, and $\mathbf{u}(t)$ is limited by the maximum velocity of the agents. Furthermore, the particular values of the bounds are not required for observer nor filter design purposes.

The following result addresses the issue of UCO of the LTV system (3), when there is $M = 1$ range source.

Theorem 3
Suppose that Assumption 2 holds. If there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geqslant t_0$ and $\boldsymbol{c} \in \mathbb{R}^4$, $\|\boldsymbol{c}\| = 1$, it is possible to choose $t^* \in [t, t+\delta]$ such that

$$\left|\Delta u_1^1(t^*)c_1^1 + \Delta u_1^2\left(t^*\right)c_1^2 + \left(t^* - t\right)\Delta u_1^1\left(t^*\right)c_2^1 + (t^* - t)\Delta u_1^2\left(t^*\right)c_2^2\right| \geqslant \alpha,$$

where $\boldsymbol{c} = \left[c_1^1 \;\, c_1^2 \;\, c_2^1 \;\, c_2^2\right]^T$, then the LTV system (3) with only one range source is UCO.

The following result establishes a sufficient condition for UCO of the LTV system (4), when there are $M \geqslant 3$ range sources.

Theorem 4
Suppose that Assumption 2 holds. If there exist positive scalar constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geqslant t_0$ and $\boldsymbol{c} \in \mathbb{R}^2$, $\|\boldsymbol{c}\| = 1$, there exist $i, j \in \{1, 2, \ldots, M\}$ and a $t^* \in [t, t+\delta]$ such that

$$\left|(p_i^1(t^*) - p_j^1(t^*))c^1 + (p_i^2(t^*) - p_j^2(t^*))c^2\right| \geqslant \alpha,$$

where $\boldsymbol{c} = \left[c^1 \;\, c^2\right]^T$, then the LTV system (4) with $M \geqslant 3$ range sources is UCO.

4. STABILITY OF THE PERTURBED KALMAN FILTER

The previous section established sufficient conditions for the observability of (1) depending on the number of range sources available to the agent. Following this, it is straightforward to design local Kalman filters for each agent that will feature GES error dynamics when the received state estimates are exact [29].

However, during operation in a formation setting, the received position estimates will not be exact as they will be produced by the local Kalman filter of the corresponding agent, thus injecting perturbations in the input of the augmented LTV system as well as in the output matrix $\mathbf{C}_B(t)$. To address this problem, this section details the stability analysis of the continuous-time Kalman filter when there are exponentially decaying perturbations in both the input and the output matrix of the system.

This section is organized as follows. First, the equations for the nominal Kalman filter and its perturbed version are laid out and assumptions on the boundedness of some of the quantities involved are detailed. Then, it is shown that uniform complete observability of the nominal system entails uniform complete observability of the system subject to the aforementioned perturbations. Following this, the deviation of the solution of the perturbed Riccati equation from the nominal solutions is studied, and it is shown that it converges exponentially fast to zero, which entails that the Kalman gain will also converge to its nominal value. Finally, the main result of the section is laid out: if the nominal LTV system is UCO, then the Kalman filter subject to exponentially decaying perturbations is a suitable state observer for the nominal LTV system, featuring exponentially convergent error dynamics.

Consider a generic LTV system

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \mathbf{A}(t)\boldsymbol{x}(t) + \mathbf{B}(t)\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \mathbf{C}(t)\boldsymbol{x}(t) \end{cases}, \tag{6}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state of the system, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is its input, and $\boldsymbol{y}(t) \in \mathbb{R}^o$ its output. $\mathbf{A}(t)$, $\mathbf{B}(t)$, and $\mathbf{C}(t)$ are matrix-valued functions of time of appropriate dimensions. It is assumed that there exist positive scalar constants α_1, α_2, and α_3 such that

$$\begin{cases} \|\mathbf{A}(t)\| \leqslant \alpha_1 \\ \|\mathbf{B}(t)\| \leqslant \alpha_2 \\ \|\mathbf{C}(t)\| \leqslant \alpha_3 \end{cases} \tag{7}$$

for all $t \geqslant t_0$. The dynamics of the Kalman filter for the LTV system (6) follow

$$\begin{cases} \dot{\hat{\boldsymbol{x}}}(t) = \mathbf{A}(t)\hat{\boldsymbol{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{K}(t)[\boldsymbol{y}(t) - \mathbf{C}(t)\hat{\boldsymbol{x}}(t)] \\ \mathbf{K}(t) = \mathbf{P}(t)\mathbf{C}^T(t)\mathbf{R}^{-1}(t) \\ \dot{\mathbf{P}}(t) = \mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T(t) + \mathbf{Q}(t) - \mathbf{P}(t)\mathbf{C}^T(t)\mathbf{R}^{-1}(t)\mathbf{C}(t)\mathbf{P}(t) \end{cases}, \tag{8}$$

in which the matrices $\mathbf{Q}(t) \succ \mathbf{0} \in \mathbb{R}^{n\times n}$ and $\mathbf{R}(t) \succ \mathbf{0} \in \mathbb{R}^{o\times o}$ are used to model process and observation noise, respectively. Suppose that there exist positive scalar constants α_4 and α_5 such that

$$\begin{cases} \alpha_4^{-1} \leqslant \|\mathbf{R}(t)\| \leqslant \alpha_4 \\ \|\mathbf{Q}(t)\| \leqslant \alpha_5 \end{cases} \tag{9}$$

for all $t \geqslant t_0$. Now, suppose that instead of using $\boldsymbol{u}(t)$ and $\mathbf{C}(t)$, the Kalman filter uses estimates of those quantities, $\hat{\boldsymbol{u}}(t)$ and $\hat{\mathbf{C}}(t)$, respectively, and assume that the errors on those estimates, $\tilde{\boldsymbol{u}}(t) := \boldsymbol{u}(t) - \hat{\boldsymbol{u}}(t)$ and $\tilde{\mathbf{C}}(t) := \mathbf{C}(t) - \hat{\mathbf{C}}(t)$, decay exponentially with time. More specifically, suppose that there exist positive scalar constants α_6, α_7, λ_1, and λ_2 such that

$$\begin{cases} \|\tilde{\boldsymbol{u}}(t)\| \leqslant \alpha_6 e^{-\lambda_1(t-t_0)} \\ \|\tilde{\mathbf{C}}(t)\| \leqslant \alpha_7 e^{-\lambda_2(t-t_0)} \end{cases} \tag{10}$$

for all $t \geqslant t_0$. Then, the Kalman filter equations become

$$\begin{cases} \dot{\hat{\boldsymbol{x}}}(t) = \mathbf{A}(t)\hat{\boldsymbol{x}}(t) + \mathbf{B}(t)\hat{\boldsymbol{u}}(t) + \hat{\mathbf{K}}(t)[\boldsymbol{y}(t) - \hat{\mathbf{C}}(t)\hat{\boldsymbol{x}}(t)] \\ \hat{\mathbf{K}}(t) = \hat{\mathbf{P}}(t)\hat{\mathbf{C}}^T(t)\mathbf{R}^{-1}(t) \\ \dot{\hat{\mathbf{P}}}(t) = \mathbf{A}(t)\hat{\mathbf{P}}(t) + \hat{\mathbf{P}}(t)\mathbf{A}^T(t) + \mathbf{Q}(t) - \hat{\mathbf{P}}(t)\hat{\mathbf{C}}^T(t)\mathbf{R}^{-1}(t)\hat{\mathbf{C}}(t)\hat{\mathbf{P}}(t) \end{cases}. \tag{11}$$

The new filter (11) will be referred to as perturbed Kalman filter from hereon for the sake of convenience.

Consider the implementation of a Kalman filter for the LTV system (4) in which both $\mathbf{u}(t)$ and $\mathbf{C}_B(t)$ are computed using the estimates for $\mathbf{p}_i(t)$, $i = 1, 2, \ldots, M$ received through communication. Then, (11) can be used to model the filter dynamics, and the assumption in (10) is verified only if the errors in the estimates received through communication decay exponentially with time, which can be guaranteed when the formation follows a hierarchical structure such as the acyclic directed graph considered in Section 5.

Remark 6
An alternative to the observability and stability analysis detailed in this work would be the use of a fully stochastic model to take the measurement noise into account in every step of the process. However, the observability and stability analysis carried out in the deterministic framework ensure that the error dynamics of the decentralized state estimator are inherently stable, while the implementation of local Kalman filters with correctly tuned noise covariance matrices **Q** and **R** for each agent ensures good filtering performance of the local sensor noise. Thus, a fully stochastic framework falls outside the scope of the present work, and it would also further complicate the already extensive calculations present in the paper.

4.1. Observability of the perturbed system

It must be shown that UCO of the nominal system (6) entails UCO of the perturbed system

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \mathbf{A}(t)\boldsymbol{x}(t) + \mathbf{B}(t)\hat{\boldsymbol{u}}(t) \\ \boldsymbol{y}(t) = \hat{\mathbf{C}}(t)\boldsymbol{x}(t) \end{cases}, \tag{12}$$

to ensure the existence of bounds on $\hat{\mathbf{P}}(t)$ and the stability of the perturbed Kalman filter's error dynamics. The following result establishes UCO of the perturbed system (12).

Lemma 3
Suppose that the LTV system (6) is UCO. Then, the perturbed system (12) is UCO.

This result will be useful to study the convergence of the perturbed Riccati equation in (11), as it allows to establish bounds for $\hat{\mathbf{P}}(t)$. To be more specific, if the LTV system (6) is UCO, then there exist positive scalar constants γ, α_8, and α_9 such that

$$\begin{cases} \alpha_8^{-1} \leqslant \|\mathbf{P}(t)\| \leqslant \alpha_8 \\ \alpha_9^{-1} \leqslant \|\hat{\mathbf{P}}(t)\| \leqslant \alpha_9 \end{cases}$$

for all $t \geqslant t_0 + \gamma$ [29].

