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On the stability of the continuous-time Kalman filter subject to exponentially decaying perturbations

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ABSTRACT

This paper details the stability analysis of the continuous-time Kalman filter dynamics for linear time-varying systems subject to exponentially decaying perturbations. It is assumed that estimates of the input, output, and matrices of the system are available, but subject to unknown perturbations which decay exponentially with time. It is shown that if the nominal system is uniformly completely observable and uniformly completely controllable, and if the state, input, and matrices of the system are bounded, then the Kalman filter built using the perturbed estimates is a suitable state observer for the nominal system, featuring exponentially convergent error dynamics.

1. Introduction

This paper details the stability analysis of the continuous-time Kalman filter subject to exponentially decaying perturbations in the input, output, and matrices of the system. The topic of state estimation subject to perturbations in the dynamics of the system has been extensively covered in the framework of robust estimation for uncertain systems, see e.g. [1] and [2]. Existing research on the subject ranges from results on quadratically stable linear time-invariant systems, see e.g. [3–5], to more general approaches on generic linear time-varying (LTV) systems such as in [6] and [7]. More recently, works such as [8–10] proposed robust filtering solutions for cases in which the matrices of the nominal system are uncertain, but known to reside in a given convex polytope. While the above-cited references consider mostly norm-bounded uncertainties, this paper considers exponentially decaying perturbations. Although this class of signals is admittedly more restrictive, it allows for arbitrarily large initial values for the uncertainties, as well as recovering the optimal Kalman filter dynamics in steady state. Another difference with respect to the existing literature regards the prior knowledge of the nominal system: while some works assume that the dynamics of the nominal system are known and others admit that they reside in a given polytope, in this paper the filter is implemented without knowing the nominal system dynamics, using instead estimates of the system matrices corrupted by the aforementioned unknown perturbations. This formulation is useful for analysis of interconnected Kalman filters in a cascade setup. While there are a number of stability results for cascade systems (see e.g. [11]), interconnecting time-varying Kalman filters in such a fashion introduces additional problems, as the estimation error of the first filter in the cascade might inject exponentially decaying errors in the dynamics of the other filters at several levels: in the input, the output, the matrices of the system model, and the computation of the filter gain and error covariance matrix. For a practical application of the results detailed in this paper, see e.g. [12], in which autonomous vehicles working in formation use state estimates from other agents to compute the system matrices needed to implement local Kalman filters. The above-cited paper includes a simplified version of the results detailed in this paper, which considers perturbations only in the input $\mathbf{u}(t)$ and output matrix $\mathbf{C}(t)$. In comparison, in this paper perturbations are introduced in most matrices of the system model (with the exception of the noise covariances $\mathbf{Q}(t)$ and $\mathbf{R}(t)$) as well as both the input and output of the system. As a result of this, the results detailed here are more general but also more complex, for the most part due to the perturbation in $\mathbf{A}(t)$ which induces errors in the state transition matrix $\boldsymbol{\Phi}(t, t_0)$.

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The rest of the paper is organized as follows. Section 2 details the problem at hand as well as the assumptions required to derive the results in subsequent sections. A generic LTV system is considered, and it is assumed that estimates of the input, output, and matrices of the system are available to implement the Kalman filter, but subject to unknown perturbations which decay exponentially with time. The two following sections detail auxiliary results which are necessary to derive the main result of this paper: in Section 3, it is shown that if the nominal system is uniformly completely observable (UCO) and uniformly completely controllable (UCC), then the perturbed system is also UCO and UCC; and in Section 4, it is shown that the solution of the Riccati equation for the Kalman filter built with the perturbed parameters converges exponentially fast to the solution of the Riccati equation for the nominal Kalman filter. Section 5 details the main result of this paper: using the previous results, it is shown that the perturbed version of the Kalman filter constitutes a suitable state observer for the nominal LTV system, featuring error dynamics that converge exponentially fast to zero. Finally, Section 6 summarizes the main conclusions of this work.

1.1. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. For a matrix $\mathbf{A}$, $\|\mathbf{A}\|$ denotes its induced 2-norm, and $\|\mathbf{A}\|_F$ denotes its Frobenius norm. The notation $\operatorname{vec}(\mathbf{A})$ denotes the vectorizing operator, which returns a vector constructed by stacking the columns of the matrix $\mathbf{A}$. For a symmetric matrix $\mathbf{P}$, $\mathbf{P} \succ \mathbf{0}$ and $\mathbf{P} \succeq \mathbf{0}$ indicate that $\mathbf{P}$ is positive definite and positive semi-definite, respectively.

