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Position and Velocity Filters for ASC/I-AUV Tandems Based on Single Range Measurements

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Abstract This paper proposes novel cooperative navigation solutions for an Intervention Autonomous Underwater Vehicle (I-AUV) working in tandem with an Autonomous Surface Craft (ASC). The I-AUV is assumed to be moving in the presence of constant unknown ocean currents, and aims to estimate its position relying on measurements of its range to the ASC. Two different scenarios are considered: in the first scenario, the ASC transmits its position and velocity to the I-AUV, and the I-AUV has access to readings of its velocity relative to the water. In the second scenario, the ASC transmits only its position, and the I-AUV has access to measurements of its velocity relative to the ASC. In both cases, it is assumed that an Attitude and Heading Reference System (AHRS) mounted on-board the I-AUV provides measurements of its attitude and angular velocity. A sufficient condition for observability and a method for designing state observers with Globally Asymptotically Stable (GAS) error dynamics are presented for both problems. Finally, simulation results are included and discussed to assess the performance of the proposed solutions in the presence of measurement noise.

Keywords Cooperative navigation · Observability of nonlinear systems · Kalman filtering

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1 Introduction

In the past decades, navigation and control of autonomous vehicles has been the subject of intensive study, fostered by the myriad of potential applications they offer. One in particular, the development of Autonomous Underwater Vehicles (AUVs) with intervention capabilities, provides a valuable asset to fields as diverse as marine rescue, scientific missions, and offshore industries, among many others. In fact, most current applications use manned submersibles or remotely operated vehicles (ROVs), as the high level of autonomy and perception required for autonomous intervention has not yet been reached. However, both of these alternatives pose strong drawbacks. The use of manned submersibles requires putting humans in harm's way, and also implies limited autonomy at a high financial cost, due to the use of a support oceanographic vessel. For ROVs operation, although danger to the pilot is not a factor, their operation over extended periods of time is still limited by the fatigue that human operators experience. Besides that, the use of a ROV will also usually require, in addition to the support vessel, an automatic Tether Management System (TMS) and a Dynamic Position (DP) system. In this context, the goal of the EU project TRIDENT[1] is to design a team of two autonomous vehicles with complementary capabilities, an Autonomous Surface Craft (ASC) and an Intervention Autonomous Underwater Vehicle (I-AUV), equipped with a dexterous manipulator, which will be used to carry out underwater manipulation tasks. This paper proposes a novel navigation solution for the I-AUV based on single range readings to the ASC, aided by auxiliary movement sensors and communication between the two robots.

Over the last few years, several approaches to the problem have been studied, see [11] for a recent, comprehensive survey on the field. One of those is Dead Reckoning (DR), whose performance is very good in the short-term but necessarily degrades over time, see [12]. Another solution is localization aided by artificial beacons that can be disposed in several configurations such as Long Base Line (LBL), Short Base Line (SBL), and Ultra-Short Base Line (USBL), see e.g. [8, 9, 13] and [15]. Although this approach yields good long-term results, it poses several operational constraints that may be overwhelming in many applications. In fact, the beacons must be previously deployed in the planned area of operation, and their position must be carefully and precisely asserted. In the context of project TRIDENT, a navigation solution was needed for the I-AUV based on range measurements to a single mobile source, the ASC. Motivated by this, this paper extends the results presented by the authors in [5], which addressed the stationary source case, to the mobile source case, and addresses the problem for two different scenarios, depending on the information that is exchanged between the agents. Besides that, the effect of the addition of a depth sensor to the I-AUV's sensor suite is studied, allowing for operation of the I-AUV at constant depth with no loss of observability. The proposed solution is also compared to the Extended Kalman Filter (EKF), as the latter is the more commonly used solution for range-based localization. Preliminary results on this subject were presented in [18], and in [17] a formation composed by multiple AUVs was considered. In recent years, several research groups have studied the problem of cooperative range-based localization. Several EKF-based solutions have been proposed: in [19], a centralized EKF solution is backed by extensive experimental results, while both [1] and [7] achieve sufficient conditions for weak local observability of the nonlinear dynamics. Other approaches include the one proposed in [2], in which successive measurements at different points of the trajectory are used to recover the present position, while [20] studies the minimum number of distinct range measurements to compute both relative position and orientation in a 2-D setting. This paper presents an alternative solution, in which the navigation system on-board the I-AUV is designed with respect to the ASC. Assuming the ASC has its own navigation system (see e.g. [16] for an experimentally validated navigation solution for the DELFIMx, an ASC associated with project TRIDENT), communication with the I-AUV will allow the latter to recover its absolute position. In comparison with the above-referenced works, the novelty of the proposed cooperative localization solution resides in two main points: the framework that is employed allows for the derivation of observability conditions that hold globally, and the proposed state estimation solution features globally asymptotically stable error dynamics, while still achieving similar filtering performance as the EKF.

This paper addresses the problem of cooperative navigation of an I-AUV/ASC tandem, in

[1] http://www.irs.uji.es/trident/

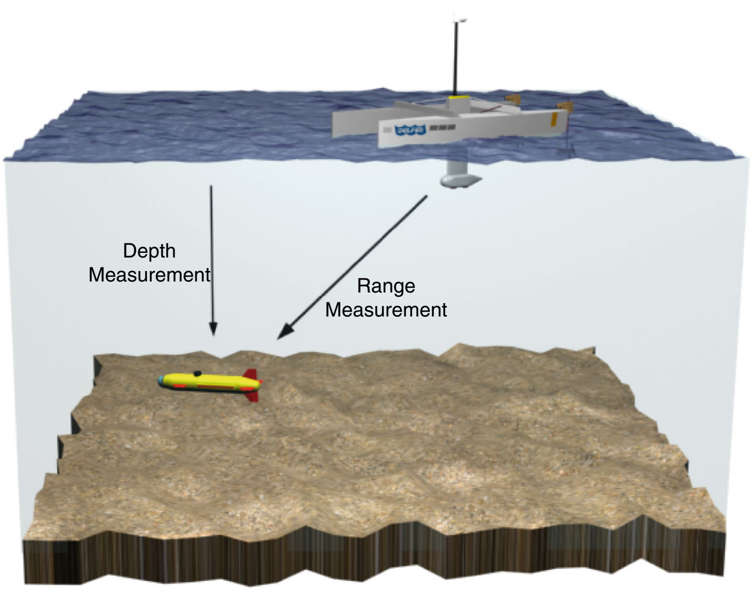
which the I-AUV estimates its position relying on measurements of its range to the ASC, aided by auxiliary sensors and transmission of relevant data between the ASC and the I-AUV (Fig. 1). Two scenarios are considered: in the first, the ASC transmits both its position and velocity to the I-AUV, which is assumed to be moving in the presence of constant unknown drifts. It is assumed that the I-AUV has access to a measurement of its velocity relative to the water, given e.g. by a Doppler Velocity Log (DVL). In the second scenario, the ASC transmits only its position to the I-AUV, which is assumed to have access to readings of its velocity relative to the ASC, given e.g. by an Acoustic Vector-Sensor Array (AVSA), see [14]. As depth measurements are easy to obtain in underwater applications, both cases are studied with and without a depth sensor, to assess the benefits brought by the additional depth measurement. The dynamics of the problem are derived from the linear motion kinematics of vehicles moving in 3-D, which are exact, and nonlinear space-state representations are presented for both scenarios. Considering this, and even though the present work is strongly motivated by the case of an I-AUV/ASC tandem in the scope of project TRIDENT, the results presented here can be extended to other classes of vehicles and/or sensors. The observability of the nonlinear systems is studied using state augmentation. By carefully defining new state variables, it is possible to derive Linear Time-Varying (LTV) systems which mimic exactly the corresponding nonlinear ones. Classical estimation and filtering theory is then applied to the LTV systems, allowing for the derivation of sufficient conditions for observability for both cases. As the observability analysis is carried out on linear systems, it allows for the design of state observers with estimation errors whose dynamics are guaranteed to be Globally Asymptotically Stable (GAS), which would not be possible using classical methods such as the EKF. Following this, a Kalman filter for the LTV systems with GAS error dynamics is proposed, and its performance is assessed in simulation.

The paper is organized as follows: Section 2 introduces the problem dynamics, while Section 3 details the observability analysis of the two nonlinear systems. Section 4 presents simulation results to assess the performance of the proposed solution and, finally, Section 5 summarizes the main conclusions of the paper.

1.1 Notation

In the paper, the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix

Fig. 1 I-AUV working in tandem with an ASC



is represented as diag $(\mathbf{A}_1, \ldots, \mathbf{A}_n)$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \times \mathbf{y}$ represents the cross product, and $\mathbf{x} \cdot \mathbf{y}$ represents the inner product.

Throughout the text, $\{I\}$ is used to denote an inertial reference coordinate frame and $\{B\}$ denotes the body-fixed coordinate frame attached to the I-AUV. Some of the notation used in the paper is summarized here for reference:

- $\mathbf{p}(t) \in \mathbb{R}^3$: inertial position of the I-AUV, in inertial coordinates
- $\mathbf{v}(t) \in \mathbb{R}^3$: velocity of the I-AUV relative to $\{I\}$, in body-fixed coordinates
- $\mathbf{R}(t) \in SO(3)$: rotation matrix from $\{B\}$ to $\{I\}$
- $\boldsymbol{\omega}(t) \in \mathbb{R}^3$: angular velocity of the I-AUV, in body-fixed coordinates
- $\mathbf{s}(t) \in \mathbb{R}^3$: inertial position of the ASC, in inertial coordinates
- $\mathbf{v}_s(t) \in \mathbb{R}^3$: velocity of the ASC relative to $\{I\}$, in inertial coordinates
- $\mathbf{r}(t) := \mathbf{R}^T(t)[\mathbf{s}(t) - \mathbf{p}(t)] \in \mathbb{R}^3$: position of the ASC relative to the I-AUV, in body-fixed coordinates
- $r(t) := \|\mathbf{r}(t)\| \in \mathbb{R}$: distance, or range, between the I-AUV and the ASC
- $z(t) := \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{p}(t)$: depth of the I-AUV, in inertial coordinates
- $\mathbf{v}_f(t) \in \mathbb{R}^3$: velocity of the fluid, in body-fixed coordinates

In the two scenarios that are considered, it is assumed that the following quantities are available to the I-AUV for observer design purposes: $\mathbf{s}(t)$, $r(t)$, $\mathbf{R}(t)$, and $\boldsymbol{\omega}(t)$.

Additionally, in the first case considered in the paper, the I-AUV has access to $\mathbf{v}_r(t) \in \mathbb{R}^3$, its velocity relative to the fluid in body-fixed coordinates, as provided by a DVL, and to $\mathbf{v}_s(t)$, communicated by the ASC. In this case, the following notation is also used:

- $\mathbf{x}_1^a(t) := \mathbf{R}(t)\mathbf{r}(t) \in \mathbb{R}^3$
- $\mathbf{x}_2^a(t) := -\mathbf{R}(t)\mathbf{v}_f(t) \in \mathbb{R}^3$
- $\mathbf{u}_a(t) := \mathbf{v}_s(t) - \mathbf{R}(t)\mathbf{v}_r(t) \in \mathbb{R}^3$

In the second case considered in the paper, the I-AUV has access to $\Delta\mathbf{v}(t) := \mathbf{R}^T(t)\mathbf{v}_s(t) - \mathbf{v}(t) \in \mathbb{R}^3$, that is, the velocity of the ASC relative to the I-AUV, expressed in body-fixed coordinates. In this case, the following notation is used:

- $\mathbf{x}_1^b(t) := \mathbf{R}(t)\mathbf{r}(t) \in \mathbb{R}^3$
- $\mathbf{u}_b(t) := \mathbf{R}(t)\Delta\mathbf{v}(t) \in \mathbb{R}^3$

2 Problem Statement

Consider an I-AUV moving underwater in the presence of constant unknown ocean currents and working in tandem with an ASC. It is assumed that the ASC has a built-in navigation system which provides accurate estimates of both its inertial position and velocity, and can communicate them to the I-AUV (using, e.g., an acoustic modem). Suppose that the I-AUV has access to a measurement of its distance, or range, $r(t) \in \mathbb{R}$ to the ASC. The problem considered in this paper is that of estimating the position of the ASC relative to the I-AUV using the range sensor as a navigation aiding device. The I-AUV can then recover its own position by comparison with the inertial position of the ASC, received through communication.