4.2. Stability of the perturbed Riccati equation

The main problem that appears when studying the stability of the perturbed Kalman filter (11) is that the Riccati equation itself will deviate from its nominal counterpart in (8) because of the perturbation in $\hat{\mathbf{C}}(t)$. Following this, define the deviation of $\hat{\mathbf{P}}(t)$ from $\mathbf{P}(t)$ as

$$\tilde{\mathbf{P}}(t) := \mathbf{P}(t) - \hat{\mathbf{P}}(t). \tag{13}$$

For the rest of this subsection, explicit time dependency of the various quantities involved in the calculations is dropped to avoid unnecessary cluttering of the notation. Computing the time derivative of (13) yields

$$\begin{aligned} \dot{\tilde{\mathbf{P}}} &= \dot{\mathbf{P}} - \dot{\hat{\mathbf{P}}} \\ &= \mathbf{AP} + \mathbf{PA}^T + \mathbf{Q} - \mathbf{PC}^T\mathbf{R}^{-1}\mathbf{CP} - \mathbf{A}\hat{\mathbf{P}} - \hat{\mathbf{P}}\mathbf{A}^T - \mathbf{Q} + \hat{\mathbf{P}}\hat{\mathbf{C}}^T\mathbf{R}^{-1}\hat{\mathbf{C}}\hat{\mathbf{P}} \\ &= \left(\mathbf{A} - \mathbf{PC}^T\mathbf{R}^{-1}\mathbf{C}\right)\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\left(\mathbf{A} - \mathbf{PC}^T\mathbf{R}^{-1}\mathbf{C}\right)^T + \tilde{\mathbf{P}}\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\tilde{\mathbf{P}} \\ &\quad + \hat{\mathbf{P}}\left[\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\tilde{\mathbf{C}} - \tilde{\mathbf{C}}^T\mathbf{R}^{-1}\mathbf{C} - \mathbf{C}^T\mathbf{R}^{-1}\tilde{\mathbf{C}}\right]\hat{\mathbf{P}}. \end{aligned} \tag{14}$$

To simplify the notation, define

$$\begin{cases} \mathbf{F}\left(\tilde{\mathbf{P}}\right) = \left(\mathbf{A} - \mathbf{PC}^T\mathbf{R}^{-1}\mathbf{C}\right)\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\left(\mathbf{A} - \mathbf{PC}^T\mathbf{R}^{-1}\mathbf{C}\right)^T + \tilde{\mathbf{P}}\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\tilde{\mathbf{P}} \\ \mathbf{G}\left(\tilde{\mathbf{C}}\right) = \hat{\mathbf{P}}\left[\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\tilde{\mathbf{C}} - \tilde{\mathbf{C}}^T\mathbf{R}^{-1}\mathbf{C} - \mathbf{C}^T\mathbf{R}^{-1}\tilde{\mathbf{C}}\right]\hat{\mathbf{P}} \end{cases}.$$

Thus, (14) can be rewritten as

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$$\dot{\tilde{\mathbf{P}}} = \mathbf{F}\left(\tilde{\mathbf{P}}\right) + \mathbf{G}\left(\tilde{\mathbf{C}}\right). \tag{15}$$

The following result establishes a sufficient condition for the exponential convergence of the dynamics of (15).

Lemma 4
Suppose that the LTV system (6) is UCO and verifies the bounds in (7), (9), and (10). Then, $\tilde{\mathbf{P}}(t)$ converges exponentially fast to the origin, in the sense that, for any given initial condition $\tilde{\mathbf{P}}(t_0)$, it is possible to choose positive scalar constants α and λ such that $\|\tilde{\mathbf{P}}(t)\| \leqslant \alpha e^{-\lambda(t-t_0)}$ for all $t \geqslant t_0$.

4.3. *Stability of the perturbed Kalman filter*

Following the previous results, it is now possible to show that the perturbed Kalman filter (11) needs a suitable state observer for the nominal LTV system (6).

Theorem 5
Suppose that:

(i) The LTV system (6) is UCO;
(ii) The bounds in (7), (9), and (10) are verified;
(iii) The state $\mathbf{x}(t)$ of the LTV system (6) is bounded for all $t \geqslant t_0$.

Then, the perturbed Kalman filter (11) is a state observer for the LTV system (6) with exponentially convergent error dynamics in the sense that, for any given initial condition $\tilde{\boldsymbol{x}}(t_0) := \boldsymbol{x}(t_0) - \hat{\boldsymbol{x}}(t_0)$, it is possible to choose positive scalar constants α and λ such that the estimation error $\tilde{\boldsymbol{x}}(t) := \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ follows $\|\tilde{\boldsymbol{x}}(t)\| \leqslant \alpha e^{-\lambda(t-t_0)}$ for all $t \geqslant t_0$.

5. STABILITY OF THE DECENTRALIZED STATE ESTIMATOR

The previous sections addressed the issues of observability of the nonlinear system (1), as well as the stability of the LTV Kalman filter subject to perturbations in the output matrix and the input of the system. These results establish the necessary foundation to address the main problem of this article: the stability of the error dynamics of the decentralized state observer composed by the interconnection of the local Kalman filters of each vehicle in the formation.

Before presenting the main result of this section, it is convenient to introduce some concepts on graph theory (see e.g., [30] and [31]), as formations of agents can be handily described by a directed graph. A directed graph, or digraph, $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ is composed by a set $\mathcal{V}$ of vertices and a set of directed edges $\mathcal{E}$, which are represented by ordered pairs of vertices. Such an edge can be expressed as $e = (a, b)$, meaning that edge e is incident on vertices a and b, directed towards b. A directed cycle in $\mathcal{G}$ is a sequence $(v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_0)$ of distinct vertices (with the exception of the first and the last) and edges of $\mathcal{G}$ such that $e_i = (v_{i-1}, v_i)$. A directed graph is called acyclic if it contains no directed cycles. If a directed graph $\mathcal{G}$ is acyclic, it can be represented graphically by a tiered drawing such as the one depicted in Figure 3, that is, the drawing is divided in K hierarchical tiers, in which tier 0 is composed of the vertices with no edges directed towards them while, for a vertex in tier $k > 0$, all directed paths ending in that vertex start in a node of a lower tier.

Now, consider the agent formation described in Section 2. This kind of formation can be associated with a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each vertex represents a distinct agent, and an edge (a, b) means that agent b measures its range to agent a. The nodes with no edges directed towards them refer to agents with access to measurements of their own absolute position.

The following result establishes a sufficient condition for exponential convergence to zero of the estimation error for the distributed state observer.

Theorem 6
Consider a formation composed by N agents, as described in Section 2, and suppose that:

1. The directed graph associated with the formation is acyclic;

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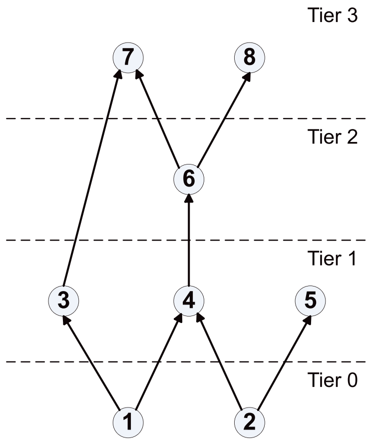

Figure 3. Example of an acyclic directed graph divided in tiers.

2. The agents in tier 0 implement GES state observers to estimate their position and velocity from the absolute position measurements available to them (see [27, 28]);
3. The agents in the other tiers implement Kalman filters for the LTV system (3) or (4), depending on the number M of range sources available to them;
4. The sufficient condition for UCO of the LTV system associated with each agent in the formation holds (Theorem 3 for the system (3) and Theorem 4 for the system (4)).

Then, the estimation error of the decentralized state observer composed by the interconnected local Kalman filters of all the agents in the formation converges exponentially fast to the origin in the sense that for any given initial condition, it is possible to choose positive scalar constants α and λ for which $\|\tilde{\mathbf{x}}_T(t)\| \leqslant \alpha e^{-\lambda(t-t_0)}$ for all $t \geqslant t_0$, where $\tilde{\mathbf{x}}_T(t) := \left[\tilde{\mathbf{x}}_1(t)\ \tilde{\mathbf{x}}_2(t)\ \ldots\ \tilde{\mathbf{x}}_N(t)\right]$ is the total estimation error in the formation.

Proof

Because the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ associated with the formation is acyclic, consider its drawing with K tiers, and notice that because the local state observer of an agent in a given tier only depends on estimates received from lower tiers, its properties can be studied disregarding the dynamics of agents in higher tiers as well as those of other agents in the same tier.

Consider an agent i in tier $k > 0$ and assume that the local observers of each agent in tiers 0 to $k-1$ have exponentially convergent error dynamics. From the necessary conditions for UCO, as described by either Theorem 3 or 4 depending on the number of available range sources, in particular, Assumption 2, it follows that the bounds in (7) hold for all $t \geqslant t_0$. Moreover, because the local observers in the previous tiers are assumed to feature exponentially convergent error dynamics, the errors in the position estimates received through communication decay exponentially fast, which implies that the bounds in (10) also hold for all $t \geqslant t_0$. Then, as long as the matrices $\mathbf{Q}$ and $\mathbf{R}$ used to tune the Kalman filter obey the bounds in (9), it follows by application of Theorem 5 that the estimation error of the Kalman filter of agent i also converges exponentially fast to the origin.

So, as the dynamics of agents in the same tier have no mutual impact, this means that because the local observers of each agent in the tiers 0 through $k-1$ have exponentially convergent error dynamics, the local observers of the agents in tier k will also feature error dynamics that converge exponentially to zero. Considering this, plus the fact that the local observers of the agents in tier 0 feature GES error dynamics, it follows by induction that all local state observers in the formation feature error dynamics that converge exponentially fast to the origin, that is, the decentralized estimator that results when considering all the local estimators as a whole has error dynamics that converge exponentially fast to the origin. □

Remark 7
The analysis in this section is restricted to acyclic formation graphs, as the introduction of algebraic loops in the global dynamics due to cycles in the graph cannot be solved using this methodology. For a solution with stability and performance guarantees even with cyclic formation graphs, using relative position measurements instead of range readings, see [32]. Nevertheless, this method can also be used to design a stable state observer when the formation graph $\mathcal{G}$ is cyclic, by removing edges from the graph until it is no longer cyclic while making sure to never remove the last edge directed towards a vertex. It seems naturally advantageous to remove as few edges as possible; therefore, this procedure could be restated as that of finding the maximum acyclic subgraph of $\mathcal{G}$ [33], with the added restriction that the last edge directed towards a vertex may not be removed (as this would obviously leave the corresponding agent without any positioning information apart from the depth measurement). This straightforward approach is then applied to the local observers by disregarding the range measurements referring to edges that were removed during this process.

6. SIMULATION RESULTS

This section presents simulation results to assess the performance of the proposed decentralized estimation solution in the presence of measurement noise. The filtering solution detailed in this paper is tested in a simulated mission scenario to assess its performance. To provide meaningful comparison terms, two alternative solutions are also simulated: a decentralized EKF implementation and a centralized EKF, both detailed further ahead in this section.

6.1. Simulation parameters

In the simulations detailed here, a formation composed by eight agents is considered. The graph associated with this formation is depicted in Figure 4. The agents are assumed to be evolving in a fluid with constant inertial velocity $\mathbf{v}_f = \begin{bmatrix}0.4 & -0.2 & 0\end{bmatrix}^T$ (m/s), and their initial positions are depicted in Figure 5.

Regarding the motion of the agents during the simulations, agent 1 follows the path depicted in Figure 5 with a constant surge velocity of 1 m/s, and agents 2 to 7 follow the same trajectory offset by their respective initial positions. Agent 8 requires a richer trajectory to guarantee observability as it has access to only one range source. To do so, it was made to perform oscillations around the nominal trajectory followed by the rest of the formations by actuating on its yaw velocity. To be more specific, denote the yaw velocity of agent 1 by $\psi_1(t)$. Then, the yaw velocity of agent 8 follows

$$\psi_8(t) = \psi_1(t) + 0.15\cos(2\pi \times 0.05 \times t)$$

throughout the course of the mission.

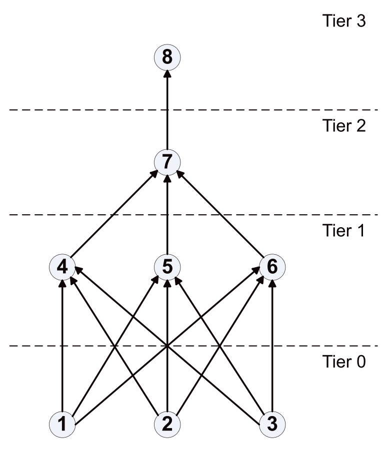

Figure 4. Directed graph associated with the formation used in the simulations.