2. Problem statement

Consider the LTV system

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \mathbf{A}(t)\boldsymbol{x}(t) + \mathbf{B}(t)\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \mathbf{C}(t)\boldsymbol{x}(t), \end{cases} \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$, $\boldsymbol{y}(t) \in \mathbb{R}^o$, and $\boldsymbol{x}_0 \in \mathbb{R}^n$ are the state, input, output, and initial condition of the system, respectively. $\mathbf{A}(t)$, $\mathbf{B}(t)$, and $\mathbf{C}(t)$ are matrix-valued functions of time of appropriate dimensions. To simplify the notation throughout the text, from hereon time-dependency of the variables is not explicitly shown ($\boldsymbol{x}$ is used instead of $\boldsymbol{x}(t)$, for example). In general, all functions and variables that appear in the text should be treated as time-varying unless explicitly stated otherwise.

It is assumed that all quantities in (1) are bounded for all time, that is, there exist positive scalar constants α_1, α_2, α_3, α_4, and α_5 such that

$$\begin{cases} \|\boldsymbol{x}\| \le \alpha_1 \\ \|\boldsymbol{u}\| \le \alpha_2 \end{cases} \quad \text{and} \quad \begin{cases} \|\mathbf{A}\| \le \alpha_3 \\ \|\mathbf{B}\| \le \alpha_4 \\ \|\mathbf{C}\| \le \alpha_5 \end{cases} \tag{2}$$

for all $t \ge t_0$.

The dynamics of the continuous-time Kalman filter for (1) follow

$$\begin{cases} \dot{\hat{\boldsymbol{x}}} = \mathbf{A}\hat{\boldsymbol{x}} + \mathbf{B}\boldsymbol{u} + \mathbf{K}[\boldsymbol{y} - \mathbf{C}\hat{\boldsymbol{x}}] \\ \mathbf{K} = \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1} \\ \dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathsf{T}} + \mathbf{D}\mathbf{Q}\mathbf{D}^{\mathsf{T}} - \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C}\mathbf{P}, \end{cases} \qquad \begin{cases} \hat{\boldsymbol{x}}(t_0) = \hat{\boldsymbol{x}}_0 \\ \mathbf{P}(t_0) = \mathbf{P}_0, \end{cases} \tag{3}$$

in which $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ is the state estimate of the filter, $\mathbf{K} \in \mathbb{R}^{n\times o}$ is the filter gain, and $\mathbf{P} \succ \mathbf{0} \in \mathbb{R}^{n\times n}$ is the estimation error covariance matrix. $\hat{\boldsymbol{x}}_0$ and $\mathbf{P}_0$ are, respectively, the initial state estimate and initial error covariance matrix. The matrices $\mathbf{D} \in \mathbb{R}^{n\times p}$, $\mathbf{Q} \succeq \mathbf{0} \in \mathbb{R}^{p\times p}$, and $\mathbf{R} \succ \mathbf{0} \in \mathbb{R}^{o\times o}$ are used to model process and observation noise. It is assumed that there exist positive scalar constants α_6, α_7, and α_8 such that

$$\begin{cases} \|\mathbf{D}\| \le \alpha_6 \\ \alpha_7^{-1} \le \|\mathbf{R}\| \le \alpha_7 \\ \|\mathbf{Q}\| \le \alpha_8 \end{cases} \tag{4}$$

for all $t \ge t_0$.

Now, suppose that for implementation of the Kalman filter (3) the nominal values of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\boldsymbol{u}$, and $\boldsymbol{y}$ are not available and that estimates of those quantities, denoted respectively as $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}$, $\hat{\boldsymbol{u}}$, and $\hat{\boldsymbol{y}}$ must be used instead. Define the errors on those estimates as

$$\begin{cases} \tilde{\mathbf{A}} := \mathbf{A} - \hat{\mathbf{A}} \\ \tilde{\mathbf{B}} := \mathbf{B} - \hat{\mathbf{B}} \\ \tilde{\mathbf{C}} := \mathbf{C} - \hat{\mathbf{C}} \\ \tilde{\mathbf{D}} := \mathbf{D} - \hat{\mathbf{D}} \\ \tilde{\boldsymbol{u}} := \boldsymbol{u} - \hat{\boldsymbol{u}} \\ \tilde{\boldsymbol{y}} := \boldsymbol{y} - \hat{\boldsymbol{y}}. \end{cases} \tag{5}$$