In addition to the range measurement, it is assumed that the I-AUV has a sensor suite mounted on-board to provide measurements regarding its movement. Attitude and angular velocity data is assumed to be provided by an Attitude and Heading Reference System (AHRS). Regarding the linear velocity of the I-AUV, two distinct cases are considered: in the first, the I-AUV uses a DVL to measure its velocity relative to the water. In the second case, the I-AUV has access to measurements of its velocity relative to the ASC, provided e.g. by an AVSA. Finally, as depth measurements are commonplace in underwater applications, each of the two aforementioned cases will be studied with and without a measurement of the depth of the I-AUV relative to the sea level, $z(t) \in \mathbb{R}$, to assess the benefits brought by the additional sensor.

2.1 Linear Motion Kinematics of the I-AUV

Let $\{I\}$ denote an inertial reference coordinate frame and $\{B\}$ a coordinate frame attached to the I-AUV, usually denominated as the body-fixed coordinate frame. The linear motion of the I-AUV can be written as

$$\dot{\mathbf{p}}(t) = \mathbf{R}(t)\mathbf{v}(t),$$

where $\mathbf{p}(t) \in \mathbb{R}^3$ denotes the inertial position of the I-AUV, $\mathbf{v}(t) \in \mathbb{R}^3$ is the velocity of the I-AUV relative to $\{I\}$ and expressed in body-fixed coordinates, and $\mathbf{R}(t) \in SO(3)$ is the rotation matrix from $\{B\}$ to $\{I\}$, which verifies

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}(\boldsymbol{\omega}(t)),$$

where $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is the angular velocity of the I-AUV, expressed in body-fixed coordinates, and $\mathbf{S}(\boldsymbol{x}) \in \mathbb{R}^{3\times 3}$ is the skew-symmetric matrix such that $\mathbf{S}(\boldsymbol{x})\,\boldsymbol{y}$ is the cross product $\boldsymbol{x} \times \boldsymbol{y}$, that is

$$\mathbf{S}(\boldsymbol{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

Let $\mathbf{s}(t) \in \mathbb{R}^3$ denote the inertial position of the ASC, and $\mathbf{v}_s(t) \in \mathbb{R}^3$ its inertial velocity. Then, the range of the I-AUV to the ASC is given by $r(t) = \|\mathbf{r}(t)\|$, where

$$\mathbf{r}(t) = \mathbf{R}^T(t)[\mathbf{s}(t) - \mathbf{p}(t)] \in \mathbb{R}^3 \tag{1}$$

is the position of the ASC relative to the I-AUV, expressed in body-fixed coordinates, precisely the quantity that the I-AUV aims to estimate. The time derivative of Eq. 1 is given by

$$\dot{\mathbf{r}}(t) = -\mathbf{S}(\boldsymbol{\omega}(t))\mathbf{r}(t) + \mathbf{R}^T(t)\mathbf{v}_s(t) - \mathbf{v}(t).$$

2.2 Nonlinear Dynamics of the Problem

For the first case, in which the I-AUV uses a DVL to measure its velocity relative to the water, let $\mathbf{v}_r(t) \in \mathbb{R}^3$ and $\mathbf{v}_f(t) \in \mathbb{R}^3$ denote the velocity of the I-AUV relative to the water and the constant unknown velocity of the water current relative to $\{I\}$, respectively, both expressed in body-fixed coordinates. Considering that the velocity of the water is constant in inertial coordinates, it is possible to further write

$$\begin{cases} \dot{\mathbf{r}}(t) = -\mathbf{S}(\boldsymbol{\omega}(t))\mathbf{r}(t) + \mathbf{R}^T(t)\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_f(t) \\ \dot{\mathbf{v}}_f(t) = -\mathbf{S}(\boldsymbol{\omega}(t))\mathbf{v}_f(t) \end{cases}.$$

Define the state transformation

$$\mathbf{T}_a(t) := \begin{bmatrix} \mathbf{R}(t) & \mathbf{0} \\ \mathbf{0} & -\mathbf{R}(t) \end{bmatrix} \in \mathbb{R}^{6\times 6},$$

which is a Lyapunov state transformation, and as such preserves stability and observability properties (see [6]). Let

$$\begin{bmatrix} \mathbf{x}_1^a(t) \\ \mathbf{x}_2^a(t) \end{bmatrix} := \mathbf{T}_a(t) \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}_f(t) \end{bmatrix},$$

and define

$$\mathbf{u}_a(t) := \mathbf{v}_s(t) - \mathbf{R}(t)\mathbf{v}_r(t) \in \mathbb{R}^3.$$

Computing the time derivatives of $\mathbf{x}_1^a(t)$ and $\mathbf{x}_2^a(t)$ and taking the range measurement $r(t)$ as an output gives the nonlinear system

$$\begin{cases} \dot{\mathbf{x}}_1^a(t) = \mathbf{x}_2^a(t) + \mathbf{u}_a(t) \\ \dot{\mathbf{x}}_2^a(t) = \mathbf{0} \\ y_1(t) = \|\mathbf{x}_1^a(t)\| \end{cases}, \tag{2}$$

where $\mathbf{x}_1^a(t),\ \mathbf{x}_2^a(t) \in \mathbb{R}^3$ are the system states, $\mathbf{u}_a(t)$ is the system input, and $y_1(t) \in \mathbb{R}$ represents the system output. With an additional depth measurement, the system becomes

$$\begin{cases} \dot{\mathbf{x}}_1^a(t) = \mathbf{x}_2^a(t) + \mathbf{u}_a(t) \\ \dot{\mathbf{x}}_2^a(t) = \mathbf{0} \\ \mathbf{y}_2(t) = \begin{bmatrix} \|\mathbf{x}_1^a(t)\| \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x}_1^a(t) \end{bmatrix} \end{cases}, \tag{3}$$

where $\mathbf{y}_2(t) \in \mathbb{R}^2$ is the system output.

Remark 1 Incorporating the velocity of the constant ocean current as a state in the dynamics of the problem may appear questionable. However, since it is assumed to be unknown, it is necessary to estimate its value, and one way to do it is to incorporate it directly in the filter design process. Moreover, as it will be seen in more detail in Section 4.1 (Filter Design), when reverting the Lyapunov state transformation to implement the state observer in the I-AUV's coordinate space, the velocity of the water surrounding it becomes a time-varying quantity, making this design choice all the more relevant.

Regarding the second case, the I-AUV has access to measurements of its velocity relative to the ASC,

$$\Delta\mathbf{v}(t) := \mathbf{R}^T(t)\mathbf{v}_s(t) - \mathbf{v}(t) \in \mathbb{R}^3,$$

expressed in body-fixed coordinates, so the time derivative of $\mathbf{r}(t)$ is reduced to

$$\dot{\mathbf{r}}(t) = -\mathbf{S}(\boldsymbol{\omega}(t))\mathbf{r}(t) + \Delta\mathbf{v}(t).$$

Now, consider another state transformation matrix

$$\mathbf{T}_b(t) := \mathbf{R}(t) \in \mathbb{R}^3,$$

which also defines a Lyapunov state transformation, given by

$$\mathbf{x}_1^b(t) := \mathbf{T}_b(t)\mathbf{r}(t),$$

and define

$$\mathbf{u}_b(t) := \mathbf{R}(t)\Delta\mathbf{v}(t).$$

Computing the time derivative of $\mathbf{x}_1^b(t)$ and taking $r(t)$ as the output gives the nonlinear system

$$\begin{cases} \dot{\mathbf{x}}_1^b(t) = \mathbf{u}_b(t) \\ y_1(t) = \|\mathbf{x}_1^b(t)\| \end{cases}, \tag{4}$$

where $\mathbf{x}_1^b(t) \in \mathbb{R}^3$ is the system state, $\mathbf{u}_b(t)$ is the system input, and $y_1(t)$ is the system output. With the depth measurement as an additional output, the system becomes

$$\begin{cases} \dot{\mathbf{x}}_1^b(t) = \mathbf{u}_b(t) \\ \mathbf{y}_2(t) = \begin{bmatrix} \|\mathbf{x}_1^b(t)\| \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x}_1^b(t) \end{bmatrix} \end{cases}. \tag{5}$$

The problems considered in this paper are the observability analysis of the nonlinear systems (2), (3), (4), and (5), and the design of state observers for those systems.

Remark 2 The nonlinear systems (4) and (5) are special cases of Eqs. 2 and 3, respectively, with $\mathbf{x}_2^a(t_0) = \mathbf{0}$, so any sufficient conditions for observability derived for the latter will also be valid for the former. Nevertheless, less restrictive sufficient conditions the observability of Eqs. 4 and 5 can be found, as it will be detailed in the following sections.

3 Observability Analysis

This section details the observability analysis of Eqs. 2, 3, 4, and 5 through state augmentation. With the proposed methodologies, it is possible to derive linear systems which capture the behavior of the nonlinear systems, and as such study their observability in a linear systems framework.

3.1 State Augmentation

In this subsection, state augmentation is performed on the nonlinear system dynamics of both cases, resulting in LTV systems. The subsection is divided in two parts, one for each case. The first part details the state augmentation of the nonlinear systems (2) and (3), while the second part deals with the state augmentation of Eqs. 4 and 5.

3.1.1 State Augmentation—First Case (Velocity Relative to the Water and Velocity of the ASC)

To derive linear systems that mimic the dynamics of the nonlinear systems (2) and (3), define three additional scalar state variables

$$\begin{cases} x_3^a(t) := r(t) \\ x_4^a(t) := \mathbf{x}_1^a(t) \cdot \mathbf{x}_2^a(t) \\ x_5^a(t) := \|\mathbf{x}_2^a(t)\|^2 \end{cases}$$

and denote by

$$\mathbf{x}_a(t) := \begin{bmatrix} \mathbf{x}_1^a(t) \\ \mathbf{x}_2^a(t) \\ x_3^a(t) \\ x_4^a(t) \\ x_5^a(t) \end{bmatrix} \in \mathbb{R}^{n_a}, \; n_a = 9,$$

the augmented state. The time derivatives of the new state variables are

$$\begin{cases} \dot{x}_3^a(t) = \dfrac{1}{r(t)}\left[x_4^a(t) + \mathbf{u}_a(t) \cdot \mathbf{x}_1^a(t)\right] \\ \dot{x}_4^a(t) = \mathbf{u}_a(t) \cdot \mathbf{x}_2^a(t) + x_5^a(t) \\ \dot{x}_5^a(t) = 0 \end{cases}.$$

Then, the dynamics of the augmented system corresponding to the nonlinear system (2) can be written as

$$\begin{cases} \dot{\mathbf{x}}_a(t) = \mathbf{A}_a(t)\mathbf{x}_a(t) + \mathbf{B}_a\mathbf{u}_a(t) \\ y_1(t) = \mathbf{C}_{a1}\mathbf{x}_a(t) \end{cases}, \tag{6}$$

where

$$\mathbf{A}_a(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \dfrac{\mathbf{u}_a^T(t)}{r(t)} & \mathbf{0} & 0 & \dfrac{1}{r(t)} & 0 \\ \mathbf{0} & \mathbf{u}_a^T(t) & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n_a \times n_a},$$

$$\mathbf{B}_a = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n_a \times 3},$$

and

$$\mathbf{C}_{a1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{1 \times n_a}.$$

Naturally, the dynamics of the augmented system corresponding to the nonlinear system (3) follow

$$\begin{cases} \dot{\mathbf{x}}_a(t) = \mathbf{A}_a(t)\mathbf{x}_a(t) + \mathbf{B}_a \mathbf{u}_a(t) \\ \mathbf{y}_2(t) = \mathbf{C}_{a2}\mathbf{x}_a(t) \end{cases}, \tag{7}$$

where $\mathbf{A}_a(t)$ and $\mathbf{B}_a$ are defined as in Eq. 6, and

$$\mathbf{C}_{a2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \\ [0\ 0\ 1] & \mathbf{0} & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{2 \times n_a}.$$

As the division by $r(t)$ in some entries of the state matrices introduced above creates a singularity when $r(t) = 0$, the following assumption is necessary for analysis purposes.