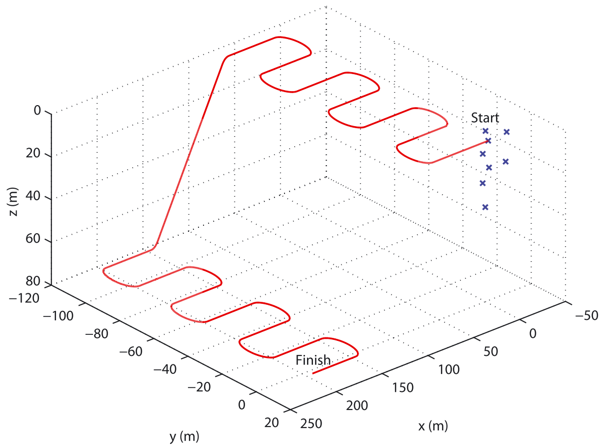

Figure 5. Initial positions of the agents and trajectory followed by the formation.

To simulate noise in the measurements, zero-mean, uncorrelated, and normally distributed perturbations were added to the range, depth, and velocity measurements, with standard deviation of 0.2 m for the range and depth measurements and 0.01 m/s for the velocity measurements. For the absolute position measurements available to agents 1 to 3, as the noise in absolute positioning systems such as the GPS is usually area-dependent, they were corrupted by additive, zero-mean white Gaussian noise with standard deviation of 0.1 m, and some correlation between the measurements was added, resulting in the following covariance matrix:

$$\mathbf{R}_{abs} = 0.01 \times \begin{bmatrix} 1 & 0.1 & 0.1 \\ 0.1 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{bmatrix} \otimes \mathbf{I}_3.$$

In addition to these perturbations, noise was also simulated in the attitude measurements required for computing the inputs of each system. To this effect, zero-mean, uncorrelated white Gaussian perturbations were added to the roll, pitch, and yaw Euler angles used to parametrize attitude, with standard deviation of 0.03° for the roll and pitch and 0.3° for the yaw.

In all simulations that were carried out, the initial value for all state estimates was set to zero regardless of the real initial value of the variables.

6.2. *Filter implementation*

For agents 1 to 3, which have access to measurements of their own inertial position, the dynamics are modeled by the linear time-invariant (LTI) system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_{\mathrm{LTI}}\mathbf{x}(t) + \mathbf{B}_{\mathrm{LTI}}\mathbf{u}(t) \\ \boldsymbol{y}(t) = \mathbf{C}_{\mathrm{LTI}}\mathbf{x}(t) \end{cases}, \tag{16}$$

in which

$$\mathbf{A}_{\mathrm{LTI}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \mathbf{B}_{\mathrm{LTI}} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}, \ \mathbf{C}_{\mathrm{LTI}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix},$$

and $\mathbf{u}(t)$ is defined as in (1). The local estimator for those agents is an LTI Kalman filter, whose dynamics follow

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_{\mathrm{LTI}}\hat{\mathbf{x}}(t) + \mathbf{B}_{\mathrm{LTI}}\mathbf{u}(t) + \mathbf{K}_{\mathrm{LTI}}(\boldsymbol{y}(t) - \mathbf{C}_{\mathrm{LTI}}\hat{\mathbf{x}}(t)),$$

in which

$$\mathbf{K}_{\mathrm{LTI}} = \mathbf{P}_{\mathrm{LTI}}\mathbf{C}_{\mathrm{LTI}}^T\mathbf{R}_{\mathrm{LTI}}^{-1},$$

where $\mathbf{P}_{\text{LTI}}$ is the positive definite solution of the algebraic Riccati equation

$$\mathbf{0} = \mathbf{A}_{\text{LTI}}\mathbf{P}_{\text{LTI}} + \mathbf{P}_{\text{LTI}}\mathbf{A}_{\text{LTI}}^T + \mathbf{Q}_{\text{ITI}} - \mathbf{P}_{\text{LTI}}\mathbf{C}_{\text{LTI}}^T\mathbf{R}_{\text{LTI}}^{-1}\mathbf{C}_{\text{LTI}}\mathbf{P}_{\text{LTI}},$$

and

$$\begin{cases} \mathbf{Q}_{\text{LTI}} = 0.01 \times \text{diag}(1, 1, 1, 0.001, 0.001, 0.001) \\ \mathbf{R}_{\text{LTI}} = \mathbf{I} \end{cases}.$$

For these three agents, no information is required from the other agents.

For the other agents in the formation, which have access to range measurements, LTV Kalman filters were implemented based on their respective augmented LTV systems. For agents 4 to 7, which have access to three range sources, the local Kalman filters are based on (4) and follow

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_L(t)\hat{\mathbf{x}}(t) + \mathbf{B}_L(t)\begin{bmatrix} \mathbf{u}(t) \\ \hat{\mathbf{p}}_1(t) \\ \hat{\mathbf{p}}_2(t) \\ \hat{\mathbf{p}}_3(t) \end{bmatrix} + \hat{\mathbf{K}}(t)[\boldsymbol{y}(t) - \hat{\mathbf{C}}_B(t)\hat{\mathbf{x}}(t)] \\ \hat{\mathbf{K}}(t) = \hat{\mathbf{P}}(t)\hat{\mathbf{C}}_B^T(t)\mathbf{R}_3^{-1} \\ \dot{\hat{\mathbf{P}}}(t) = \mathbf{A}_L(t)\hat{\mathbf{P}}(t) + \hat{\mathbf{P}}(t)\boldsymbol{A}_L^T(t) + \mathbf{Q}_3 - \hat{\mathbf{P}}(t)\hat{\mathbf{C}}_B^T(t)\mathbf{R}_3^{-1}\hat{\mathbf{C}}_B(t)\hat{\mathbf{P}}(t) \end{cases},$$

in which

$$\begin{cases} \mathbf{R}_3 = 0.01 \times \text{diag}(1, 1, 1, 0.001, 0.001, 0.001, 1, 1, 1) \\ \mathbf{Q}_3 = \text{diag}(1, 1, 1, 1, 2, 2, 2) \end{cases}.$$

In this case, the agent needs to receive information through communication: updated position estimates and inputs of the range sources (agents 1 to 3 for agents 4, 5, and 6, and agents 4 to 6 for agent 7).

Finally, for agent 8, because it only has access to one range source, the local Kalman filter is based on (3), and its dynamics follow

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_L(t)\hat{\mathbf{x}}(t) + \mathbf{B}_L(t)\begin{bmatrix} \mathbf{u}(t) \\ \hat{\mathbf{p}}_7(t) \end{bmatrix} + \hat{\mathbf{K}}(t)[\boldsymbol{y}(t) - \mathbf{C}_A\hat{\mathbf{x}}(t)] \\ \hat{\mathbf{K}}(t) = \hat{\mathbf{P}}(t)\boldsymbol{C}_A^T\mathbf{R}_1^{-1} \\ \dot{\hat{\mathbf{P}}}(t) = \mathbf{A}_L(t)\hat{\mathbf{P}}(t) + \hat{\mathbf{P}}(t)\boldsymbol{A}_L^T(t) + \mathbf{Q}_1 - \hat{\mathbf{P}}(t)\boldsymbol{C}_A^T\mathbf{R}_1^{-1}\mathbf{C}_A\hat{\mathbf{P}}(t) \end{cases},$$

in which

$$\begin{cases} \mathbf{R}_1 = 0.01 \times \text{diag}(1, 1, 1, 0.001, 0.001, 0.001, 1) \\ \mathbf{Q}_1 = \mathbf{I}_2 \end{cases}.$$

In this case, the agent needs to receive updated position estimates from agent 7, as well as its input.

6.3. Extended Kalman filter-based solutions

Both EKF-based implementations (centralized and decentralized) are based on the original nonlinear, non-augmented dynamics (1).

For the decentralized case, agents 1 to 3, with measurements of their own position, implement the LTI Kalman filter (16). The other agents in the formation implement an EKF based on their local dynamics (1). For example, for agent 4, the filter dynamics follow

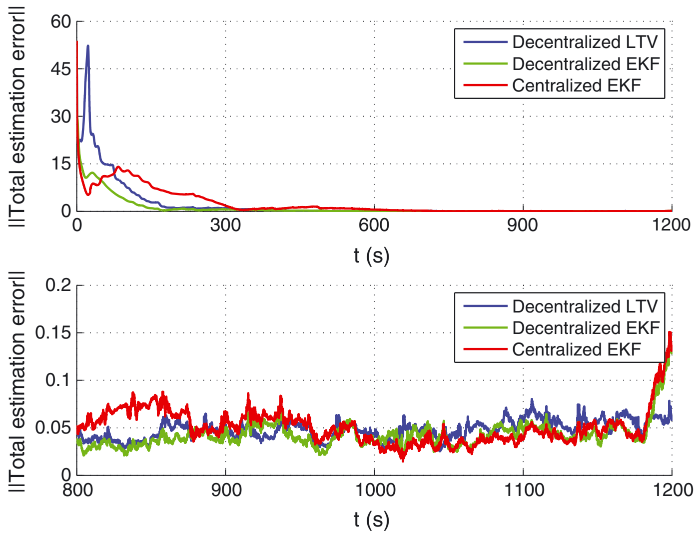

Figure 6. Initial evolution and detailed view of the norm of the total estimation error in the formation.

Table I. Mean of the steady-state total estimation error in the formation.

Decentralized LTV	Decentralized EKF	Centralized EKF
6.17×10^{-2}	5.40×10^{-2}	4.83×10^{-2}

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_{\text{LTI}}\hat{\mathbf{x}}(t) + \mathbf{B}_{\text{LTI}}\mathbf{u}(t) + \hat{\mathbf{K}}(t)[\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)] \\ \hat{\mathbf{K}}(t) = \hat{\mathbf{P}}(t)\hat{\mathbf{C}}_{\text{EKF}}^T(t)\mathbf{R}_{\text{LTI}}^{-1} \\ \dot{\hat{\mathbf{P}}}(t) = \mathbf{A}_{\text{LTI}}\hat{\mathbf{P}}(t) + \hat{\mathbf{P}}(t)\mathbf{A}_{\text{LTI}}^T + \mathbf{Q}_{\text{LTI}} - \hat{\mathbf{P}}(t)\hat{\mathbf{C}}_{\text{EKF}}^T(t)\mathbf{R}_{\text{LTI}}^{-1}\hat{\mathbf{C}}_{\text{EKF}}(t)\hat{\mathbf{P}}(t) \end{cases},$$

in which

$$\hat{\mathbf{C}}_{\text{EKF}}(t) = \left.\frac{\partial \boldsymbol{y}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}(t)} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} & \mathbf{0} \\ \frac{\hat{\mathbf{x}}_1^T(t) - \hat{\mathbf{p}}_1^T(t)}{r_{14}(t)} & \mathbf{0} \\ \frac{\hat{\mathbf{x}}_1^T(t) - \hat{\mathbf{p}}_2^T(t)}{r_{24}(t)} & \mathbf{0} \\ \frac{\hat{\mathbf{x}}_1^T(t) - \hat{\mathbf{p}}_3^T(t)}{r_{34}(t)} & \mathbf{0} \end{bmatrix}$$

and

$$\hat{y}(t) = \begin{bmatrix} \hat{x}_1^3(t) \\ \|\hat{\mathbf{p}}_1(t) - \hat{\mathbf{x}}_1(t)\| \\ \|\hat{\mathbf{p}}_2(t) - \hat{\mathbf{x}}_1(t)\| \\ \|\hat{\mathbf{p}}_3(t) - \hat{\mathbf{x}}_1(t)\| \end{bmatrix}.$$

In this case, agent 4 needs to receive update position estimates from agents 1, 2, and 3 through communication.