It is assumed that these errors decay exponentially fast with time, that is, there exist positive scalar constants α_9, α_{10}, α_{11}, α_{12}, α_{13}, α_{14}, λ_1, λ_2, λ_3, λ_4, λ_5, and λ_6 such that

$$\begin{cases} \|\tilde{\mathbf{A}}\| \le \alpha_9 e^{-\lambda_1(t-t_0)} \\ \|\tilde{\mathbf{B}}\| \le \alpha_{10} e^{-\lambda_2(t-t_0)} \\ \|\tilde{\mathbf{C}}\| \le \alpha_{11} e^{-\lambda_3(t-t_0)} \\ \|\tilde{\mathbf{D}}\| \le \alpha_{12} e^{-\lambda_4(t-t_0)} \\ \|\tilde{\boldsymbol{u}}\| \le \alpha_{13} e^{-\lambda_5(t-t_0)} \\ \|\tilde{\boldsymbol{y}}\| \le \alpha_{14} e^{-\lambda_6(t-t_0)} \end{cases} \tag{6}$$

for all $t \ge t_0$. Note that, as a result of the boundedness of $\mathbf{A}$ and $\hat{\mathbf{A}}$, the norm of the associated state transition matrices, $\mathbf{\Phi}(t, t_0)$ and $\hat{\mathbf{\Phi}}(t, t_0)$ respectively, can also be bounded for finite intervals. More specifically, for any given $T > 0$, there exist positive scalar constants α_{15} and α_{16} such that

$$\begin{cases} \|\mathbf{\Phi}(t + t^*, t)\| \le \alpha_{15} \\ \|\hat{\mathbf{\Phi}}(t + t^*, t)\| \le \alpha_{16} \end{cases} \tag{7}$$

for all $0 \le t^* \le T$ and $t \ge t_0$.

Using the estimates that are available, the Kalman filter equations become

$$\begin{cases} \dot{\hat{\boldsymbol{x}}} = \hat{\mathbf{A}}\hat{\boldsymbol{x}} + \hat{\mathbf{B}}\hat{\boldsymbol{u}} + \hat{\mathbf{K}}[\hat{\boldsymbol{y}} - \hat{\mathbf{C}}\hat{\boldsymbol{x}}] \\ \hat{\mathbf{K}} = \hat{\mathbf{P}}\hat{\mathbf{C}}^{\mathsf{T}}\mathbf{R}^{-1} \\ \dot{\hat{\mathbf{P}}} = \hat{\mathbf{A}}\hat{\mathbf{P}} + \hat{\mathbf{P}}\hat{\mathbf{A}}^{\mathsf{T}} + \hat{\mathbf{D}}\mathbf{Q}\hat{\mathbf{D}}^{\mathsf{T}} - \hat{\mathbf{P}}\hat{\mathbf{C}}^{\mathsf{T}}\mathbf{R}^{-1}\hat{\mathbf{C}}\hat{\mathbf{P}}. \end{cases} \tag{8}$$

Note that, due to the perturbations in the parameters of the system, both the filter gain $\hat{\mathbf{K}}$ and the solution of the Riccati equation $\hat{\mathbf{P}}$ will deviate from their nominal counterparts $\mathbf{K}$ and $\mathbf{P}$. The new filter equations (8) will be referred to as Perturbed Kalman Filter (PKF) equations from hereon for the sake of convenience.

The observability Gramian associated with the pair $(\mathbf{A}, \mathbf{C})$ is defined as

$$\mathcal{W}_O(t_1, t_2) = \int_{t_1}^{t_2} \mathbf{\Phi}^{\mathsf{T}}(\sigma, t_1)\mathbf{C}^{\mathsf{T}}(\sigma)\mathbf{R}^{-1}(\sigma)\mathbf{C}(\sigma)\mathbf{\Phi}(\sigma, t_1)\, d\sigma,$$

and the controllability Gramian associated with the pair $(\mathbf{A}, \mathbf{D})$ follows

$$\mathcal{W}_C(t_1, t_2) = \int_{t_1}^{t_2} \mathbf{\Phi}(t_1, \sigma)\mathbf{D}(\sigma)\mathbf{Q}(\sigma)\mathbf{D}^{\mathsf{T}}(\sigma)\mathbf{\Phi}^{\mathsf{T}}(t_1, \sigma)\, d\sigma.$$

The pair $(\mathbf{A}, \mathbf{C})$ is said to be UCO if and only if there exist positive scalar constants δ_1 and γ_1 such that

$$\gamma_1^{-1} \le \mathbf{d}^{\mathsf{T}}\mathcal{W}_O(t, t + \delta_1)\mathbf{d} \le \gamma_1$$

for all $t \geq t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. Similarly, the pair $(\mathbf{A}, \mathbf{D})$ is said to be UCC if and only if there exist positive scalar constants δ_2 and γ_2 such that

$$\gamma_2^{-1} \leq \mathbf{d}^\mathsf{T} \mathcal{W}_C(t, t+\delta_2)\mathbf{d} \leq \gamma_2$$

for all $t \geq t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. If the pair $(\mathbf{A}, \mathbf{C})$ is UCO and the pair $(\mathbf{A}, \mathbf{D})$ is UCC, then the state estimate $\hat{\boldsymbol{x}}$ of the Kalman filter (3) converges globally exponentially fast to the state $\boldsymbol{x}$ of the nominal LTV system (1), see [13].