Assumption 1 *The range measurements obey $r(t) > \epsilon$, $\forall t$, where ϵ is a positive scalar constant.*

In practical terms, this is a very mild assumption, as its violation would translate in a collision between the I-AUV and the ASC. On the other hand, in potential real-world applications very small values for $r(t)$ could cause numerical problems in software implementations of the proposed filtering algorithms. In most cases this is not a real concern, as the I-AUV will always be kept at a safe distance from the ASC to avoid any collisions. However, if such numerical problems are a concern, using $r^2(t)$ instead instead of $r(t)$ in the computations will get rid of the potential singularities at the expense of filtering performance, see [3].

Remark 3 Notice that there is equivalence between Eqs. 6 and 2, and between Eqs. 7 and 3, if only if the algebraic restrictions

$$\begin{cases} x_3^a(t) = \|\mathbf{x}_1^a(t)\| \\ x_4^a(t) = \mathbf{x}_1^a(t) \cdot \mathbf{x}_2^a(t) \\ x_5^a(t) = \|\mathbf{x}_2^a(t)\|^2 \end{cases} \tag{8}$$

are verified. Thus, observability of Eqs. 6 and 7 do not automatically entail the observability of Eqs. 2 and 3, respectively. The approach taken in this paper to derive sufficient conditions for observability for the nonlinear systems (2) and (3) can be described succinctly as follows: first, derive a sufficient condition for observability of the augmented LTV system or, in other words, a condition that guarantees that the initial condition of the LTV system is uniquely defined. Then, prove equivalence between the nonlinear system and its augmented LTV counterpart. To do so, nothing can be assumed about the initial condition of the nonlinear system. Instead, one must compare what is already known to be equivalent between the two systems, that is, the outputs, and show that if the sufficient condition for observability is met, then the initial conditions of both systems are equal, and the algebraic restrictions in Eq. 8 are verified. This method is detailed in the rest of this section.

3.1.2 State Augmentation—Second Case (Velocity Relative to the ASC)

Regarding the second case, the systems are simpler, and therefore only one additional scalar state variable is needed,

$$x_2^b(t) := r(t),$$

and the augmented state is defined as

$$\mathbf{x}_b(t) := \begin{bmatrix} \mathbf{x}_1^b(t) \\ x_2^b(t) \end{bmatrix} \in \mathbb{R}^{n_b}, n_b = 4.$$

The time derivative of $x_2^b(t)$ is given by

$$\dot{x}_2^b(t) = \frac{\mathbf{u}_b(t) \cdot \mathbf{x}_1^b(t)}{r(t)},$$

thus the dynamics of the augmented system corresponding to the nonlinear system (4) can be written in the form

$$\begin{cases} \dot{\mathbf{x}}_b(t) = \mathbf{A}_b(t)\mathbf{x}_b(t) + \mathbf{B}_b\mathbf{u}_b(t) \\ y_1(t) = \mathbf{C}_{b1}\mathbf{x}_b(t) \end{cases}, \tag{9}$$

where

$$\mathbf{A}_b(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \frac{\mathbf{u}_b^T(t)}{r(t)} & 0 \end{bmatrix} \in \mathbb{R}^{n_b \times n_b}, \quad \mathbf{B}_b = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n_b \times 3},$$

and

$$\mathbf{C}_{b1} = \begin{bmatrix} \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^{1 \times n_b}.$$

The dynamics of the augmented system corresponding to the nonlinear system (5) follow

$$\begin{cases} \dot{\mathbf{x}}_b(t) = \mathbf{A}_b(t)\mathbf{x}_b(t) + \mathbf{B}_b\mathbf{u}_b(t) \\ \mathbf{y}_2(t) = \mathbf{C}_{b2}\mathbf{x}_b(t) \end{cases}, \tag{10}$$

where $\mathbf{A}_b(t)$ and $\mathbf{B}_b$ are defined as in Eq. 9, and

$$\mathbf{C}_{b2} = \begin{bmatrix} \mathbf{0} & 1 \\ [0\ 0\ 1] & 0 \end{bmatrix} \in \mathbb{R}^{2 \times n_b}.$$

Note that there is nothing in Eqs. 9 and 10 imposing $x_2^b(t) = \|\mathbf{x}_1^b(t)\|$, so care must be taken when extrapolating conclusions from the observability of the LTV systems (9) and (10) to the observability of the nonlinear systems (4) and (5), respectively.

Remark 4 Even though $\mathbf{A}_a(t)$ from Eqs. 6 and 7, as well as $\mathbf{A}_b(t)$ from Eqs. 9 and 10, depend on the system input and output, they can still be regarded as LTV systems for analysis purposes, as all quantities involved are known, see [5, Lemma 1].

3.2 Observability of the LTV Systems

This subsection details the analysis of the observability of the LTV systems derived in the previous subsection. The subsection is divided in two parts: in the first part, sufficient conditions for the observability of the LTV systems (6) and (7) are presented, while the second part details sufficient conditions for the observability of the LTV system (9) and (10).

3.2.1 Observability of the LTV Systems—First Case (Velocity Relative to the Water and Velocity of the ASC)

Before showing sufficient conditions for the observability of Eqs. 6 and 7, it is convenient to compute the transition matrix associated with $\mathbf{A}_a(t)$. To simplify the derivation of results, define

$$\mathbf{u}_a^{[1]}(t,t_0) := \int_{t_0}^{t} \mathbf{u}_a(\sigma)d\sigma = \begin{bmatrix} u_{a1}^{[1]}(t,t_0) \\ u_{a2}^{[1]}(t,t_0) \\ u_{a3}^{[1]}(t,t_0) \end{bmatrix} \in \mathbb{R}^3.$$

The transition matrix for $\mathbf{A}_a(t)$ is given by the Peano-Baker series

$$\begin{aligned} \boldsymbol{\phi}_a(t,t_0) &:= \mathbf{I} + \int_{t_0}^{t} \mathbf{A}_a(s_1)ds_1 \\ &+ \int_{t_0}^{t} \mathbf{A}_a(s_1)\int_{t_0}^{s_1} \mathbf{A}_a(s_2)ds_2ds_1 \\ &+ \int_{t_0}^{t} \mathbf{A}_a(s_1)\int_{t_0}^{s_1} \mathbf{A}_a(s_2)\int_{t_0}^{s_2} \mathbf{A}_a(s_3)ds_3ds_2ds_1 + \dots . \end{aligned} \tag{11}$$

As all terms of Eq. 11 after the first three are identically zero, it can be shown that $\boldsymbol{\phi}_a(t,t_0)$ follows

$$\boldsymbol{\phi}_a(t,t_0) = \begin{bmatrix} \mathbf{I} & (t-t_0)\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \int_{t_0}^{t} \frac{\mathbf{u}_a^T(\sigma)}{r(\sigma)}d\sigma & \int_{t_0}^{t} \frac{[\mathbf{u}_a^{[1]}(\sigma,t_0)]^T + (\sigma-t_0)\mathbf{u}_a^T(\sigma)}{r(\sigma)}d\sigma & 1 & \int_{t_0}^{t} \frac{1}{r(\sigma)}d\sigma & \int_{t_0}^{t} \frac{\sigma-t_0}{r(\sigma)}d\sigma \\ \mathbf{0} & \left[\mathbf{u}_a^{[1]}(t,t_0)\right]^T & 0 & 1 & t-t_0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 1 \end{bmatrix}.$$

The observability Gramian for the pair $(\mathbf{A}_a(t), \mathbf{C}_{a1})$ is given by

$$\mathcal{W}_{a1}(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\phi}_a^T(t, t_0)\mathbf{C}_{a1}^T\mathbf{C}_{a1}\boldsymbol{\phi}_a(t, t_0)dt.$$

Note that, attending to the structure of $\boldsymbol{\phi}_a(t, t_0)$ and $\mathbf{C}_{a1}$, the above expression can be reduced to

$$\mathcal{W}_{a1}(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\psi}_{a1}^T(t, t_0)\boldsymbol{\psi}_{a1}(t, t_0)dt, \tag{12}$$

where

$$\boldsymbol{\psi}_{a1}(t,t_0)=\left[\int_{t_0}^{t} \frac{\mathbf{u}_a^T(\sigma)}{r(\sigma)}d\sigma \;\; \int_{t_0}^{t} \frac{[\mathbf{u}_a^{[1]}(\sigma,t_0)]^T+(\sigma-t_0)\mathbf{u}_a^T(\sigma)}{r(\sigma)}d\sigma \;\; 1 \;\; \int_{t_0}^{t}\frac{1}{r(\sigma)}d\sigma \;\; \int_{t_0}^{t}\frac{\sigma-t_0}{r(\sigma)}d\sigma\right].$$

Similarly, the observability Gramian for the pair $(\mathbf{A}_a(t), \mathbf{C}_{a2})$ can be shown to follow

$$\mathcal{W}_{a2}(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\psi}_{a2}^T(t, t_0)\boldsymbol{\psi}_{a2}(t, t_0)dt,$$

where

$$\boldsymbol{\psi}_{a2}(t, t_0) = \begin{bmatrix} \boldsymbol{\psi}_{a1}(t, t_0) \\ \boldsymbol{\phi}_{az}(t, t_0) \end{bmatrix},$$

in which

$$\boldsymbol{\phi}_{az}(t, t_0) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & (t-t_0) & 0 & 0 & 0 \end{bmatrix}.$$

The following result presents a sufficient condition for the observability of the LTV system (6).

Theorem 1 *Suppose that the set of functions*

$$\begin{aligned}\mathcal{F}_{a1} = \Big\{&(t-t_0),\ (t-t_0)^2,\ u_{a1}^{[1]}(t,t_0),\ u_{a2}^{[1]}(t,t_0),\\ &u_{a3}^{[1]}(t,t_0),\ (t-t_0)u_{a1}^{[1]}(t,t_0),\\ &(t-t_0)u_{a2}^{[1]}(t,t_0),\ (t-t_0)u_{a3}^{[1]}(t,t_0)\Big\}\end{aligned} \tag{13}$$

is linearly independent on $[t_0, t_f]$, $t_0 < t_f$. *Then, the linear time-varying system* (6) *is observable on* $[t_0, t_f]$ *in the sense that, given the system input* $\mathbf{u}_a(t)$, $t \in [t_0, t_f]$, *and the system output* $y_1(t)$, $t \in [t_0, t_f]$, *the initial condition is uniquely defined.*

Proof Suppose that the LTV system (6) is not observable on $[t_0, t_f]$. Then, the observability Gramian $\mathcal{W}_{a1}(t_0, t_f)$ is not positive definite and therefore there exists a constant $\mathbf{d} \in \mathbb{R}^{n_a}$, $\|\mathbf{d}\| = 1$ such that

$$\mathbf{d}^T\mathcal{W}_{a1}(t_0, t)\mathbf{d} = 0,\ \forall t \in [t_0, t_f]. \tag{14}$$

Let

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} \in \mathbb{R}^{n_a}, \quad \text{with } \mathbf{d}_1 = \begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \end{bmatrix}$$

$$\text{and } \mathbf{d}_2 = \begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \end{bmatrix}.$$

Expanding Eq. 14 and using Eq. 12 gives

$$\int_{t_0}^{t} [\boldsymbol{\psi}_{a1}(\sigma, t_0)\mathbf{d}]^T\boldsymbol{\psi}_{a1}(\sigma, t_0)\mathbf{d}\ d\sigma = 0,\ \forall t \in [t_0, t_f],$$

and it follows that

$$\boldsymbol{\psi}_{a1}(t, t_0)\mathbf{d} = 0,\ \forall t \in [t_0, t_f]. \tag{15}$$

But

$$\boldsymbol{\psi}_{a1}(t_0, t_0)\mathbf{d} = d_3,$$

so for Eq. 15 to hold, it must be $d_3 = 0$. From Eq. 15, it also follows that

$$\frac{d}{dt}\boldsymbol{\psi}_{a1}(t, t_0)\mathbf{d} = 0,\ \forall t \in [t_0, t_f],$$

yielding

$$\begin{aligned}0 = \mathbf{u}_a(t)\cdot\mathbf{d}_1 + [\mathbf{u}_a^{[1]}(t,t_0) + (t-t_0)\mathbf{u}_a(t)]\cdot\mathbf{d}_2\\ + d_4 + (t-t_0)d_5,\ \forall t \in [t_0, t_f].\end{aligned} \tag{16}$$

Integrating both sides of Eq. 16 gives

$$\begin{aligned}0 = \mathbf{u}_a^{[1]}(t,t_0)\cdot\mathbf{d}_1 + (t-t_0)\mathbf{u}_a^{[1]}(t,t_0)\cdot\mathbf{d}_2 + (t-t_0)d_4\\ + \frac{(t-t_0)^2}{2}d_5,\ \forall t \in [t_0, t_f],\end{aligned}$$

which implies that the set of functions $\mathcal{F}_{a1}$ is not linearly independent on $[t_0, t_f]$. Then, if the set of functions $\mathcal{F}_{a1}$ is linearly independent on $[t_0, t_f]$, the observability Gramian must be positive definite, and therefore Eq. 6 is observable, see [5, Lemma 1]. □

Remark 5 It may not be obvious what the abovementioned linearly independence condition entails from a practical point of view. However, expanding the integral

$$\mathbf{u}_a^{[1]}(t, t_0) = \int_{t_0}^{t} \mathbf{u}_a(\sigma) d\sigma = \int_{t_0}^{t} \mathbf{v}_s(\sigma) - \mathbf{R}(\sigma)\mathbf{v}_r(\sigma) d\sigma,$$

it is clear that linear independence between the three components of $\mathbf{u}_a^{[1]}(t, t_0)$ means that the 3D motion of the I-AUV relative to the ASC must be sufficiently rich in order to guarantee observability. This condition imposes restrictions on the relative movement of the vehicles, but it is needed when doing localization with single range readings. In solutions with multiple sources, such as USBL and LBL navigation systems, the presence of a sufficient number of landmarks allows the use of triangulation to recover the position. When there is only one landmark, a rich enough trajectory is required to perform some kind of implicit triangulation over time and estimate the position of the vehicle, as well as isolating the effect of the unknown current from the vehicle's actuation. A similar observation can be made for the subsequent cases. For example, the presence of a depth measurement in the next result allows the I-AUV to operate at constant depth, as richness of trajectory is only needed in the horizontal plane.