The centralized EKF solution estimates the state variables in (1) for each agent in the formation in a single EKF, tuned with the covariance matrices

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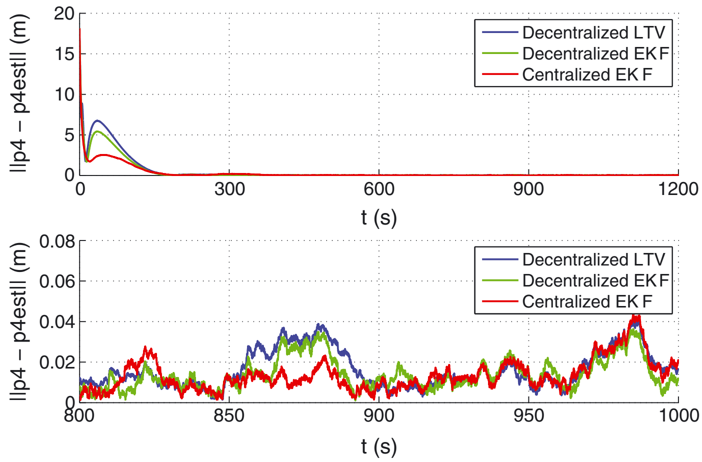

Figure 7. Initial evolution and detailed view of the norm of the position estimation error at agent 4.

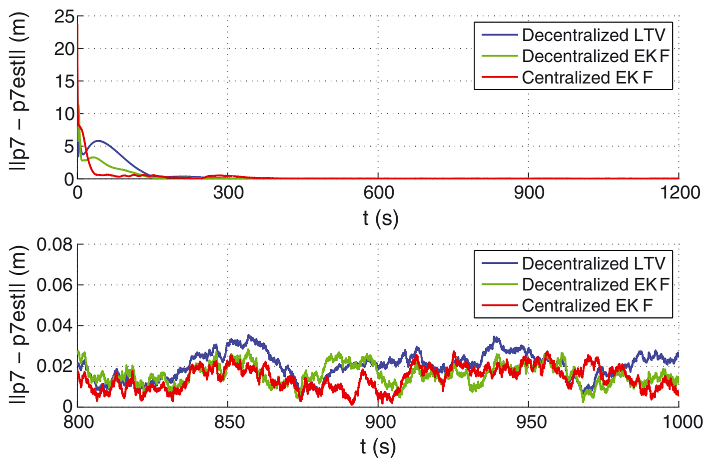

Figure 8. Initial evolution and detailed view of the estimation error for the first coordinate of the position at agent 7.

$$\begin{cases} \mathbf{R}_{\mathrm{CEN}} = \mathrm{diag}\left(\mathbf{R}_{abs}, \mathbf{I}_{18}\right) \\ \mathbf{Q}_{\mathrm{CEN}} = \mathbf{I}_8 \otimes \mathbf{Q}_{LTI} \end{cases}.$$

6.4. *Performance comparison*

Figure 6 depicts the initial evolution and a detailed view of the norm of the total estimation error in the formation over the course of the simulation. In the computation of this norm, the errors relative to the augmented state variables in the decentralized LTV solution are omitted for fairness of comparison. As it can be seen, after large transients caused by the errors in the initial estimates, the estimation error converges to the vicinity of zero (not converging to zero only because of the noise present in the measurements) and, qualitatively, the three competing solutions appear to offer similar filtering performance. To complement the graphical data, Table I details the mean of the norm of the steady-state estimation error for the three filtering solutions, averaged over 1000 runs of the simulation. The data shows that the centralized EKF outperforms both decentralized solutions, and that the decentralized EKF implementation achieves a better overall performance than the decentralized LTV solution.

However, looking at individual agents in the formation paints a different picture. Figures 7, 8, and 9 depict the initial evolution and detailed view of the norm of estimation error for the position and velocity at agents 4, 7, and 8, respectively, and Table II details the measured average norm of the

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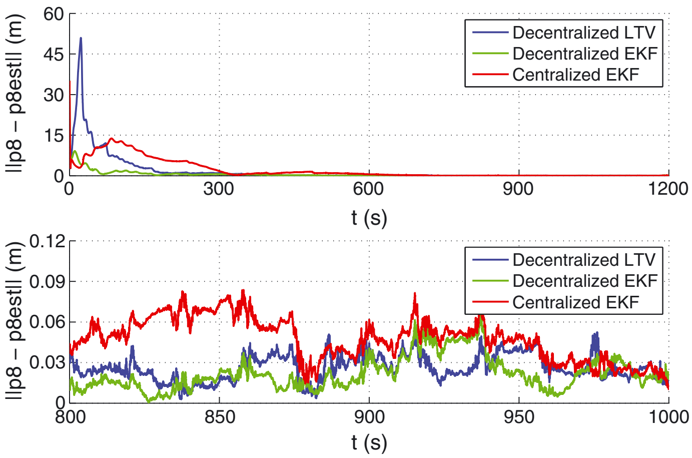

Figure 9. Initial evolution and detailed view of the estimation error for the first coordinate of the position at agent 8.

Table II. Average steady-state error norm of the position and velocity estimates at agents 4, 7, and 8.

Variable	Decentralized LTV	Decentralized EKF	Centralized EKF
$\Vert\mathbf{p}_4(t)\Vert$ (m)	1.48×10^{-2}	1.50×10^{-2}	1.36×10^{-2}
$\Vert\mathbf{v}_{f4}(t)\Vert$ (m/s)	2.80×10^{-4}	3.15×10^{-4}	2.81×10^{-4}
$\Vert\mathbf{p}_7(t)\Vert$ (m)	1.61×10^{-2}	1.68×10^{-2}	1.47×10^{-2}
$\Vert\mathbf{v}_{f7}(t)\Vert$ (m/s)	3.18×10^{-4}	3.72×10^{-4}	2.96×10^{-4}
$\Vert\mathbf{p}_8(t)\Vert$ (m)	4.96×10^{-2}	3.95×10^{-2}	3.47×10^{-2}
$\Vert\mathbf{v}_{f8}(t)\Vert$ (m/s)	4.21×10^{-4}	4.36×10^{-4}	3.69×10^{-4}

estimation error for the same agents, averaged over 1000 runs of the simulation. As the data shows, the decentralized LTV solution performs as well or better than the decentralized EKF in agents 4 and 7, while falling behind at agent 8.

Looking at the overall results, it can be concluded that, even though EKF-based solutions perform well and have shown good results in practical applications, the decentralized LTV solution presented in this paper is a viable alternative with similar performance, with the added value of global stability guarantees.

7. CONCLUSIONS

The problem of decentralized position and velocity estimation in formations of autonomous vehicles was addressed in this paper. A limited number of vehicles in the formation have access to absolute position measurements, while the rest must rely on range measurements to neighboring agents, local sensor data, and limited communication capabilities to estimate their own position and velocity. A method for designing local state observers for each agent in the formation that rely only on locally available information was presented, and the stability of the continuous-time LTV Kalman filter subject to exponentially decaying perturbations in some variables was analyzed. Building on this, the stability of the error dynamics of the resulting decentralized state observer was established for acyclic formations. Simulation results were presented and discussed to validate the proposed solution, as well as assessing its performance under the influence of measurement noise.

APPENDIX A: PROOFS FOR THE RESULTS OF SECTION 3

Proof for Lemma 1
The observability Gramian of the LTV system (3) is defined as

$$\mathcal{W}_A(t_0,t)=\int_{t_0}^{t}[\mathbf{C}_A\mathbf{\Phi}_L(\sigma,t_0)]^T\mathbf{C}_A\mathbf{\Phi}_L(\sigma,t_0)\,d\sigma.$$

If the system is not observable, then there exists a $\mathbf{d}\in\mathbb{R}^7\neq\mathbf{0}$, $\mathbf{d}:=\begin{bmatrix}\mathbf{d}_1^T & \mathbf{d}_2^T & d_3\end{bmatrix}^T$, with $\mathbf{d}_1,\mathbf{d}_2\in\mathbb{R}^3$, and $d_3\in\mathbb{R}$, such that

$$\mathbf{d}^T\mathcal{W}_A(t_0,t)\mathbf{d}=0$$

for all $t\in[t_0,t_f]$. This implies that

$$\mathbf{C}_A\mathbf{\Phi}_L(t,t_0)\mathbf{d}=\mathbf{0}$$

for all $t\in[t_0,t_f]$. From this, it follows that

$$\int_{t_0}^{t}\frac{\Delta\mathbf{u}_1(\sigma)\cdot\mathbf{d}_1}{r_1(\sigma)}\,d\sigma+\int_{t_0}^{t}\frac{(\sigma-t_0)\Delta\mathbf{u}_1(\sigma)\cdot\mathbf{d}_2}{r_1(\sigma)}\,d\sigma-d_3=0 \tag{17}$$

and

$$d_1^3+(t-t_0)d_2^3=0 \tag{18}$$

for all $t\in[t_0,t_f]$. Making $t=t_0$ in (18) shows that $d_{13}=0$, and it follows that d_{23} must also be 0. Taking (17) and making $t=t_0$ shows that $d_3=0$. Taking its time derivative yields

$$\Delta u_1^1(t)d_1^1+\Delta u_1^2(t)d_1^2+(t-t_0)\Delta u_1^1(t)d_2^1+(t-t_0)\Delta u_1^2(t)d_2^2=0 \tag{19}$$

for all $t\in[t_0,t_f]$. Then, because the functions of the set (5) are assumed to be linearly independent on $[t_0,t_f]$, (19) implies that $d_1^1=d_1^2=d_2^1=d_2^2=0$. Then, $\mathbf{d}=\mathbf{0}$, which is a contradiction. Thus, the system is observable on $[t_0,t_f]$. □

Proof for Lemma 2
The observability Gramian of the LTV system (4) is defined as

$$\mathcal{W}_B(t_0,t):=\int_{t_0}^{t}[\mathbf{C}_B(\sigma)\mathbf{\Phi}_L(\sigma,t_0)]^T\mathbf{C}_B(\sigma)\mathbf{\Phi}_L(\sigma,t_0)\,d\sigma.$$

Following a similar reasoning as in Theorem 1, if (4) is not observable, then there exists a $\mathbf{d}\in\mathbb{R}^{6+M}\neq\mathbf{0}$, $\mathbf{d}:=\begin{bmatrix}\mathbf{d}_1^T & \mathbf{d}_2^T & \mathbf{d}_3^T\end{bmatrix}^T$, with $\mathbf{d}_1,\mathbf{d}_2\in\mathbb{R}^3$, and $\mathbf{d}_3\in\mathbb{R}^M$, such that

$$\mathbf{d}^T\mathcal{W}_B(t_1,t_2)\mathbf{d}=0$$

for all $t_1,t_2\in[t_0,t_f]$, $t_2\geqslant t_1$. This implies that

$$\mathbf{C}_B(t_2)\mathbf{\Phi}_L(t_2,t_1)\mathbf{d}=\mathbf{0} \tag{20}$$

for all $t_1,t_2\in[t_0,t_f]$, $t_2\geqslant t_1$. Expanding (20) yields

$$\int_{t_1}^{t_2}\frac{\Delta\mathbf{u}_j(\sigma)\cdot\mathbf{d}_1}{r_j(\sigma)}d\sigma+\int_{t_1}^{t_2}\frac{(\sigma-t_1)\Delta\mathbf{u}_j(\sigma)\cdot\mathbf{d}_2}{r_j(\sigma)}d\sigma-d_3^j=0,\ j=1,2,\ldots,M, \tag{21}$$