The problem considered in this paper is the stability analysis of the PKF, that is, to show that, if the nominal pairs $(\mathbf{A}, \mathbf{C})$ and $(\mathbf{A}, \mathbf{D})$ are UCO and UCC respectively, then the state estimate of the PKF (8) also converges exponentially fast to the state of the nominal LTV system (1). The approach detailed in this paper is divided into three sub-problems:

1. Show that the perturbed pairs $(\hat{\mathbf{A}}, \hat{\mathbf{C}})$ and $(\hat{\mathbf{A}}, \hat{\mathbf{D}})$ are also UCO and UCC, respectively, which allows to establish useful bounds on $\hat{\mathbf{P}}$;
2. Show that the solution $\hat{\mathbf{P}}$ of the perturbed Riccati equation in (8) converges to the solution $\mathbf{P}$ of the nominal Riccati equation in (3), which entails that the perturbed Kalman gain $\hat{\mathbf{K}}$ also converges to the nominal $\mathbf{K}$;
3. Finally, prove that the state estimate $\hat{\boldsymbol{x}}$ of the PKF (8) converges exponentially fast to the state $\boldsymbol{x}$ of the nominal system (1).

3. Uniform complete observability and uniform complete controllability

This section focuses on showing that uniform complete observability of the nominal pair $(\mathbf{A}, \mathbf{C})$ entails uniform complete observability of the perturbed pair $(\hat{\mathbf{A}}, \hat{\mathbf{C}})$, and that uniform complete controllability of the nominal pair $(\mathbf{A}, \mathbf{D})$ entails uniform complete controllability of the perturbed pair $(\hat{\mathbf{A}}, \hat{\mathbf{D}})$.

The following proposition is needed to prove the main results of this section:

Proposition 1. *Define the deviation of* $\hat{\boldsymbol{\Phi}}(t, t_0)$ *from* $\boldsymbol{\Phi}(t, t_0)$ *as*

$$\tilde{\boldsymbol{\Phi}}(t, t_0) := \boldsymbol{\Phi}(t, t_0) - \hat{\boldsymbol{\Phi}}(t, t_0). \tag{9}$$

For any given constant $\delta > 0$, *there exist positive scalar constants* α_{17} *and* λ_7 *such that*

$$\|\tilde{\boldsymbol{\Phi}}(t+\delta, t)\| \leq \alpha_{17} e^{-\lambda_7 (t-t_0)} \tag{10}$$

for all $t \geq t_0$.

Proof. Differentiating $\tilde{\boldsymbol{\Phi}}(t+\tau, t)$ with respect to τ yields

$$\begin{aligned}\frac{\partial \tilde{\boldsymbol{\Phi}}(t+\tau, t)}{\partial \tau} &= \frac{\partial \boldsymbol{\Phi}(t+\tau, t)}{\partial \tau} - \frac{\partial \hat{\boldsymbol{\Phi}}(t+\tau, t)}{\partial \tau} \\ &= \mathbf{A}(t+\tau)\boldsymbol{\Phi}(t+\tau, t) - \hat{\mathbf{A}}(t+\tau)\hat{\boldsymbol{\Phi}}(t+\tau, t)\end{aligned} \tag{11}$$

for all $t \geq t_0$. Using (9) in (11), it can be shown that

$$\frac{\partial \tilde{\boldsymbol{\Phi}}(t+\tau, t)}{\partial \tau} = \mathbf{A}(t+\tau)\tilde{\boldsymbol{\Phi}}(t+\tau, t) + \tilde{\mathbf{A}}(t+\tau)\hat{\boldsymbol{\Phi}}(t+\tau, t) \tag{12}$$

for all $t \geq t_0$. Solving (12) with the initial condition $\tilde{\boldsymbol{\Phi}}(t, t) = \mathbf{0}$ yields, for any given $\delta > 0$,

$$\tilde{\boldsymbol{\Phi}}(t+\delta, t) = \int_0^\delta \boldsymbol{\Phi}(t+\delta, t+\tau)\tilde{\mathbf{A}}(t+\tau)\hat{\boldsymbol{\Phi}}(t+\tau, t)\, d\tau \tag{13}$$