The following result presents a sufficient condition for the observability of Eq. 7.

Corollary 1 *Suppose that the set of functions*

$$\mathcal{F}_{a2} = \Big\{(t - t_0), (t - t_0)^2, u_{a1}^{[1]}(t, t_0), u_{a2}^{[1]}(t, t_0), (t - t_0)u_{a1}^{[1]}(t, t_0), (t - t_0)u_{a2}^{[1]}(t, t_0)\Big\} \tag{17}$$

is linearly independent on $[t_0, t_f], t_0 < t_f$. Then, the linear time-varying system (7) is observable on $[t_0, t_f]$ in the sense that, given the system input $\mathbf{u}_a(t)$, $t \in [t_0, t_f]$, and the system output $\mathbf{y}_2(t)$, $t \in [t_0, t_f]$, the initial condition is uniquely defined.

Proof Suppose that the LTV system (7) is not observable on $[t_0, t_f]$. Then, the observability Gramian $\boldsymbol{\mathcal{W}}_{a2}(t_0, t_f)$ is not positive definite and therefore there exists a constant $\mathbf{d} \in \mathbb{R}^{n_a}$, $\|\mathbf{d}\| = 1$ such that

$$\mathbf{d}^T \boldsymbol{\mathcal{W}}_{a2}(t_0, t)\mathbf{d} = 0, \ \forall\, t \in [t_0, t_f].$$

It follows that

$$\begin{cases} \boldsymbol{\psi}_{a1}(t, t_0)\mathbf{d} = 0 \\ \boldsymbol{\phi}_{az}(t, t_0)\mathbf{d} = 0 \end{cases}, \ \forall t \in [t_0, t_f]. \tag{18}$$

Note that

$$\boldsymbol{\phi}_{az}(t_0, t_0)\mathbf{d} = d_{13},$$

so it follows that $d_{13} = 0$. From Eq. 18, it also follows that

$$\frac{d}{dt}\boldsymbol{\phi}_{az}(t, t_0)\mathbf{d} = 0, \ \forall t \in [t_0, t_f],$$

which means that $d_{23} = 0$. By proceeding as in the proof of Theorem 1, it is straightforward to show that $d_3 = 0$, and that

$$0 = \mathbf{u}_a^{[1]}(t, t_0) \cdot \mathbf{d}_1 + (t - t_0)\mathbf{u}_a^{[1]}(t, t_0) \cdot \mathbf{d}_2 + (t - t_0)d_4 + \frac{(t - t_0)^2}{2} d_5, \ \forall t \in [t_0, t_f]. \tag{19}$$

In this case, since $d_{13} = d_{23} = 0$, Eq. 19 can be rewritten as

$$0 = u_{a1}^{[1]}(t, t_0)d_{11} + u_{a2}^{[1]}(t, t_0)d_{12} + (t - t_0)u_{a1}^{[1]}(t, t_0)d_{21} + (t - t_0)u_{a2}^{[1]}(t, t_0)d_{22} + (t - t_0)d_4 + \frac{(t - t_0)^2}{2} d_5,$$

which implies that the set of functions $\mathcal{F}_{a2}$ is not linearly independent on $[t_0, t_f]$. Then, if the set of functions $\mathcal{F}_{a2}$ is linearly independent on $[t_0, t_f]$, the observability Gramian must be positive definite, and therefore Eq. 7 is observable, see [5, Lemma 1]. □

3.2.2 Observability of the LTV Systems—Second Case (Velocity Relative to the ASC)

Before providing sufficient conditions for the observability of Eqs. 9 and 10, it is convenient to compute the transition matrix associated with $\mathbf{A}_b(t)$, which is given by

$$\boldsymbol{\phi}_b(t, t_0) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \int_{t_0}^{t} \frac{\mathbf{u}_b^T(\sigma)}{r(\sigma)} d\sigma & 1 \end{bmatrix}.$$

Similarly to the previous case, the observability Gramian of the LTV system (9) can be simplified to

$$\mathcal{W}_{b1}(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\psi}_{b1}^T(t, t_0)\boldsymbol{\psi}_{b1}(t, t_0)dt, \quad (20)$$

where

$$\boldsymbol{\psi}_{b1}(t, t_0) = \begin{bmatrix} \int_{t_0}^{t} \frac{\mathbf{u}_b^T(\sigma)}{r(\sigma)} d\sigma & 1 \end{bmatrix},$$

and the observability Gramian of the LTV system (10) follows

$$\mathcal{W}_{b2}(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\psi}_{b2}^T(t, t_0)\boldsymbol{\psi}_{b2}(t, t_0)dt,$$

in which

$$\boldsymbol{\psi}_{b2}(t, t_0) = \begin{bmatrix} \boldsymbol{\psi}_{b1}(t, t_0) \\ \boldsymbol{\phi}_{bz}(t, t_0) \end{bmatrix},$$

where

$$\boldsymbol{\phi}_{bz} = \begin{bmatrix} 0 \; 0 \; 1 \; 0 \end{bmatrix}.$$

To simplify the derivation of results, let

$$\mathbf{u}_b^{[1]}(t, t_0) := \int_{t_0}^{t} \mathbf{u}_b(\sigma)d\sigma = \begin{bmatrix} u_{b1}^{[1]}(t, t_0) \\ u_{b2}^{[1]}(t, t_0) \\ u_{b3}^{[1]}(t, t_0) \end{bmatrix} \in \mathbb{R}^3.$$

The following theorem provides a sufficient condition for the observability of Eq. 9.

Theorem 2 *Suppose that the set of functions*

$$\mathcal{F}_{b1} = \left\{ u_{b1}^{[1]}(t, t_0), u_{b2}^{[1]}(t, t_0), u_{b3}^{[1]}(t, t_0) \right\} \quad (21)$$

is linearly independent on $[t_0, t_f]$, $t_0 < t_f$. *Then, the LTV system* (9) *is observable on* $[t_0, t_f]$ *in the sense that, given the system input* $\mathbf{u}_b(t)$, $t \in [t_0, t_f]$, *and the system output* $y_1(t)$, $t \in [t_0, t_f]$, *the initial condition is uniquely defined.*

Proof Following the same train of thought as in Theorem 1, suppose that Eq. 9 is not observable on $[t_0, t_f]$. Then, the observability Gramian $\mathcal{W}_{b1}(t_0, t_f)$ is not positive definite and therefore there exists a constant $\mathbf{d} \in \mathbb{R}^{n_b}$, $\|\mathbf{d}\| = 1$ such that

$$\mathbf{d}^T \mathcal{W}_{b1}(t_0, t)\mathbf{d} = 0, \; \forall t \in [t_0, t_f]. \quad (22)$$

Let

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ d_2 \end{bmatrix} \in \mathbb{R}^{n_b}, \quad \text{with } \mathbf{d}_1 = \begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \end{bmatrix}.$$

Expanding Eq. 22 and using Eq. 20 gives

$$\int_{t_0}^{t} [\boldsymbol{\psi}_{b1}(\sigma, t_0)\mathbf{d}]^T \boldsymbol{\psi}_{b1}(\sigma, t_0)\mathbf{d} \, d\sigma = 0, \; \forall t \in [t_0, t_f],$$

and it follows that

$$\boldsymbol{\psi}_{b1}(t, t_0)\mathbf{d} = \mathbf{0}, \; \forall t \in [t_0, t_f]. \quad (23)$$

But, as

$$\boldsymbol{\psi}_{b1}(t_0, t_0)\mathbf{d} = d_2,$$

for Eq. 23 to hold it must be $d_2 = 0$. From 23, it also follows that

$$\frac{d}{dt}\boldsymbol{\psi}_{b1}(t, t_0)\mathbf{d} = 0, \; \forall t \in [t_0, t_f],$$

yielding

$$0 = \mathbf{u}_b(t) \cdot \mathbf{d}_1, \; \forall t \in [t_0, t_f]. \quad (24)$$

Integrating both sides of Eq. 24 gives

$$0 = \mathbf{u}_b^{[1]}(t, t_0) \cdot \mathbf{d}_1, \; \forall t \in [t_0, t_f],$$

which implies that the set of functions $\mathcal{F}_{b1}$ is not linearly independent on $[t_0, t_f]$. Then, if the set of functions $\mathcal{F}_{b1}$ is linearly independent on $[t_0, t_f]$, the observability Gramian must be positive definite, and therefore Eq. 9 is observable, see [5, Lemma 1]. □

The following result provides a sufficient condition for the observability of Eq. 10.

Corollary 2 *Suppose that the set of functions*

$$\mathcal{F}_{b2} = \left\{u_{b1}^{[1]}(t, t_0), u_{b2}^{[1]}(t, t_0)\right\} \tag{25}$$

is linearly independent on $[t_0, t_f]$, $t_0 < t_f$. *Then, the LTV system* (10) *is observable on* $[t_0, t_f]$ *in the sense that, given the system input* $\mathbf{u}_b(t)$, $t \in [t_0, t_f]$, *and the system output* $\mathbf{y}_2(t)$, $t \in [t_0, t_f]$, *the initial condition is uniquely defined.*

Proof Suppose that the LTV system (10) is not observable on $[t_0, t_f]$. Then, the observability Gramian $\mathcal{W}_{b2}(t_0, t_f)$ is not positive definite and therefore there exists a constant $\mathbf{d} \in \mathbb{R}^{n_b}$, $\|\mathbf{d}\| = 1$ such that

$$\mathbf{d}^T\mathcal{W}_{b2}(t_0, t)\mathbf{d} = 0, \ \forall t \in [t_0, t_f].$$

It follows that

$$\begin{cases} \boldsymbol{\psi}_{b1}(t, t_0)\mathbf{d} = 0 \\ \boldsymbol{\phi}_{bz}(t, t_0)\mathbf{d} = 0 \end{cases}, \ \forall t \in [t_0, t_f]. \tag{26}$$

But, as

$$\boldsymbol{\phi}_{bz}(t_0, t_0)\mathbf{d} = d_{13},$$

for Eq. 26 to hold it must be $d_{13} = 0$. By proceeding as in the proof of Theorem 2, it is straightforward to show that $d_3 = 0$, and that

$$0 = \mathbf{u}_b^{[1]}(t, t_0) \cdot \mathbf{d}_1, \ \forall t \in [t_0, t_f]. \tag{27}$$

In this case, since $d_{13} = 0$, Eq. 27 can be rewritten as

$$0 = u_{b1}^{[1]}(t, t_0)d_{11} + u_{b2}^{[1]}(t, t_0)d_{12}, \ \forall t \in [t_0, t_f],$$

which implies that the set of functions $\mathcal{F}_{b2}$ is not linearly independent on $[t_0, t_f]$. Then, if the set of functions $\mathcal{F}_{b2}$ is linearly independent on $[t_0, t_f]$, the observability Gramian must be positive definite, and therefore Eq. 10 is observable, see [5, Lemma 1]. □

3.3 Observability of the Nonlinear Systems

This subsection details the analysis of the observability of the nonlinear systems (2), (3), (4), and (5).