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$$d_{13} + (t_2 - t_1)d_{23} = 0, \tag{22}$$

and

$$\begin{aligned}\mathbf{0} = \mathbf{C}_1(t_2)\mathbf{d}_1 + \mathbf{C}_2 &\begin{bmatrix} -\int_{t_1}^{t_2} \frac{\Delta \mathbf{u}_1^T(\sigma)}{r_1(\sigma)} d\sigma \\ \vdots \\ -\int_{t_1}^{t_2} \frac{\Delta \mathbf{u}_M^T(\sigma)}{r_M(\sigma)} d\sigma \end{bmatrix} \mathbf{d}_1 + (t_2 - t_1)\mathbf{C}_1(t_2)\mathbf{d}_2 \\ + \mathbf{C}_2 &\begin{bmatrix} -\int_{t_1}^{t_2} \frac{(\sigma - t_1)\Delta \mathbf{u}_1^T(\sigma)}{r_1(\sigma)} d\sigma \\ \vdots \\ -\int_{t_1}^{t_2} \frac{(\sigma - t_1)\Delta \mathbf{u}_M^T(\sigma)}{r_M(\sigma)} d\sigma \end{bmatrix} \mathbf{d}_2 + \mathbf{C}_2\mathbf{d}_3\end{aligned} \tag{23}$$

for all $t_1, t_2 \in [t_0, t_f]$, $t_2 \geqslant t_1$. Proceeding as in the previous case, (21) and (22) imply that $d_1^3 = d_2^3 = 0$ and $\mathbf{d}_3 = \mathbf{0}$. If Assumption 1 is verified, making $t_2 = t_1 = t_{nc}$ in (23) yields

$$\mathbf{C}_1(t_{nc})\mathbf{d}_1 = \mathbf{0},$$

that is,

$$\begin{cases} \left(p_1^1(t_{nc}) - p_2^1(t_{nc})\right)d_1^1 = \left(p_1^2(t_{nc}) - p_2^2(t_{nc})\right) d_1^2 = 0 \\ \left(p_1^1(t_{nc}) - p_3^1(t_{nc})\right)d_1^1 = \left(p_1^2(t_{nc}) - p_3^2(t_{nc})\right) d_2^2 = 0 \\ \vdots \\ \left(p_{M-1}^1(t_{nc}) - p_M^1(t_{nc})\right)d_1^1 = \left(p_{M-1}^2(t_{nc}) - p_M^2(t_{nc})\right) d_1^2 = 0 \end{cases}$$

which, under Assumption 1, implies that it must be $d_1^1 = d_1^2 = 0$. Taking the time derivative of (23) and making $t_2 = t_1 = t_{nc}$ yields

$$\mathbf{C}_1(t_{nc})\mathbf{d}_2 = 0. \tag{24}$$

Following the same argument, (24) means that, under Assumption 1, $d_2^1 = d_2^2 = 0$. So $\mathbf{d} = 0$, which is a contradiction. Therefore, the LTV system (4) is observable on $[t_0, t_f]$. □

Proof for Theorem 1

Let $\mathbf{x}(t_0) := \left[\mathbf{x}_1^T(t_0)\ \mathbf{x}_2^T(t_0)\right]^T$ be the initial state of the nonlinear system (1) and $\mathbf{w}(t_0) := \left[\mathbf{w}_1^T(t_0)\ \mathbf{w}_2^T(t_0)\ w_3(t_0)\right]^T$ the initial state of the LTV system (3). Because the functions of the set (5) are assumed to be linearly independent, the initial state of the LTV system (3) is uniquely defined. Notice that $z(t_0) = x_1^3(t_0) = w_1^3(t_0)$ and that $r_1(t_0) = w_3(t_0) = \|\mathbf{p}_1(t_0) - \mathbf{x}_1(t_0)\|$. The comparison of $z(t)$ for the nonlinear system and the LTV system yields

$$(t - t_0)\left[x_2^3(t_0) - w_2^3(t_0)\right] = 0,$$

so it follows that $x_2^3(t_0) = w_2^3(t_0)$. On the other hand, comparing $[r_1(t)]^2$ for the nonlinear system and the LTV system yields

$$\begin{aligned}2\Delta \mathbf{u}_1^{[1]}(t, t_0) \cdot [\mathbf{w}_1(t_0) - \mathbf{x}_1(t_0)] + 2(t - t_0)\Delta \mathbf{u}_1^{[1]}(t, t_0) \cdot [\mathbf{x}_2(t_0) - \mathbf{w}_2(t_0)] \\ - 2\Delta \mathbf{u}_1^{[2]}(t, t_0) \cdot [\mathbf{x}_2(t_0) - \mathbf{w}_2(t_0)] = 0.\end{aligned} \tag{25}$$

Taking the time derivative of (25) yields

$$2\Delta \mathbf{u}_1(t) \cdot [\mathbf{w}_1(t_0) - \mathbf{x}_1(t_0)] + 2(t - t_0)\Delta \mathbf{u}_1(t) \cdot [\mathbf{x}_2(t_0) - \mathbf{w}_2(t_0)] = 0. \tag{26}$$

Clearly, because the functions of the set (5) are linearly independent, (26) implies that $x_1^1(t_0) = w_1^1(t_0)$, $x_1^2(t_0) = w_1^2(t_0)$, $x_2^1(t_0) = w_2^1(t_0)$, and $x_2^2(t_0) = w_2^2(t_0)$. So the initial state of the nonlinear system (1) with one range measurement matches the initial state of the LTV system (3), which means that the initial state of the nonlinear system is uniquely defined. Moreover, equivalence between the states of both systems also implies that a state observer for the LTV system (3) will act simultaneously as a state observer for the nonlinear system (1) and that if it features GES error dynamics for (3), the error on the estimates corresponding to the state variables of (1) will converge exponentially fast to zero regardless of the initial conditions.

□

Proof for Theorem 2

Let $\mathbf{x}(t_0) := \left[\mathbf{x}_1^T(t_0)\ \mathbf{x}_2^T(t_0)\right]^T$ be the initial state of the nonlinear system (1) and $\mathbf{w}(t_0) := \left[\mathbf{w}_1^T(t_0)\ \mathbf{w}_2^T(t_0)\ \mathbf{w}_3^T(t_0)\right]^T$ the initial state of the LTV system (4). Because Assumption 1 holds, the initial state of the LTV system (4) is uniquely defined. As in the previous case, comparison of $z(t_0)$, $r_j(t_0)$ and the time evolution of $z(t)$ in both systems show that $x_1^3(t_0) = w_1^3(t_0)$, $x_2^3(t_0) = w_2^3(t_0)$, and $r_j(t_0) = w_3^j(t_0) = \|\mathbf{p}_j(t_0) - \mathbf{x}_1(t_0)\|$. Denote each augmented output as $c_{jk}(t)$, where j and k refer to the range measurements used in its construction. By comparing the quantities $c_{jk}(t)\left[r_j(t) + r_k(t)\right]$ for the nonlinear system (1) and the LTV system (4), it can be shown that, under Assumption 1,

$$
\begin{aligned}
0 = {} & 2[\mathbf{p}_j(t_{nc}) - \mathbf{p}_k(t_{nc})] \cdot [\mathbf{w}_1(t_{nc}) - \mathbf{x}_1(t_{nc})] \\
& + 2(t - t_{nc})[\mathbf{p}_j(t_{nc}) - \mathbf{p}_k(t_{nc})] \cdot [\mathbf{w}_2(t_{nc}) - \mathbf{x}_2(t_{nc})] \\
& - 2\left[\mathbf{u}_j^{[2]}(t, t_{nc}) - \mathbf{u}_k^{[2]}(t, t_{nc})\right] \cdot [\mathbf{x}_2(t_{nc}) - \mathbf{w}_2(t_{nc})]
\end{aligned} \tag{27}
$$

for $j, k = 1, 2, \ldots, M,\ j \neq k$ and all $t \geqslant t_{nc}$. The time derivative of (27) follows

$$
2[\mathbf{p}_j(t_{nc}) - \mathbf{p}_k(t_{nc})] \cdot [\mathbf{w}_2(t_{nc}) - \mathbf{x}_2(t_{nc})] - 2\left[\mathbf{u}_j^{[1]}(t, t_{nc}) - \mathbf{u}_k^{[1]}(t_{nc})\right] \cdot [\mathbf{x}_2(t_{nc}) - \mathbf{w}_2(t_{nc})] = 0, \tag{28}
$$

for all $t \geqslant t_{nc}$ and $j, k = 1, 2, \ldots, M,\ j \neq k$. Now making $t = t_{nc}$ in (28) yields

$$
\begin{cases}
(p_1^1(t_{nc}) - p_2^1(t_{nc}))(w_2^1(t_{nc}) - x_2^1(t_{nc})) = 0 \\
(p_1^2(t_{nc}) - p_2^2(t_{nc}))(w_2^2(t_{nc}) - x_2^2(t_{nc})) = 0 \\
(p_1^1(t_{nc}) - p_3^1(t_{nc}))(w_2^1(t_{nc}) - x_2^1(t_{nc})) = 0 \\
(p_1^2(t_{nc}) - p_3^2(t_{nc}))(w_2^2(t_{nc}) - x_2^2(t_{nc})) = 0 \\
\vdots \\
(p_{M-1}^1(t_{nc}) - p_M^1(t_{nc}))(w_2^1(t_{nc}) - x_2^1(t_{nc})) = 0 \\
(p_{M-1}^2(t_{nc}) - p_M^2(t_{nc}))(w_2^2(t_{nc}) - x_2^2(t_{nc})) = 0
\end{cases}. \tag{29}
$$

If Assumption 1 is verified, then (29) implies that it must be $x_2^1(t_{nc}) = w_2^1(t_{nc})$ and $x_2^2(t_{nc}) = w_2^2(t_{nc})$, which also means that $x_2^1(t_0) = w_2^1(t_0)$ and $x_2^2(t_0) = w_2^2(t_0)$. Making $t = t_{nc}$ in (27), it follows by the same argument that $x_1^1(t_{nc}) = w_1^1(t_{nc})$ and $x_1^2(t_{nc}) = w_1^2(t_{nc})$. Noticing that

$$
\mathbf{x}_1(t_0) = \mathbf{x}_1(t_{nc}) - (t_{nc} - t_0)\mathbf{x}_2(t_0) - \mathbf{u}^{[1]}(t_{nc}, t_0)
$$

and

$$
\mathbf{w}_1(t_0) = \mathbf{w}_1(t_{nc}) - (t_{nc} - t_0)\mathbf{w}_2(t_0) - \mathbf{u}^{[1]}(t_{nc}, t_0),
$$

it is clear that $x_1^1(t_0) = w_1^1(t_0)$ and $x_1^2(t_0) = w_1^2(t_0)$. Then, as the initial state of the nonlinear system (1) matches the initial state of the LTV system (4), it is uniquely defined, which means the nonlinear system is observable, and that a state observer for the LTV system (4) featuring GES error dynamics will also be a state observer for the nonlinear system (1) with $M \geqslant 3$ range measurements, whose error converges exponentially fast to zero for all initial conditions. □

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The following proposition [28, Proposition 4.2] is needed for the two proofs that follow.