3.3.1 Observability of the Nonlinear Systems—First Case (Velocity Relative to the Water and Velocity of the ASC)

The following theorem provides a sufficient condition for the observability of the nonlinear system (2), as well as a practical result that can be used in the design of state observers for that system.

Theorem 3 *Suppose that the set of functions* (13) *is linearly independent on* $[t_0, t_f]$, $t_0 < t_f$. *Then, the nonlinear system* (2) *is observable on* $[t_0, t_f]$ *in the sense that, given the system input* $\mathbf{u}_a(t)$, $t \in [t_0, t_f]$, *and the system output* $y_1(t)$, $t \in [t_0, t_f]$, *the initial condition is uniquely defined. Moreover, a state observer with globally asymptotically stable error dynamics for the LTV system* (6) *is also a state observer for the nonlinear system* (2), *with globally asymptotically stable error dynamics.*

Proof Let

$$\begin{bmatrix} \mathbf{x}_1^a(t_0) \\ \mathbf{x}_2^a(t_0) \end{bmatrix}, \text{ with } \mathbf{x}_1^a(t_0) = \begin{bmatrix} x_{11}^a(t_0) \\ x_{12}^a(t_0) \\ x_{13}^a(t_0) \end{bmatrix} \text{ and}$$

$$\mathbf{x}_2^a(t_0) = \begin{bmatrix} x_{21}^a(t_0) \\ x_{22}^a(t_0) \\ x_{23}^a(t_0) \end{bmatrix},$$

be the initial state of the nonlinear system (2). The square of $r(t)$ follows

$$\begin{aligned}
[r(t)]^2 &= \|\mathbf{x}_1^a(t)\|^2 \\
&= \|\mathbf{x}_1^a(t_0) + (t - t_0)\mathbf{x}_2^a(t_0) + \mathbf{u}_a^{[1]}(t, t_0)\|^2 \\
&= \|\mathbf{x}_1^a(t_0)\|^2 + 2\mathbf{u}_a^{[1]}(t, t_0) \cdot \mathbf{x}_1^a(t_0) \\
&\quad + 2(t - t_0)\mathbf{u}_a^{[1]}(t, t_0) \cdot \mathbf{x}_2^a(t_0) \\
&\quad + 2(t - t_0)\mathbf{x}_1^a(t_0) \cdot \mathbf{x}_2^a(t_0) + (t - t_0)^2\|\mathbf{x}_2^a(t_0)\|^2 \\
&\quad + \|\mathbf{u}_a^{[1]}(t, t_0)\|^2.
\end{aligned} \tag{28}$$

Since the set of functions (13) is assumed linearly independent on $[t_0, t_f]$ it follows, from Theorem 1, that the LTV system (6) is observable on $[t_0, t_f]$. Thus, given $y_1(t)$ and $\mathbf{u}_a(t)$ for $t \in [t_0, t_f]$, the initial state of Eq. 6 is determined uniquely. Let

$$\begin{bmatrix} \mathbf{w}_1^a(t_0) \\ \mathbf{w}_2^a(t_0) \\ w_3^a(t_0) \\ w_4^a(t_0) \\ w_5^a(t_0) \end{bmatrix}, \text{ with } \mathbf{w}_1^a(t_0) = \begin{bmatrix} w_{11}^a(t_0) \\ w_{12}^a(t_0) \\ w_{13}^a(t_0) \end{bmatrix} \text{ and}$$

$$\mathbf{w}_2^a(t_0) = \begin{bmatrix} w_{21}^a(t_0) \\ w_{22}^a(t_0) \\ w_{23}^a(t_0) \end{bmatrix},$$

be the initial state of the linear system (6). Note that the state $w_3^a(t)$ is measured, thus $w_3^a(t_0) = r(t_0) = \|\mathbf{x}_1^a(t_0)\|$. As $[r(t)]^2 = [w_3^a(t)]^2$, it follows that

$$\frac{d}{dt}[r(t)]^2 = 2w_3^a(t)\dot{w}_3^a(t) = 2\mathbf{u}_a^T(t)\mathbf{w}_1^a(t) + 2w_4^a(t).$$

Since

$$\begin{cases} \mathbf{w}_1^a(t) = \mathbf{w}_1^a(t_0) + (t - t_0)\mathbf{w}_2^a(t_0) + \mathbf{u}_a^{[1]}(t, t_0) \\ w_4^a(t) = w_4^a(t_0) + \mathbf{u}_a^{[1]}(t, t_0) \cdot \mathbf{w}_2^a(t_0) + (t - t_0)w_5^a(t_0) \end{cases},$$

it is straightforward to show that

$$\begin{aligned} [r(t)]^2 = [w_3^a(t)]^2 = [w_3^a(t_0)]^2 &+ 2\mathbf{u}_a^{[1]}(t, t_0) \cdot \mathbf{w}_1^a(t_0) \\ &+ 2(t - t_0)\mathbf{u}_a^{[1]}(t, t_0) \cdot \mathbf{w}_2^a(t_0) \\ &+ 2(t - t_0)w_4^a(t_0) + (t - t_0)^2 w_5^a(t_0) \\ &+ \|\mathbf{u}_a^{[1]}(t, t_0)\|^2. \end{aligned} \tag{29}$$

Comparison between Eqs. 28 and 29 yields

$$\begin{aligned} 0 = 2\mathbf{u}_a^{[1]}(t, t_0) \cdot [\mathbf{x}_1^a(t_0) - \mathbf{w}_1^a(t_0)] &+ 2(t - t_0)\mathbf{u}_a^{[1]}(t, t_0) \\ \cdot [\mathbf{x}_2^a(t_0) - \mathbf{w}_2^a(t_0)] & \\ + 2(t - t_0)\big[\mathbf{x}_1^a(t_0) \cdot \mathbf{x}_2^a(t_0) - w_4^a(t_0)\big] & \\ + (t - t_0)^2\big[\|x_2^a(t_0)\|^2 - w_5^a(t_0)\big], & \end{aligned} \tag{30}$$

for all $t \in [t_0, t_f]$. Since the set of functions $\mathcal{F}_{a1}$ is assumed linearly independent, Eq. 30 implies that

$$\begin{cases} \mathbf{x}_1^a(t_0) = \mathbf{w}_1^a(t_0) \\ \mathbf{x}_2^a(t_0) = \mathbf{w}_2^a(t_0) \\ \mathbf{x}_1^a(t_0) \cdot \mathbf{x}_2^a(t_0) = w_4^a(t_0) \\ \|\mathbf{x}_2^a(t_0)\|^2 = w_5^a(t_0) \end{cases}.$$

This concludes the proof, as both the initial state of the nonlinear system (2) is uniquely determined and the initial state of the linear system (6) matches the initial state of the nonlinear system (2). □

The following theorem provides a sufficient condition for the observability of the nonlinear system (3), as well as a practical result that can be used in the design of state observers for that system.

Corollary 3 *Suppose that the set of functions (17) is linearly independent on $[t_0, t_f]$, $t_0 < t_f$. Then, the nonlinear system (3) is observable on $[t_0, t_f]$ in the sense that, given the system input $\mathbf{u}_a(t)$, $t \in [t_0, t_f]$, and the system output $\mathbf{y}_2(t)$, $t \in [t_0, t_f]$, the initial condition is uniquely defined. Moreover, a state observer with globally asymptotically stable error dynamics for the LTV system (7) is also a state observer for the nonlinear system (3), with globally asymptotically stable error dynamics.*

Proof Proceeding as in the proof of Theorem 3, it can be shown that

$$\begin{aligned} 0 = 2\mathbf{u}_a^{[1]}(t, t_0) \cdot \big[\mathbf{x}_1^a(t_0) - \mathbf{w}_1^a(t_0)\big] &+ 2(t - t_0)\mathbf{u}_a^{[1]}(t, t_0) \\ \cdot \big[\mathbf{x}_2^a(t_0) - \mathbf{w}_2^a(t_0)\big] & \\ + 2(t - t_0)\big[\mathbf{x}_1^a(t_0) \cdot \mathbf{x}_2^a(t_0) - w_4^a(t_0)\big] & \\ + (t - t_0)^2\big[\|x_2^a(t_0)\|^2 - w_5^a(t_0)\big], & \end{aligned} \tag{31}$$

for all $t \in [t_0, t_f]$. As the states $w_3^a(t)$ and $w_{13}^a(t)$ are measured, it follows that $w_3^a(t_0) = r(t_0) = \|\mathbf{x}_1^a(t_0)\|$ and $w_{13}^a(t_0) = z(t_0) = x_{13}^a(t_0)$. For the nonlinear system, $z(t)$ follows

$$\begin{aligned} z(t) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x}_1^a(t) \\ &= x_{13}^a(t_0) + (t - t_0)x_{23}^a(t_0) + u_{a3}^{[1]}(t, t_0) \end{aligned} \tag{32}$$

and, for the LTV system, it satisfies

$$r(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{w}_1^a(t)$$
$$= w_{13}^a(t_0) + (t - t_0) w_{23}^a(t_0) + u_{a3}^{[1]}(t, t_0). \quad (33)$$

From the comparison between Eqs. 32 and 33, it follows that $x_{23}^a(t_0) = w_{23}^a(t_0)$. Then, Eq. 31 can be rewritten as

$$\begin{aligned} 0 = {} & 2u_{a1}^{[1]}(t, t_0)\big(x_{11}^a(t_0) - w_{11}^a(t_0)\big) \\ & + 2u_{a2}^{[1]}(t, t_0)\big(x_{12}^a(t_0) - w_{12}^a(t_0)\big) \\ & + 2(t - t_0)u_{a1}^{[1]}(t, t_0)\big(x_{21}^a(t_0) - w_{21}^a(t_0)\big) \\ & + 2(t - t_0)u_{a2}^{[1]}(t, t_0)\big(x_{22}^a(t_0) - w_{22}^a(t_0)\big) \\ & + 2(t - t_0)\big[\mathbf{x}_1^a(t_0) \cdot \mathbf{x}_2^a(t_0) - w_4^a(t_0)\big] \\ & + (t - t_0)^2\big[\|x_2^a(t_0)\|^2 - w_5^a(t_0)\big], \end{aligned} \quad (34)$$

for all $t \in [t_0, t_f]$. Since the set of functions $\mathcal{F}_{a2}$ is assumed linearly independent, Eq. 34 implies that

$$\begin{cases} \mathbf{x}_1^a(t_0) = \mathbf{w}_1^a(t_0) \\ \mathbf{x}_2^a(t_0) = \mathbf{w}_2^a(t_0) \\ \mathbf{x}_1^a(t_0) \cdot \mathbf{x}_2^a(t_0) = w_4^a(t_0) \\ \|\mathbf{x}_2^a(t_0)\|^2 = w_5^a(t_0) \end{cases}.$$

This concludes the proof, as both the initial state of the nonlinear system (3) is uniquely determined and the initial state of the linear system (7) matches the initial state of the nonlinear system (3). □

3.3.2 Observability of the Nonlinear Systems—Second Case (Velocity Relative to the ASC)

The following theorem provides a sufficient condition for the observability of the nonlinear system (4), as well as a practical result that can be used in the design of state observers for that system.