Proposition 1
Let $\mathbf{f}(t) : [t_0, t_f] \subset \mathbb{R} \to \mathbb{R}^n$ be a continuous and i-times continuously differentiable function on $\ell := [t_0, t_f]$, $T := t_f - t_0 > 0$ and such that

$$\mathbf{f}(t_0) = \dot{\mathbf{f}}(t_0) = \ldots = \mathbf{f}^{(i-1)}(t_0) = \mathbf{0}.$$

Further, assume that

$$\max_{t \in \ell} \left\| \mathbf{f}^{(i+1)}(t) \right\| \leqslant C.$$

If

$$\exists \alpha > 0,\ t_1 \in \ell : \left\| \mathbf{f}^{(i)}(t_1) \right\| \geqslant \alpha,$$

then

$$\exists\, 0 < \delta \leqslant T,\ \beta > 0 : \|\mathbf{f}(t_0 + \delta)\| \geqslant \beta.$$

Proof for Theorem 3
That there is a positive constant c_M such that

$$\mathbf{d}^T \mathcal{W}_A(t, t+\delta)\mathbf{d} \leqslant c_M,$$

for all $t \geqslant t_0$ and $\mathbf{d} \in \mathbb{R}^7$, $\|\mathbf{d}\| = 1$, is evident as the observability Gramian is the integral of a continuous bounded function on all intervals $[t, t+\delta]$, $t \geqslant t_0$.

Now consider

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ d_3 \end{bmatrix},\ \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^3,\ d_3 \in \mathbb{R},$$

with $\|\mathbf{d}\| = 1$ and notice that

$$\mathbf{C}_A \mathbf{\Phi}_L(t, t)\mathbf{d} = \begin{bmatrix} d_1^3 \\ d_3 \end{bmatrix}.$$

Suppose that there is a positive scalar constant ε_1 such that $\left\| \begin{bmatrix} d_1^3 & d_3 \end{bmatrix} \right\| \geqslant \varepsilon_1$. Then,

$$\|\mathbf{C}_A \mathbf{\Phi}_L(t, t)\mathbf{d}\| \geqslant \varepsilon_1.$$

By applying Proposition 1, it follows that there exists a positive scalar constant c_{m1} such that

$$\mathbf{d}^T \mathcal{W}_A(t, t+\delta)\mathbf{d} \geqslant c_{m1}$$

for all $t \geqslant t_0$, which means the LTV system (3) is UCO. Suppose now that $\left| d_2^3 \right| \geqslant \varepsilon_2$ for some $\varepsilon_2 > 0$. Then,

$$\left\| \frac{d}{d\tau} \mathbf{C}_A \mathbf{\Phi}_L(\tau, t)\mathbf{d} \bigg|_{\tau = t} \right\| \geqslant \left| d_2^3 \right| \geqslant \varepsilon_2.$$

By applying Proposition 1, it follows that, for some $\delta^* \in]0, \delta[$ and $\varepsilon_3 > 0$,

$$\left\| \mathbf{C}_A \mathbf{\Phi}_L(t + \delta^*, t)\mathbf{d} - \begin{bmatrix} d_1^3 \\ d_3 \end{bmatrix} \right\| \geqslant 2\varepsilon_3 \tag{30}$$

for all $t \geqslant t_0$. Now, if $\left\| \begin{bmatrix} d_1^3 & d_3 \end{bmatrix} \right\| \geqslant \varepsilon_3$, then the LTV system (3) is UCO. On the other hand, if $\left\| \begin{bmatrix} d_1^3 & d_3 \end{bmatrix} \right\| \leqslant \varepsilon_3$, (30) implies that

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$$\|\mathbf{C}_A\mathbf{\Phi}_L(t+\delta^*,t)\mathbf{d}\| \geqslant \varepsilon_3$$

for all $t \geqslant t_0$, and by application of Proposition 1, it is clear that there exists a positive scalar constant c_{m2} such that

$$\mathbf{d}^T\mathcal{W}_A(t,t+\delta)\mathbf{d} \geqslant c_{m2}$$

for all $t \geqslant t_0$, and the LTV system (3) is UCO.

Finally, suppose that there exists a positive scalar constant ε_4 such that $\left\|\left[d_1^1\ d_1^2\ d_2^1\ d_2^2\right]\right\| \geqslant 2\varepsilon_4$, and note that if it is not the case, then UCO immediately follows from the previous arguments as $\mathbf{d}$ is a unit vector. From the assumptions of the theorem, for all $t \geqslant t_0$, there exists a $t^* \in [t, t+\delta]$ such that

$$\left|\Delta u_1^1(t^*)d_1^1 + \Delta u_1^2(t^*)d_1^2 + (t^*-t)\Delta u_1^1(t^*)d_2^1 + (t^*-t)\Delta u_1^2(t^*)d_2^2\right| \geqslant 2\alpha\varepsilon_4$$

for all $t \geqslant t_0$. Now assume that $\left|\Delta u_1^3(t^*)d_1^3 + (t^*-t)\Delta u_1^3(t^*)d_2^3\right| < \alpha\varepsilon_4$. If this is not verified, UCO immediately follows from the previous arguments. On the other hand, if it is in fact verified, one can write

$$|\Delta\mathbf{u}_1(t^*)\cdot\mathbf{d}_1 + (t^*-t)\Delta\mathbf{u}_1(t^*)\cdot\mathbf{d}_2| \geqslant \alpha\varepsilon_4,$$

which in turn implies that

$$\left\|\left.\frac{d}{d\tau}\mathbf{C}_A\mathbf{\Phi}_L(\tau,t)\mathbf{d}\right|_{\tau=t^*}\right\| \geqslant \frac{\alpha\varepsilon_4}{y_M}.$$

Then, applying Proposition 1 twice, it can be concluded that there exists a positive scalar constant c_{m3} such that

$$\mathbf{d}^T\mathcal{W}_A(t,t+\delta)\mathbf{d} \geqslant c_{m3}$$

for all $t \geqslant t_0$, and the LTV system (3) is UCO, which concludes the proof. □

Proof for Theorem 4

As in the previous result, because the observability Gramian is the integral of a continuous bounded function on all intervals $[t, t+\delta]$, $t \geqslant t_0$, it immediately follows that there is a positive scalar constant c_M such that

$$\mathbf{d}^T\mathcal{W}_B(t,t+\delta)\mathbf{d} \leqslant c_M$$

for all $t \geqslant t_0$ and $\mathbf{d} \in \mathbb{R}^{6+M}$, $\|\mathbf{d}\| = 1$. Now, consider a

$$\mathbf{d} = \begin{bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\\ \mathbf{d}_3\end{bmatrix},\ \mathbf{d}_1,\ \mathbf{d}_2 \in \mathbb{R}^3,\ \mathbf{d}_3 \in \mathbb{R}^M,$$

with $\|\mathbf{d}\| = 1$ and, to simplify the notation in the computations that follow, let

$$\mathbf{C}_B(\tau)\mathbf{\Phi}_L(\tau,t)\mathbf{d} =: \begin{bmatrix}\Theta_1(\tau,t)\\ \mathbf{\Theta}_2(\tau,t)\\ \mathbf{\Theta}_3(\tau,t)\end{bmatrix} \in \mathbb{R}^{1+(M+1)M/2},$$

in which

$$\Theta_1(\tau,t) := d_1^3 + (\tau-t)d_2^3 \in \mathbb{R},$$

$$\mathbf{\Theta}_2(\tau,t) := -\boldsymbol{\beta}_1(\tau,t)\mathbf{d}_1 - \boldsymbol{\beta}_2(\tau,t)\mathbf{d}_2 + \mathbf{d}_3 \in \mathbb{R}^M,$$

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and

$$\boldsymbol{\Theta}_3(\tau,t) := \mathbf{C}_1(\tau)\mathbf{d}_1 - \mathbf{C}_2\boldsymbol{\beta}_1(\tau,t)\mathbf{d}_1 + (\tau-t)\mathbf{C}_1(\tau)\mathbf{d}_2 - \mathbf{C}_2\boldsymbol{\beta}_2(\tau,t)\mathbf{d}_2 + \mathbf{C}_2\mathbf{d}_3 \in \mathbb{R}^{M(M-1)/2}, \quad (31)$$

with

$$\boldsymbol{\beta}_1(\tau,t) := \begin{bmatrix} \int_t^\tau \frac{\Delta\mathbf{u}_1^T(\sigma)}{r_1(\sigma)}\,d\sigma \\ \vdots \\ \int_t^\tau \frac{\Delta\mathbf{u}_M^T(\sigma)}{r_M(\sigma)}\,d\sigma \end{bmatrix} \in \mathbb{R}^{M\times 3}$$

and

$$\boldsymbol{\beta}_2(\tau,t) := \begin{bmatrix} \int_t^\tau \frac{(\sigma-t)\Delta\mathbf{u}_1^T(\sigma)}{r_1(\sigma)}\,d\sigma \\ \vdots \\ \int_t^\tau \frac{(\sigma-t)\Delta\mathbf{u}_M^T(\sigma)}{r_M(\sigma)}\,d\sigma \end{bmatrix} \in \mathbb{R}^{M\times 3}.$$

Suppose that there exists a positive scalar constant ε_1 such that $\left\|\begin{bmatrix} d_1^3 & \mathbf{d}_3^{\mathbf{T}} \end{bmatrix}\right\| \geqslant \varepsilon_1$. Then,

$$\left\|\begin{bmatrix} \Theta_1(t,t) \\ \boldsymbol{\Theta}_2(t,t) \end{bmatrix}\right\| = \left\|\begin{bmatrix} d_1^3 \\ \mathbf{d}_3 \end{bmatrix}\right\| \geqslant \varepsilon_1$$

for all $t \geqslant t_0$, and, by application of Proposition 1, it follows that there exists a positive scalar constant c_{m1} such that

$$\mathbf{d}^T \mathcal{W}_B(t,t+\delta)\mathbf{d} \geqslant c_{m1}$$

for all $t \geqslant t_0$, which means the LTV system (4) is UCO. Suppose now that there exists a positive scalar constant ε_2 such that $\left|d_2^3\right| \geqslant \varepsilon_2$. Then,

$$\left|\left.\frac{d}{d\tau}\Theta_1(\tau,t)\right|_{\tau=t}\right| = \left|d_2^3\right| \geqslant \varepsilon_2$$

for all $t \geqslant t_0$. Through application of Proposition 1, one can conclude that, for some $\delta^* \in]0,\delta[$ and $\varepsilon_3 > 0$,

$$\left|\Theta_1(t+\delta^*,t) - d_1^3\right| \geqslant 2\varepsilon_3 \quad (32)$$

for all $t \geqslant t_0$. Now, if $\left|d_1^3\right| \geqslant \varepsilon_3$ then, following the previous argument, the LTV system (4) is guaranteed to be UCO. However, if $\left|d_1^3\right| < \varepsilon_3$, (32) implies that

$$|\Theta_1(t+\delta^*,t)| \geqslant \varepsilon_3$$

for all $t \geqslant t_0$, and by applying Proposition 1 it follows that there exists some positive scalar constant c_{m2} such that

$$\mathbf{d}^T \mathcal{W}_B(t,t+\delta)\mathbf{d} \geqslant c_{m2}$$

for all $t \geqslant t_0$, and the LTV system (4) is UCO. Now suppose that there exists a positive scalar constant ε_4 such that $\left\|\begin{bmatrix} d_1^1 & d_1^2 \end{bmatrix}\right\| \geqslant 4\varepsilon_4$. Then for all $t \geqslant t_0$, there exist $i, j \in \{1,2,\ldots,M\}$ and a $t^* \in [t, t+\delta]$ for which the inequality

$$\left|\left(p_i^1(t^*) - p_j^1(t^*)\right)d_1^1 + \left(p_i^2(t^*) - p_j^2(t^*)\right)d_1^2\right| \geqslant 4\alpha\varepsilon_4 \quad (33)$$

holds. Fixing $\tau = t = t^*$ in (31) yields

$$\mathbf{\Theta}_3(t^*, t^*) = \mathbf{C}_1(t^*)\mathbf{d}_1 + \mathbf{C}_2\mathbf{d}_3. \tag{34}$$