Theorem 4 *Suppose that the set of functions (21) is linearly independent on $[t_0, t_f]$, $t_0 < t_f$. Then, the nonlinear system (4) is observable on $[t_0, t_f]$ in the sense that, given the system input $\mathbf{u}_b(t)$, $t \in [t_0, t_f]$, and the system output $y_1(t)$, $t \in [t_0, t_f]$, the initial condition is uniquely defined. Moreover, a state observer with globally asymptotically stable error dynamics for the LTV system (9) is also a state observer for the nonlinear system (4), with globally asymptotically stable error dynamics.*

Proof Let

$$\mathbf{x}_1^b(t_0) = \begin{bmatrix} x_{11}^b(t_0) \\ x_{12}^b(t_0) \\ x_{13}^b(t_0) \end{bmatrix}$$

be the initial state of the nonlinear system (4). The square of $r(t)$ follows

$$\begin{aligned} [r(t)]^2 &= \|\mathbf{x}_1^b(t)\|^2 = \|\mathbf{x}_1^b(t_0) + \mathbf{u}_b^{[1]}(t, t_0)\|^2 \\ &= \|\mathbf{x}_1^b(t_0)\|^2 + 2\mathbf{u}_b^{[1]}(t, t_0) \cdot \mathbf{x}_1^b(t_0) \\ &\quad + \|\mathbf{u}_b^{[1]}(t, t_0)\|^2. \end{aligned} \quad (35)$$

Since the set of functions (21) is assumed linearly independent on $[t_0, t_f]$ it follows, from Theorem 2, that the LTV system (9) is observable on $[t_0, t_f]$. Thus, given $y_1(t)$ and $\mathbf{u}_b(t)$ for $t \in [t_0, t_f]$, the initial state of (9) is determined uniquely. Let

$$\begin{bmatrix} \mathbf{w}_1^b(t_0) \\ w_2^b(t_0) \end{bmatrix}, \text{ with } \mathbf{w}_1^b(t_0) = \begin{bmatrix} w_{11}^b(t_0) \\ w_{12}^b(t_0) \\ w_{13}^b(t_0) \end{bmatrix},$$

be the initial state of the linear system (9). Note that the state $w_2^b(t)$ is measured, thus $w_2^b(t_0) = r(t_0) = \|\mathbf{x}_1^b(t_0)\|$. As $[r(t)]^2 = [w_2^b(t)]^2$, it follows that

$$\frac{d}{dt}[r(t)]^2 = \frac{d}{dt}\big[w_2^b(t)\big]^2 = 2\mathbf{u}_b(t) \cdot \mathbf{w}_1^b(t).$$

Since

$$\mathbf{w}_1^b(t) = \mathbf{w}_1^b(t_0) + \mathbf{u}_b^{[1]}(t, t_0),$$

it is straightforward to show that

$$\begin{aligned} [r(t)]^2 &= \big[w_2^b(t_0)\big]^2 + 2\mathbf{u}_b^{[1]}(t, t_0) \cdot \mathbf{w}_1^b(t_0) \\ &\quad + \|\mathbf{u}_b^{[1]}(t, t_0)\|^2. \end{aligned} \quad (36)$$

Comparison between Eqs. 35 and 36 yields

$$0 = \mathbf{u}_b^{[1]}(t, t_0) \cdot \big[\mathbf{x}_1^b(t_0) - \mathbf{w}_1^b(t_0)\big], \ \forall t \in [t_0, t_f]. \quad (37)$$

Since the set of functions $\mathcal{F}_{b1}$ is assumed linearly independent, Eq. 37 implies that $\mathbf{x}_1(t_0) =$

$\mathbf{w}_1(t_0)$. This concludes the proof, as both the initial state of the nonlinear system (4) is uniquely determined and the initial state of the linear system (9) matches the initial state of the nonlinear system (4). □

The following theorem provides a sufficient condition for the observability of the nonlinear system (3), as well as a practical result that can be used in the design of state observers for that system.

Corollary 4 *Suppose that the set of functions (25) is linearly independent on $[t_0, t_f]$, $t_0 < t_f$. Then, the nonlinear system (5) is observable on $[t_0, t_f]$ in the sense that, given the system input $\mathbf{u}_a(t)$, $t \in [t_0, t_f]$, and the system output $\mathbf{y}_2(t)$, $t \in [t_0, t_f]$, the initial condition is uniquely defined. Moreover, a state observer with globally asymptotically stable error dynamics for the LTV system (10) is also a state observer for the nonlinear system (5), with globally asymptotically stable error dynamics.*

Proof Proceeding as in the proof of Theorem 4, it can be shown that

$$0 = \mathbf{u}_b^{[1]}(t, t_0) \cdot \left[\mathbf{x}_1^b(t_0) - \mathbf{w}_1^b(t_0)\right], \ \forall t \in [t_0, t_f]. \quad (38)$$

As the states $w_2^b(t)$ and $w_{13}^b(t)$ are measured, it follows that $w_2^b(t_0) = r(t_0) = \|\mathbf{x}_1^b(t_0)\|$ and $w_{13}^b(t_0) = z(t_0) = x_{13}^b(t_0)$. Then, Eq. 38 can be rewritten as

$$\begin{aligned} 0 &= u_{b1}^{[1]}(t, t_0)\left(x_{11}^b(t_0) - w_{11}^b(t_0)\right) \\ &\quad + u_{b2}^{[1]}(t, t_0)\left(x_{12}^b(t_0) - w_{12}^b(t_0)\right), \ \forall t \in [t_0, t_f]. \quad (39) \end{aligned}$$

Since the set of functions $\mathcal{F}_{b2}$ is assumed linearly independent, Eq. 39 implies that $\mathbf{x}_1(t_0) = \mathbf{w}_1(t_0)$. This concludes the proof, as both the initial state of the nonlinear system (5) is uniquely determined and the initial state of the linear system (10) matches the initial state of the nonlinear system (5). □

4 Simulation Results

This section provides simulation results to assess the behavior and performance of the filtering solutions proposed in the paper.

4.1 Filter Design

While the observability analysis was carried out in the inertial coordinate frame, it is possible to design GAS state estimators in the original coordinate space of the AUV by applying the appropriate state transformation. This allows to avoid the algebraic transformation of sensor data to inertial coordinates, which can greatly amplify the effect of the noise in the attitude measurements, see [4].

For the first case, that is, for the LTV systems (6) and (7), define an augmented state transformation

$$\mathcal{T}_a(t) := \begin{bmatrix} \mathbf{R}(t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{R}(t) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n_a \times n_a},$$

and define new state variables $\boldsymbol{\chi}_a(t) \in \mathbb{R}^{n_a}$ such that

$$\boldsymbol{\chi}_a(t) = \mathcal{T}_a^T(t)\mathbf{x}_a(t).$$

Then, it is straightforward to show that the dynamics of the new state $\boldsymbol{\chi}_a(t)$ can be described by the LTV system

$$\begin{cases} \dot{\boldsymbol{\chi}}_a(t) = \mathcal{A}_a(t)\boldsymbol{\chi}_a(t) + \mathbf{B}_a\mathbf{u}_{ar}(t) \\ y_1(t) = \mathbf{C}_{a1}\boldsymbol{\chi}_a(t) \end{cases}, \quad (40)$$

where $\mathbf{B}_a$ and $\mathbf{C}_{a1}$ are defined as in Eq. 6,

$$\mathcal{A}_a(t) = \begin{bmatrix} -\mathbf{S}(\boldsymbol{\omega}(t)) & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S}(\boldsymbol{\omega}(t)) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \dfrac{\mathbf{u}_{ar}^T(t)}{r(t)} & \mathbf{0} & 0 & \dfrac{1}{r(t)} & 0 \\ \mathbf{0} & -\mathbf{u}_{ar}^T(t) & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathbf{u}_{ar}(t) = \mathbf{R}^T(t)\mathbf{v}_s(t) - \mathbf{v}_r(t).$$

If there is a depth measurement, the system becomes

$$\begin{cases} \dot{\boldsymbol{\chi}}_a(t) = \mathcal{A}_a(t)\boldsymbol{\chi}_a(t) + \mathbf{B}_a\mathbf{u}_{ar}(t) \\ \mathbf{y}_2(t) = \mathcal{C}_{a2}(t)\boldsymbol{\chi}_a(t) \end{cases}, \quad (41)$$

in which

$$\mathcal{C}_{a2}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{R}(t) & \mathbf{0} & 0 & 0 & 0 \end{bmatrix}.$$

Notice that, as $\mathcal{T}_a(t)$ is a Lyapunov state transformation, the stability and observability properties of Eqs. 6 and 7 also apply to Eqs. 40 and 41, respectively, see [6]. Then, using Kalman filtering theory, it is straightforward to design GAS state observers for Eqs. 40 and 41, see [10].

The same can be done for the second case, that is, for the LTV systems (9) and (10). Define an augmented state transformation

$$\mathcal{T}_b(t) := \begin{bmatrix} \mathbf{R}(t) & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^{n_b \times n_b},$$

and define new state variables $\chi_b(t) \in \mathbb{R}^{n_b}$ such that

$$\chi_b(t) = \mathcal{T}_b^T(t)\mathbf{x}_b(t).$$

Then, it is straightforward to show that the dynamics of the new state $\chi_b(t)$ can be described by the LTV system

$$\begin{cases} \dot{\chi}_b(t) = \mathcal{A}_b(t)\chi_b(t) + \mathbf{B}_b\mathbf{u}_{br}(t) \\ y_1(t) = \mathbf{C}_{b1}\chi_a(t) \end{cases}, \tag{42}$$

where $\mathbf{B}_b$ and $\mathbf{C}_{b1}$ are defined as in Eq. 9,

$$\mathcal{A}_b(t) = \begin{bmatrix} -\mathbf{S}(\boldsymbol{\omega}(t)) & \mathbf{0} \\ \dfrac{\mathbf{u}_{br}^T(t)}{r(t)} & 0 \end{bmatrix},$$

and

$$\mathbf{u}_{br}(t) = \Delta\mathbf{v}(t).$$

If there is a depth measurement, the system becomes

$$\begin{cases} \dot{\chi}_b(t) = \mathcal{A}_b(t)\chi_b(t) + \mathbf{B}_b\mathbf{u}_{br}(t) \\ \mathbf{y}_2(t) = \mathcal{C}_{b2}(t)\chi_b(t) \end{cases}, \tag{43}$$

in which

$$\mathcal{C}_{b2}(t) = \begin{bmatrix} \mathbf{0} & 1 \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{R}(t) & 0 \end{bmatrix}.$$

Notice that, as $\mathcal{T}_b(t)$ is also a Lyapunov state transformation, it is straightforward to design GAS state observers for Eqs. 42 and 43 by applying Kalman filtering theory.

Remark 6 During the analysis and filter design process, the velocity of the current affecting the I-AUV is assumed to be constant. Evidently, this will never be the case in real-world applications. However, as the proposed Kalman filter assumes that the system is driven by random noise, it is also possible to estimate slowly time-varying ocean currents by correctly tuning the covariance matrices associated with the Kalman filter.

4.2 Mission Scenario

In the simulations that were carried out, Kalman filters based on the LTV systems (40), (41), (42), and (43) were implemented in very similar scenarios. To provide a meaningful comparison term, an Extended Kalman Filter (EKF) was implemented

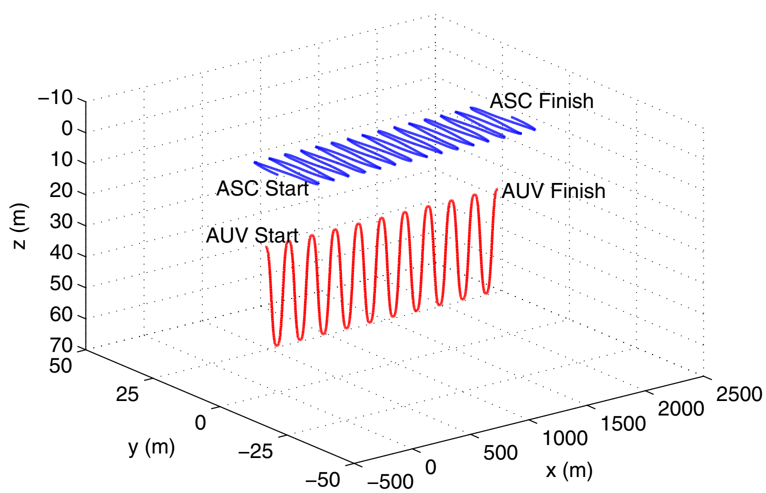
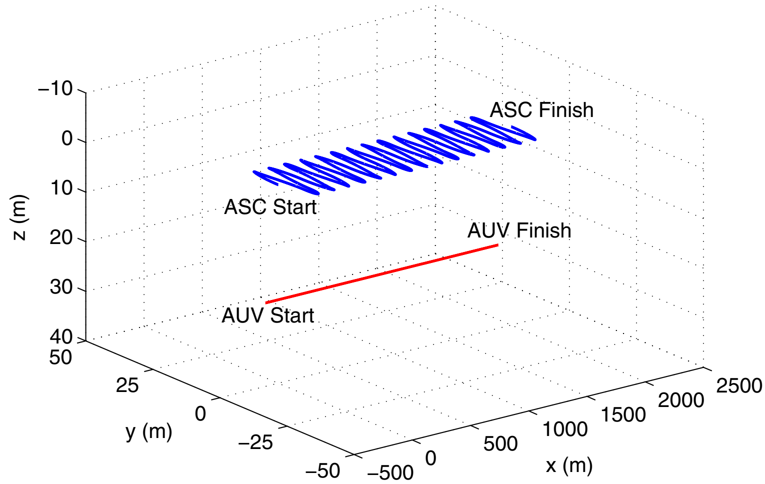


Fig. 2 Trajectories for the state estimators without depth measurements

Fig. 3 Trajectories for the state estimators with depth measurements

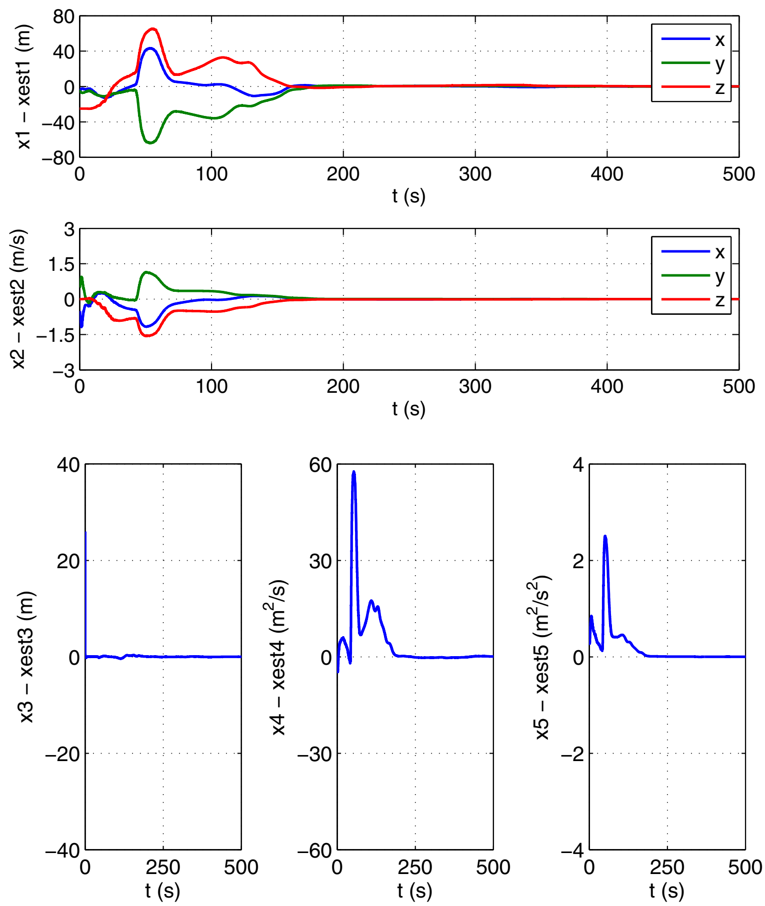


for the nonlinear system

$$\begin{cases} \dot{\mathbf{r}}(t) = -\mathbf{S}(\boldsymbol{\omega}(t))\mathbf{r}(t) - \mathbf{v}_f(t) + \mathbf{u}_{ar}(t) \\ \dot{\mathbf{v}}_f(t) = -\mathbf{S}(\boldsymbol{\omega}(t))\mathbf{v}_f(t) \\ y_1(t) = \|\mathbf{r}(t)\| \end{cases},$$

which estimates the same quantities as the Kalman filter based on the LTV system (40). The I-AUV starts at $\mathbf{p}(0) = \begin{bmatrix}5 & 5 & 25\end{bmatrix}^T$ (m), and is assumed to be moving in a fluid with velocity

Fig. 4 Initial convergence of the estimation error variables for the Kalman filter based on the LTV system (40)



$\mathbf{v}_f = \left[-0.5 \;\; 0.5 \;\; 0\right]^T$ (m/s) relative to $\{I\}$, expressed in inertial coordinates. The ASC starts at $\mathbf{s}(0) = \left[0 \;\; 0 \;\; 0\right]^T$ (m), and performs rich trajectories to guarantee observability. Two different trajectories for the I-AUV were considered: the first, depicted in Fig. 2, was used for the state estimators based on the systems (40) and (42). The second, depicted in Fig. 3, was used for the state estimators based on the systems (41) and (43), which have a depth measurement, thus not requiring vertical movement to guarantee observability.

The measurement noise was simulated by adding a zero-mean, uncorrelated and normally distributed perturbation to the velocity, range, and depth measurements, with standard deviations of 0.01 m/s for the velocity measurements relating to the first case ($\mathbf{v}_r(t)$ and $\mathbf{v}_s(t)$), 0.02 m/s for the velocity measurements relating to the second case ($\Delta\mathbf{v}(t)$), and 0.2 m for both the range and depth measurements. In addition to the perturbations in the position and velocity measurements, noise was also simulated in the attitude and angular velocity measurements required for the implementation of the Kalman filters in the original coordinate space, as provided by an Attitude and Heading Reference System. The angular velocity measurements were corrupted by zero-mean uncorrelated white Gaussian noise, with standard deviation of 0.05°/s. The attitude is usually parametrized by roll, pitch, and yaw Euler angles, and as such noise in the attitude measurements was simulated by adding zero-mean, uncorrelated white Gaussian perturbations to the roll, pitch, and yaw, with standard deviation of 0.03° for the roll and pitch, and 0.3° for the yaw. To tune

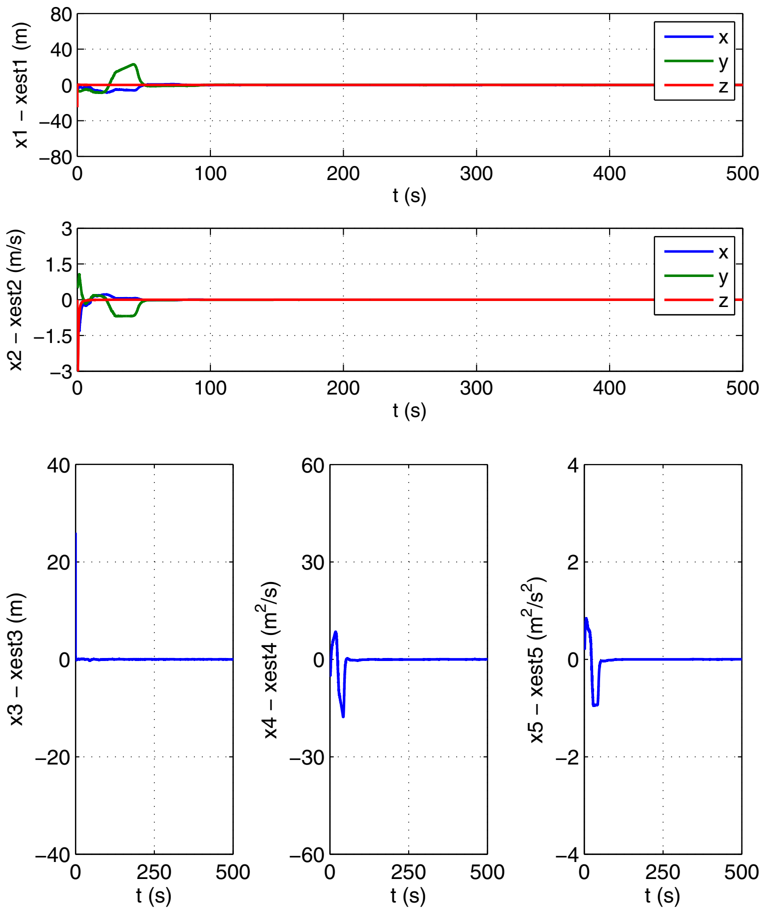


Fig. 5 Initial convergence of the estimation error variables for the Kalman filter based on the LTV system (41)

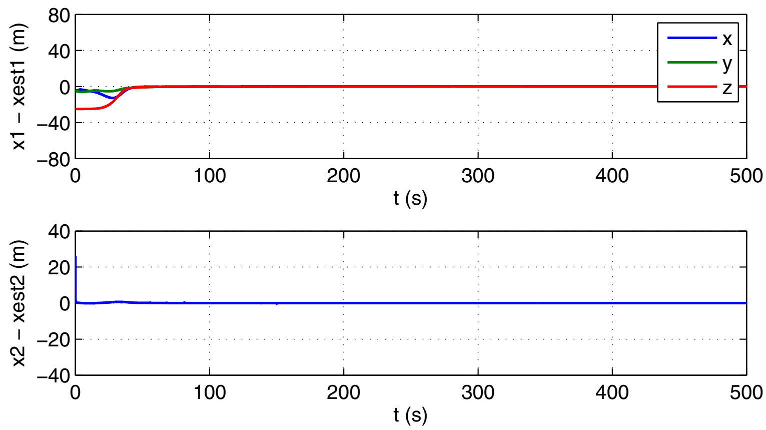


Fig. 6 Initial convergence of the estimation error variables for the Kalman filter based on the LTV system (42)

the Kalman filters, the covariance matrices were set to

$$\begin{cases} \mathbf{Q}_a = \ 0.01 \times \operatorname{diag}\big(1,\ 1,\ 1,\ 0.001,\ 0.001, \\ \qquad\qquad\qquad 0.001,\ 1,\ 1,\ 0.001\big) \\ R_{a1} = 1,\ \mathbf{R}_{a2} = \mathbf{I} \end{cases}$$

for the Kalman filters based on the systems (40) and (41), and

$$\begin{cases} \mathbf{Q}_b = 0.01 \times \mathbf{I} \\ R_{b1} = 1,\ \mathbf{R}_{b2} = \mathbf{I} \end{cases}$$

for the Kalman filters based on the systems (42) and (43).

4.3 Results

Figures 4, 5, 6 and 7 detail the initial evolution of the estimation error variables for the Kalman filters based on the LTV systems (40), (41), (42) and (43), respectively, and Fig. 8 details the initial evolution of the estimation error variables for the EKF. The large transients that can be observed are caused by mismatches in the initial conditions.

To better assess the performance of the proposed state estimators, Figs. 9, 10, 11, 12 and 13 depict the detailed evolution of the estimation error variables after the initial transients have settled. As it can be seen, even in the presence of sensor noise with realistic intensity, the achieved values remain confined to small intervals around zero and excellent filtering performance is achieved. To complement the graphical data, the Monte Carlo method was applied. The simulation was carried out 1,000 times with different, randomly generated noise signals, and significant statistical data was extracted from the results. The results are depicted in Tables 1 and 2,

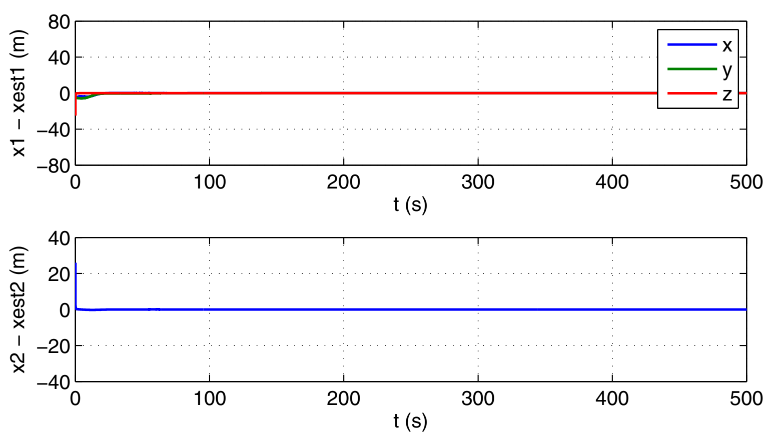


Fig. 7 Initial convergence of the estimation error variables for the Kalman filter based on the LTV system (43)