Now if

$$\left\| \mathbf{C}_1(t^*) \begin{bmatrix} 0 \\ 0 \\ d_1^3 \end{bmatrix} + \mathbf{C}_2\mathbf{d}_3 \right\| \geqslant \frac{\alpha\varepsilon_4}{y_M},$$

it follows from the previous arguments that the LTV system (4) is UCO. On the other hand, if

$$\left\| \mathbf{C}_1(t^*) \begin{bmatrix} 0 \\ 0 \\ d_1^3 \end{bmatrix} + \mathbf{C}_2\mathbf{d}_3 \right\| < \frac{\alpha\varepsilon_4}{y_M},$$

then (33) and (34) imply that

$$\|\mathbf{\Theta}_3(t^*, t^*)\| \geqslant \frac{\alpha\varepsilon_4}{y_M}$$

and Proposition 1 entails that there exists a positive scalar constant c_{m4} such that

$$\mathbf{d}^T \mathcal{W}_B(t, t+\delta)\mathbf{d} \geqslant c_{m3}$$

for all $t \geqslant t_0$, and thus the LTV system (4) is UCO.

Finally, suppose that there exists a positive scalar constant ε_5 such that $\left\| \left[d_2^1 \;\; d_2^2 \right] \right\| \geqslant 4\varepsilon_5$ and note that if this is not verified, then from the previous discussion, the LTV system (4) is guaranteed to be UCO. Then, for all $t \geqslant t_0$, there exist $i, j \in \{1, 2, \ldots, M\}$ and a $t^* \in [t, t+\delta]$ for which the inequality

$$\left| \left(p_i^1(t^*) - p_j^1(t^*) \right) d_2^1 + \left(p_i^2(t^*) - p_j^2(t^*) \right) d_2^2 \right| \geqslant 4\alpha\varepsilon_5 \tag{35}$$

holds. Differentiating $\mathbf{\Theta}_3(\tau, t^*)$ once and fixing $\tau = t^*$ yields

$$\left.\frac{d}{d\tau}\mathbf{\Theta}_3(\tau, t^*)\right|_{\tau=t^*} = \left.\frac{d}{d\tau}\left[\mathbf{C}_1(\tau)\mathbf{d}_1 - \mathbf{C}_2\boldsymbol{\beta}_1(\tau, t^*)\mathbf{d}_1\right]\right|_{\tau=t^*} + \mathbf{C}_1(t^*)\mathbf{d}_2. \tag{36}$$

Now if

$$\left\| \left.\frac{d}{d\tau}\left[\mathbf{C}_1(\tau)\mathbf{d}_1 - \mathbf{C}_2\boldsymbol{\beta}_1(\tau, t^*)\mathbf{d}_1\right]\right|_{\tau=t^*} + \mathbf{C}_1(t^*)\begin{bmatrix} 0 \\ 0 \\ d_2^3 \end{bmatrix} \right\| \geqslant \frac{\alpha\varepsilon_5}{y_M},$$

it follows from the assumptions of the theorem and the previous arguments that the LTV system (4) is UCO. On the other hand, if

$$\left\| \left.\frac{d}{d\tau}\left[\mathbf{C}_1(\tau)\mathbf{d}_1 - \mathbf{C}_2\boldsymbol{\beta}_1(\tau, t^*)\mathbf{d}_1\right]\right|_{\tau=t^*} + \mathbf{C}_1(t^*)\begin{bmatrix} 0 \\ 0 \\ d_2^3 \end{bmatrix} \right\| < \frac{\alpha\varepsilon_5}{y_M},$$

then (35) and (36) imply that

$$\left\| \left.\frac{d}{d\tau}\mathbf{\Theta}_3(\tau, t^*)\right|_{\tau=t^*} \right\| \geqslant \frac{\alpha\varepsilon_5}{y_M}.$$

By applying Proposition 1 twice in the same fashion as in the previous case, it follows that there exists some positive scalar constant c_{m4} such that

$$\mathbf{d}^T \mathcal{W}_B(t, t+\delta)\mathbf{d} \geqslant c_{m4}$$

for all $t \geqslant t_0$, and the LTV system (4) is UCO, which concludes the proof. □

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APPENDIX B: PROOFS FOR THE RESULTS OF SECTION 4

Proof for Lemma 3
The observability Gramian of the LTV system (6) is given by

$$\mathcal{W}_O(t_0,t)=\int_{t_0}^{t}\mathbf{\Phi}^T(\sigma,t_0)\mathbf{C}^T(\sigma)\mathbf{C}(\sigma)\mathbf{\Phi}(\sigma,t_0)\,d\sigma,$$

in which $\mathbf{\Phi}(t,t_0)$ denotes the transition matrix associated with $\mathbf{A}(t)$. As the system (6) is assumed to be UCO, it follows that there exist positive scalar constants δ, γ_1, γ_2, and γ_3 such that

$$\begin{cases}\gamma_1\leqslant \mathbf{d}^T\mathcal{W}_O(t,t+\delta)\mathbf{d}\leqslant\gamma_2\\ \|\mathbf{\Phi}(t+\delta,t)\|\leqslant\gamma_3\end{cases}\tag{37}$$

holds for all $t\geqslant t_0$ and $\mathbf{d}\in\mathbb{R}^n$, $\|\mathbf{d}\|=1$ [29].

The observability Gramian for the perturbed system (12) follows

$$\hat{\mathcal{W}}_O(t_0,t)=\int_{t_0}^{t}\mathbf{\Phi}^T(\sigma,t_0)\hat{\mathbf{C}}^T(\sigma)\hat{\mathbf{C}}(\sigma)\mathbf{\Phi}(\sigma,t_0)\,d\sigma,$$

or, expanding the terms $\hat{\mathbf{C}}(\sigma)$,

$$\hat{\mathcal{W}}_O(t_0,t)=\mathcal{W}_O(t_0,t)+\mathbf{\Gamma}_1(t_0,t)-\mathbf{\Gamma}_2(t_0,t),$$

with

$$\begin{cases}\mathbf{\Gamma}_1(t_0,t)=\int_{t_0}^{t}\mathbf{\Phi}^T(\sigma,t_0)\tilde{\mathbf{C}}^T(\sigma)\tilde{\mathbf{C}}(\sigma)\mathbf{\Phi}(\sigma,t_0)\,d\sigma\succeq\mathbf{0}\\ \mathbf{\Gamma}_2(t_0,t)=\int_{t_0}^{t}\mathbf{\Phi}^T(\sigma,t_0)\left[\tilde{\mathbf{C}}^T(\sigma)\mathbf{C}(\sigma)+\mathbf{C}^T(\sigma)\tilde{\mathbf{C}}(\sigma)\right]\mathbf{\Phi}(\sigma,t_0)\,d\sigma\end{cases}.\tag{38}$$

The norm of $\mathbf{\Gamma}_2(t,t-\sigma)$ satisfies

$$\begin{aligned}\|\mathbf{\Gamma}_2(t,t+\delta)\|&=\left\|\int_{t}^{t+\delta}\mathbf{\Phi}^T(\sigma,t)\left[\tilde{\mathbf{C}}^T(\sigma)\mathbf{C}(\sigma)+\mathbf{C}^T(\sigma)\tilde{\mathbf{C}}(\sigma)\right]\mathbf{\Phi}(\sigma,t)\,d\sigma\right\|\\&\leqslant\int_{t}^{t+\delta}\left\|\mathbf{\Phi}^T(\sigma,t)\right\|\left\|\left[\tilde{\mathbf{C}}^T(\sigma)\mathbf{C}(\sigma)+\mathbf{C}^T(\sigma)\tilde{\mathbf{C}}(\sigma)\right]\right\|\|\mathbf{\Phi}(\sigma,t)\|\,d\sigma\\&\leqslant 2\int_{t}^{t+\delta}\|\mathbf{\Phi}^T(\sigma,t)\|^2\left\|\tilde{\mathbf{C}}(\sigma)\right\|\|\mathbf{C}(\sigma)\|\,d\sigma.\end{aligned}$$

Then using the bounds in (7), (10), and (37), it follows that there is a positive scalar constant γ_4 such that

$$\|\mathbf{\Gamma}_2(t,t+\delta)\|\leqslant\gamma_4 e^{-\lambda_2(t-t_0)}$$

for all $t\geqslant t_0$ or, equivalently,

$$\mathbf{d}^T\mathbf{\Gamma}_2(t,t+\delta)\mathbf{d}\leqslant\gamma_4 e^{-\lambda_2(t-t_0)}\tag{39}$$

for all $t\geqslant t_0$ and $\mathbf{d}\in\mathbb{R}^n$, $\|\mathbf{d}\|=1$. Thus, (38) and (39) imply that

$$\mathbf{d}^T\hat{\mathcal{W}}_O(t,t+\delta)\mathbf{d}\geqslant\gamma_1-\gamma_4 e^{-\lambda_2(t-t_0)}$$

for all $t\geqslant t_0$ and $\mathbf{d}\in\mathbb{R}^n$, $\|\mathbf{d}\|=1$. Then for any $0<\gamma_5<\gamma_1$, there exists a $\delta^*\geqslant\delta$ such that

$$\mathbf{d}^T\hat{\mathcal{W}}_O(t,t+\delta^*)\mathbf{d}\geqslant\gamma_5$$

for all $t\geqslant t_0$ and $\mathbf{d}\in\mathbb{R}^n$, $\|\mathbf{d}\|=1$. As the perturbed observability Gramian is an integral of bounded functions of time, it also follows that there is a $\gamma_6>0$ such that

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$$\mathbf{d}^T \hat{\mathcal{W}}_O(t, t+\delta^*)\mathbf{d} \leqslant \gamma_6$$

for all $t \geqslant t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. Thus, the perturbed system (12) is UCO. □

Proof for Lemma 4
First, note that when there are no errors in $\hat{\mathbf{C}}(t)$, the dynamics of $\tilde{\mathbf{P}}$ follow

$$\dot{\tilde{\mathbf{P}}} = \mathbf{F}(\tilde{\mathbf{P}}), \tag{40}$$

whose origin is GES [34]. Defining

$$\begin{cases} \tilde{\mathbf{P}}_v = \mathrm{vec}(\tilde{\mathbf{P}}) \\ \mathbf{F}_v(\tilde{\mathbf{P}}) = \mathrm{vec}\left(\mathbf{F}(\tilde{\mathbf{P}})\right) \\ \mathbf{G}_v(\tilde{\mathbf{C}}) = \mathrm{vec}\left(\mathbf{G}(\tilde{\mathbf{C}})\right) \end{cases},$$

the systems (15) and (40) can be represented respectively as

$$\dot{\tilde{\mathbf{P}}}_v = \mathbf{F}_v\left(\tilde{\mathbf{P}}\right) + \mathbf{G}_v\left(\tilde{\mathbf{C}}\right) \tag{41}$$

and

$$\dot{\tilde{\mathbf{P}}}_v = \mathbf{F}_v\left(\tilde{\mathbf{P}}_v\right). \tag{42}$$

Then as the origin of (42) is GES, there exist positive constants c_1, c_2, c_3, and c_4 and a Lyapunov function $V : (0, \infty) \times \mathbb{R}^{n^2} \to \mathbb{R}$ that satisfy

$$\begin{cases} c_1 \left\|\tilde{\mathbf{P}}_v\right\|^2 \leqslant V\left(t, \tilde{\mathbf{P}}_v\right) \leqslant c_2 \left\|\tilde{\mathbf{P}}_v\right\|^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \tilde{\mathbf{P}}_v}\mathbf{F}_v\left(\tilde{\mathbf{P}}\right) \leqslant -c_3 \left\|\tilde{\mathbf{P}}_v\right\|^2 \\ \left\|\frac{\partial V}{\partial \tilde{\mathbf{P}}_v}\right\| \leqslant c_4 \left\|\tilde{\mathbf{P}}_v\right\| \end{cases}$$

for all $t \geqslant t_0$ [35, Theorem 4.14]. On the other hand, there exists a $t_1 \geqslant t_0$ for which the term $\mathbf{G}\left(\tilde{\mathbf{C}}\right)$ can be bounded as follows:

$$\begin{aligned} \left\|\mathbf{G}\left(\tilde{\mathbf{C}}\right)\right\| &= \left\|\hat{\mathbf{P}}\left[\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\tilde{\mathbf{C}} - \tilde{\mathbf{C}}^T\mathbf{R}^{-1}\mathbf{C} - \mathbf{C}^T\mathbf{R}^{-1}\tilde{\mathbf{C}}\right]\hat{\mathbf{P}}\right\| \\ &\leqslant \left\|\hat{\mathbf{P}}\right\|^2 \left\|\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\tilde{\mathbf{C}} - \tilde{\mathbf{C}}^T\mathbf{R}^{-1}\mathbf{C} - \mathbf{C}^T\mathbf{R}^{-1}\tilde{\mathbf{C}}\right\| \\ &\leqslant \alpha_9^2\left(\left\|\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\tilde{\mathbf{C}}\right\| + 2\left\|\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\mathbf{C}\right\|\right) \\ &\leqslant \alpha_4^{-1}\alpha_9^2\left\|\tilde{\mathbf{C}}\right\|^2 + 2\alpha_3\alpha_4^{-1}\alpha_9^2\left\|\tilde{\mathbf{C}}\right\| \end{aligned} \tag{43}$$

for all $t \geqslant t_1$. Now define $\tilde{\mathbf{C}}_v := \mathrm{vec}\left(\tilde{\mathbf{C}}\right)$ and note that

$$\begin{cases} \left\|\tilde{\mathbf{C}}\right\| \leqslant \left\|\tilde{\mathbf{C}}\right\|_F = \left\|\tilde{\mathbf{C}}_v\right\| \\ \left\|\mathbf{G}_v(\tilde{\mathbf{C}})\right\| = \left\|\mathbf{G}(\tilde{\mathbf{C}})\right\|_F \leqslant \sqrt{n}\left\|\mathbf{G}(\tilde{\mathbf{C}})\right\| \end{cases}. \tag{44}$$

Using (44) in (43), it is clear that there exist positive constants α_{10} and α_{11} such that

$$\left\|\mathbf{G}_v(\tilde{\mathbf{C}})\right\| \leqslant \alpha_{10}\left\|\tilde{\mathbf{C}}_v\right\| + \alpha_{11}\left\|\tilde{\mathbf{C}}_v\right\|^2$$

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for all $t \geqslant t_1$. Then the derivative of V along the trajectories of (15) follows

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \tilde{\mathbf{P}}_v}\mathbf{F}_v\left(\tilde{\mathbf{P}}\right) + \frac{\partial V}{\partial \tilde{\mathbf{P}}_v}\mathbf{G}_v\left(\tilde{\mathbf{C}}\right) \leqslant -c_3\left\|\tilde{\mathbf{P}}_v\right\|^2 + c_4\left\|\tilde{\mathbf{P}}_v\right\|\left(\alpha_{10}\left\|\tilde{\mathbf{C}}_v\right\| + \alpha_{11}\left\|\tilde{\mathbf{C}}_v\right\|^2\right).$$

Choosing any θ such that $0 < \theta < 1$, it is straightforward to show that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \tilde{\mathbf{P}}_v}\mathbf{F}_v\left(\tilde{\mathbf{P}}\right) + \frac{\partial V}{\partial \tilde{\mathbf{P}}_v}\mathbf{G}_v\left(\tilde{\mathbf{C}}\right) \leqslant -c_3(1-\theta)\left\|\tilde{\mathbf{P}}_v\right\|^2, \ \forall \ \left\|\tilde{\mathbf{P}}_v\right\| \geqslant \frac{c_4}{c_3\theta}\left(\alpha_{10}\left\|\tilde{\mathbf{C}}_v\right\| + \alpha_{11}\left\|\tilde{\mathbf{C}}_v\right\|^2\right),$$

and thus, (41) is input-to-state stable (ISS) with $\tilde{\mathbf{C}}_v$ as input on the interval $[t_1, +\infty]$ [35, Theorem 4.19]. As (10) implies that $\tilde{\mathbf{C}}_v$ decays exponentially with time, it follows that $\tilde{\mathbf{P}}(t)$ also does decay exponentially fast for $t \geqslant t_1$ [35, Lemma 4.7]. Then, because neither $\mathbf{P}(t)$ nor $\hat{\mathbf{P}}(t)$ (and consequently $\tilde{\mathbf{P}}(t)$) grow unbounded between t_0 and t_1 [36], it follows that there exist positive scalar constants α and λ such that $\left\|\tilde{\mathbf{P}}(t)\right\| \leqslant \alpha e^{-\lambda(t-t_0)}$, which concludes the proof. □

Proof for Theorem 5

First, note that as a consequence of Lemma 4, the deviation of $\hat{\mathbf{K}}(t)$ from the nominal Kalman gain $\mathbf{K}(t)$, defined as $\tilde{\mathbf{K}}(t) := \mathbf{K}(t) - \hat{\mathbf{K}}(t)$, converges exponentially fast to zero. Using (6) and (11), it can be shown that the time derivative of $\tilde{\mathbf{x}}(t)$ follows

$$\dot{\tilde{\boldsymbol{x}}}(t) = \mathbf{f}\left(t, \tilde{\boldsymbol{x}}(t)\right) + \tilde{\mathbf{f}}\left(t, \tilde{\boldsymbol{x}}(t), \tilde{\mathbf{C}}(t), \tilde{\mathbf{K}}(t)\right) + \boldsymbol{g}\left(t, \tilde{\boldsymbol{u}}(t), \tilde{\mathbf{C}}(t), \tilde{\mathbf{K}}(t)\right), \tag{45}$$

where

$$\mathbf{f}(t, \tilde{\boldsymbol{x}}(t)) = \left(\mathbf{A}(t) - \mathbf{K}(t)\mathbf{C}(t)\right)\tilde{\boldsymbol{x}}(t)$$

are the error dynamics of the nominal Kalman filter (8),

$$\tilde{\mathbf{f}}\left(t, \tilde{\boldsymbol{x}}(t), \tilde{\mathbf{C}}(t), \tilde{\mathbf{K}}(t)\right) = \left[\mathbf{K}(t)\tilde{\mathbf{C}}(t) + \tilde{\mathbf{K}}(t)\mathbf{C}(t) - \tilde{\mathbf{K}}(t)\tilde{\mathbf{C}}(t)\right]\tilde{\boldsymbol{x}}(t)$$

and

$$\boldsymbol{g}\left(t, \tilde{\boldsymbol{u}}(t), \tilde{\mathbf{C}}(t), \tilde{\mathbf{K}}(t)\right) = \left[\tilde{\mathbf{K}}(t)\tilde{\mathbf{C}}(t) - \mathbf{K}(t)\tilde{\mathbf{C}}(t)\right]\boldsymbol{x}(t) + \mathbf{B}(t)\tilde{\boldsymbol{u}}(t).$$

Notice that this last term is identically equal to zero when $\tilde{\boldsymbol{u}}(t) = \mathbf{0}$ and $\tilde{\boldsymbol{C}}(t) = \mathbf{0}$, and that it does not depend on the observer error $\tilde{\boldsymbol{x}}(t)$. Now consider the system

$$\dot{\tilde{\boldsymbol{x}}}(t) = \mathbf{f}(t, \tilde{\boldsymbol{x}}(t)) + \tilde{\mathbf{f}}\left(t, \tilde{\boldsymbol{x}}(t), \tilde{\mathbf{C}}(t), \tilde{\mathbf{K}}(t)\right). \tag{46}$$

This system can be regarded as a perturbation of the system $\dot{\tilde{\boldsymbol{x}}}(t) = \mathbf{f}(t, \tilde{\boldsymbol{x}}(t))$, which, under the assumptions of the theorem, is GES. Then, because the terms that multiply $\tilde{\boldsymbol{x}}(t)$ in $\tilde{\mathbf{f}}\left(t, \tilde{\boldsymbol{x}}(t), \tilde{\mathbf{C}}(t), \tilde{\mathbf{K}}(t)\right)$ converge exponentially fast to zero, there is a $t_1 \geqslant t_0$ for which the perturbed system (46) is also GES on the interval $(t_1, +\infty)$ [35, Lemma9.1]. Following this, because the system of error dynamics (45) is GES when $\tilde{\mathbf{u}}(t) = \mathbf{0}$ and $\tilde{\mathbf{C}}(t) = \mathbf{0}$ and, under the assumptions of the theorem, $\mathbf{B}(t)$, $\boldsymbol{x}(t)$, and $\mathbf{K}(t)$ are bounded, then for $t \geqslant t_1$, it is also ISS with $\tilde{\boldsymbol{u}}(t)$ and $\tilde{\mathbf{C}}(t)$ as inputs [35, Lemma 4.6]. Then, because $\tilde{\boldsymbol{u}}(t)$ and $\tilde{\mathbf{C}}(t)$ converge exponentially fast to zero, the error dynamics (45) also converge exponentially fast to the origin for $t \geqslant t_1$ [35, Lemma 4.7]. Finally, as under the assumptions of the theorem neither $\boldsymbol{x}(t)$ nor $\hat{\boldsymbol{x}}(t)$ can grow unbounded between t_0 and t_1, it follows that there exist positive scalar constants α and λ such that $\|\tilde{\boldsymbol{x}}(t)\| \leqslant \alpha e^{-\lambda(t-t_0)}$ for all $t \geqslant t_0$. □

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