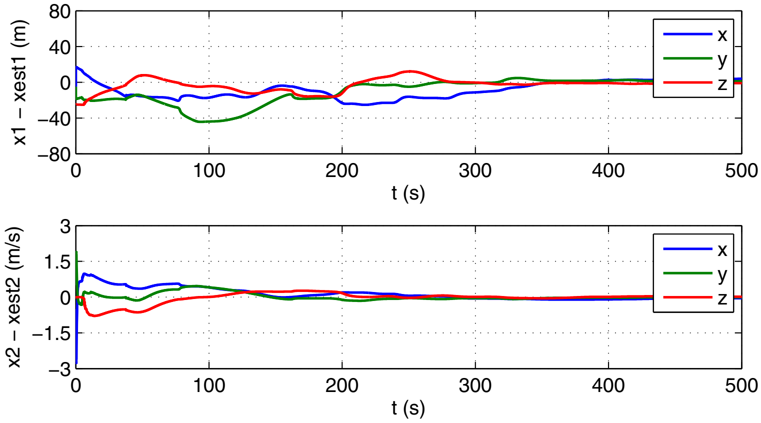


Fig. 8 Initial convergence of the estimation error variables for the Extended Kalman Filter

which detail the measured standard deviations of the steady-state estimation error variables, averaged over the 1,000 simulations. As expected, the filters based on the systems with additional depth measurements perform better than the ones based on systems with only range measure-

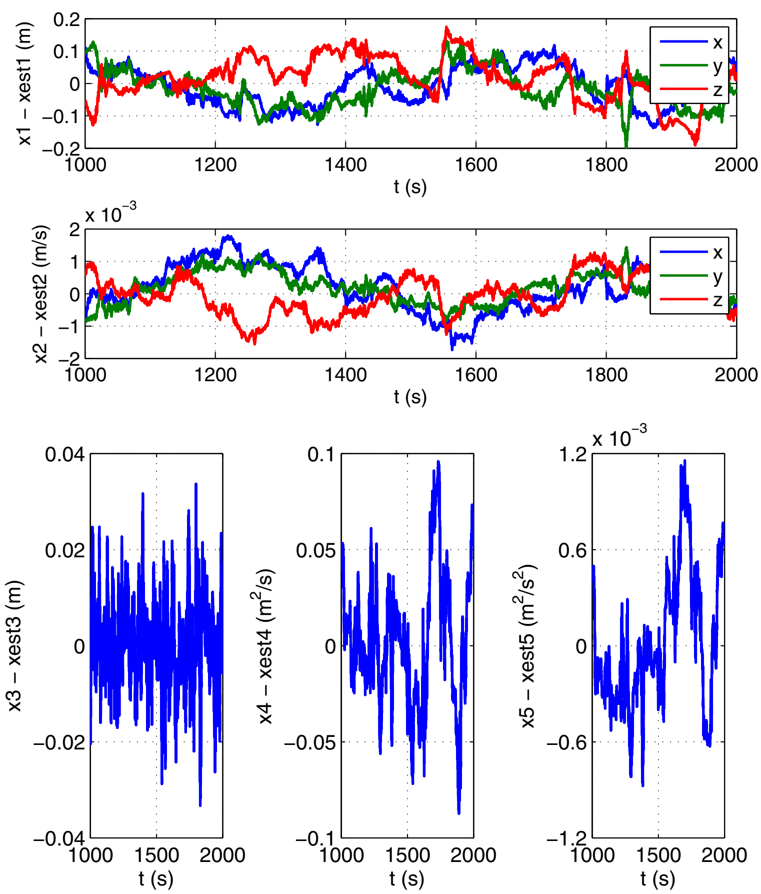


Fig. 9 Steady-state behavior of the estimation error variables for the Kalman filter based on the LTV system (40)

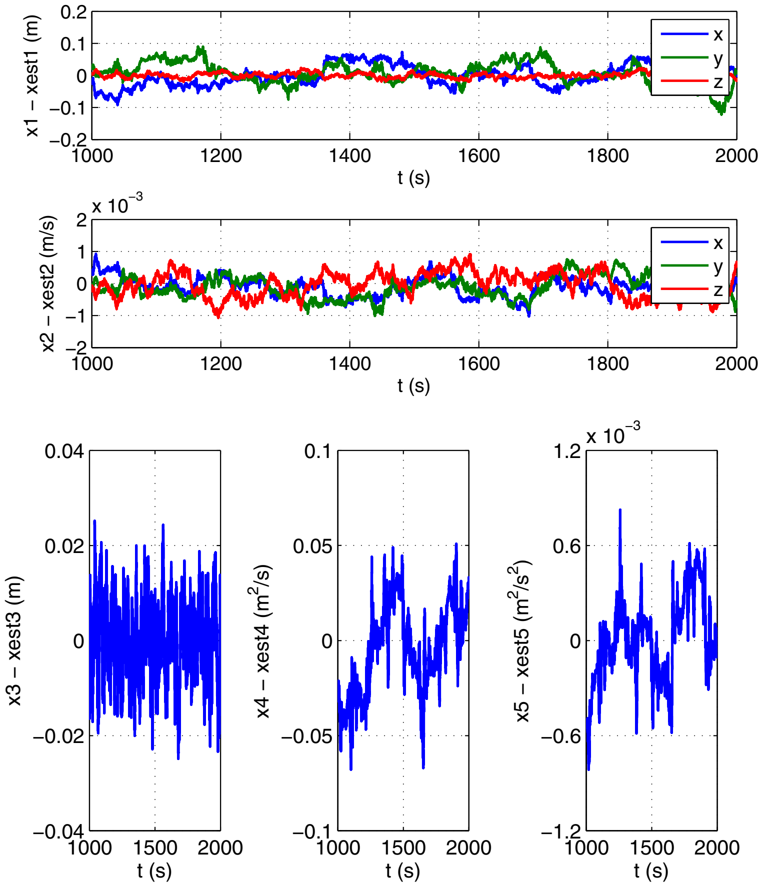


Fig. 10 Steady-state behavior of the estimation error variables for the Kalman filter based on the LTV system (41)

ments. Comparing the results for the Kalman filter based on the LTV system (40) and for the EKF shows that the latter achieves slightly better results in steady-state. However, it does so at the expense of globally asymptotic stability guarantees, and takes significantly more time to converge than the Kalman filter for the LTV system (40).

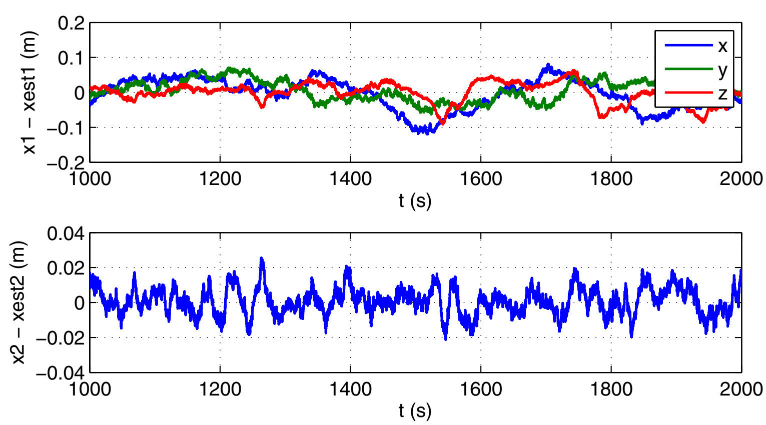


Fig. 11 Steady-state behavior of the estimation error variables for the Kalman filter based on the LTV system (42)

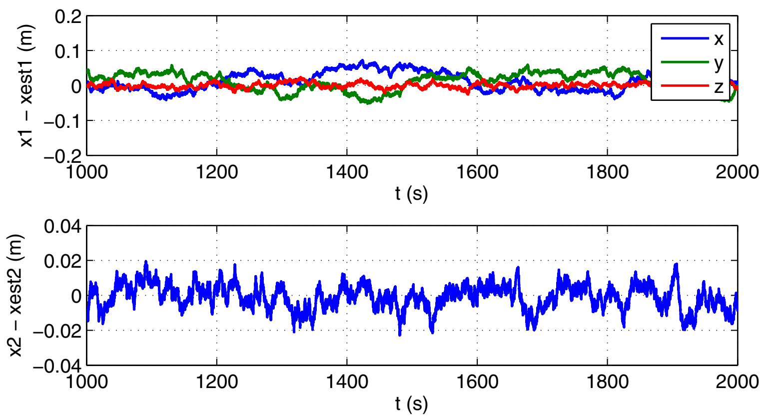


Fig. 12 Steady-state behavior of the estimation error variables for the Kalman filter based on the LTV system (43)

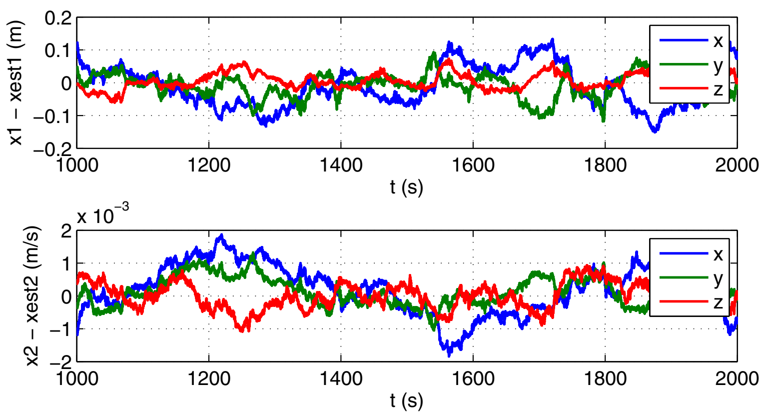


Fig. 13 Steady-state behavior of the estimation error variables for the Extended Kalman Filter

Table 1 Standard deviation of the steady-state estimation error variables for the Kalman filters based on the LTV systems (40) and (41) and for the EKF

Variable	σ, KF for Eq. 40	σ, KF for Eq. 41	σ, EKF
$x_{11}^a(t)$ [m]	4.68×10^{-2}	3.14×10^{-2}	4.17×10^{-2}
$x_{12}^a(t)$ [m]	5.51×10^{-2}	3.25×10^{-2}	4.34×10^{-2}
$x_{13}^a(t)$ [m]	5.80×10^{-2}	6.74×10^{-3}	2.07×10^{-2}
$x_{21}^a(t)$ [m/s]	5.14×10^{-4}	4.15×10^{-4}	4.85×10^{-4}
$x_{22}^a(t)$ [m/s]	4.87×10^{-4}	4.29×10^{-4}	4.77×10^{-4}
$x_{23}^a(t)$ [m/s]	6.20×10^{-4}	3.58×10^{-4}	4.13×10^{-4}
$x_3^a(t)$ [m]	8.65×10^{-3}	8.51×10^{-3}	–
$x_4^a(t)$ [m^2/s]	3.37×10^{-2}	2.11×10^{-2}	–
$x_5^a(t)$ [m^2/s^2]	3.32×10^{-4}	2.78×10^{-4}	–

Table 2 Standard deviation of the steady-state estimation error variables for the Kalman filters based on the LTV systems (42) and (43)

Variable	σ, KF for Eq. 42	σ, KF for Eq. 43
$x_{11}^b(t)$ [m]	3.38×10^{-2}	2.31×10^{-2}
$x_{12}^b(t)$ [m]	3.45×10^{-2}	2.23×10^{-2}
$x_{13}^b(t)$ [m]	2.43×10^{-2}	6.56×10^{-3}
$x_2^b(t)$ [m]	7.71×10^{-3}	7.17×10^{-3}

5 Conclusions

This paper proposed novel cooperative navigation solutions for an I-AUV working in tandem with an ASC. The I-AUV was assumed to be moving in the presence of constant unknown ocean currents, and to have access to measurements of its range to the ASC to estimate its position. Two different scenarios were considered:

1. In the first scenario, the ASC transmits its position and velocity to the I-AUV, and the I-AUV has access to readings of its velocity relative to the water, provided by a DVL;
2. In the second case, the ASC transmits only its position, and the I-AUV has access to measurements of its velocity relative to the ASC.

A sufficient condition for observability and a method for designing state observers with GAS error dynamics were presented for both cases. Additionally, it was verified through its effect on the observability conditions that the addition of a depth sensor allows for operation of the I-AUV at constant depth. Finally, simulation results were presented and discussed to assess the performance of the proposed solutions in the presence of measurement noise.

Future work will focus mainly on experimental validation of the proposed solution. Ideally, the experimental trials would mirror the scenario that is proposed in the paper, that is, an I-AUV/ASC tandem communicating exclusively through the acoustic channel. However, at least for preliminary trials, a more likely approach would be to use two ASCs instead, as the possibility of having GPS on-board both vehicles would provide accurate positioning data to compare with the estimates of the range-based filter.

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