for all $t \geq t_0$. Taking the norm of (13), it follows that

$$\begin{aligned}&\|\tilde{\boldsymbol{\Phi}}(t+\delta, t)\| \\ &\quad \leq \left\| \int_0^\delta \boldsymbol{\Phi}(t+\delta, t+\tau)\tilde{\mathbf{A}}(t+\tau)\hat{\boldsymbol{\Phi}}(t+\tau, t)\, d\tau \right\| \\ &\quad \leq \int_0^\delta \|\boldsymbol{\Phi}(t+\delta, t+\tau)\|\, \|\tilde{\mathbf{A}}(t+\tau)\|\, \|\hat{\boldsymbol{\Phi}}(t+\tau, t)\|\, d\tau\end{aligned} \tag{14}$$

for all $t \geq t_0$. Substituting (6) and (7) in (14) yields

$$\begin{aligned}\|\tilde{\boldsymbol{\Phi}}(t+\delta, t)\| &\leq \int_0^\delta \alpha_9 \alpha_{15} \alpha_{16} e^{-\lambda_1 (t+\tau-t_0)}\, d\tau \\ &= \alpha_9 \alpha_{15} \alpha_{16} e^{-\lambda_1 (t-t_0)} \int_0^\delta e^{-\lambda_1 \tau}\, d\tau \\ &= \frac{\alpha_9 \alpha_{15} \alpha_{16}}{\lambda_1}(1 - e^{-\lambda_1 \delta}) e^{-\lambda_1 (t-t_0)}\end{aligned}$$

for all $t \geq t_0$. Thus, there exist positive scalar constants α_{17} and λ_7 such that (10) is verified for all $t \geq t_0$. □

The following result establishes uniform complete observability of the perturbed pair $(\hat{\mathbf{A}}, \hat{\mathbf{C}})$.

Lemma 1. *Suppose that the pair* $(\mathbf{A}, \mathbf{C})$ *of the nominal system* (1) *is UCO. Then, the perturbed pair* $(\hat{\mathbf{A}}, \hat{\mathbf{C}})$ *is UCO.*

Proof. The observability Gramian of the pair $(\mathbf{A}, \mathbf{C})$, repeated here for the sake of clarity, is given by

$$\mathcal{W}_O(t_1, t_2) = \int_{t_1}^{t_2} \boldsymbol{\Phi}^\mathsf{T}(\sigma, t_1)\mathbf{C}^\mathsf{T}(\sigma)\mathbf{R}^{-1}(\sigma)\mathbf{C}(\sigma)\boldsymbol{\Phi}(\sigma, t_1)\, d\sigma.$$

As the pair $(\mathbf{A}, \mathbf{C})$ is assumed to be UCO, it follows that there exist positive scalar constants δ_1 and γ_1 such that

$$\gamma_1^{-1} \leq \mathbf{d}^\mathsf{T} \mathcal{W}_O(t, t+\delta_1)\mathbf{d} \leq \gamma_1 \tag{15}$$

holds for all $t \geq t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. The observability Gramian for the perturbed pair $(\hat{\mathbf{A}}, \hat{\mathbf{C}})$ follows

$$\hat{\mathcal{W}}_O(t_1, t_2) = \int_{t_1}^{t_2} \hat{\boldsymbol{\Phi}}^\mathsf{T}(\sigma, t_1)\hat{\mathbf{C}}^\mathsf{T}(\sigma)\mathbf{R}^{-1}(\sigma)\hat{\mathbf{C}}(\sigma)\hat{\boldsymbol{\Phi}}(\sigma, t_1)\, d\sigma,$$

or, using (5) and (9),

$$\hat{\mathcal{W}}_O(t_1, t_2) = \mathcal{W}_O(t_1, t_2) + \boldsymbol{\Gamma}_1(t_1, t_2) - \boldsymbol{\Gamma}_2(t_1, t_2) - \boldsymbol{\Gamma}_3(t_1, t_2),$$

with

$$\begin{aligned}\boldsymbol{\Gamma}_1(t_1, t_2) \;=\; &\int_{t_1}^{t_2} \Big[\boldsymbol{\Phi}^\mathsf{T}(\sigma, t_1)\tilde{\mathbf{C}}^\mathsf{T}(\sigma)\mathbf{R}^{-1}(\sigma)\tilde{\mathbf{C}}(\sigma)\boldsymbol{\Phi}(\sigma, t_1) \\ &+ \tilde{\boldsymbol{\Phi}}^\mathsf{T}(\sigma, t_1)\hat{\mathbf{C}}^\mathsf{T}(\sigma)\mathbf{R}^{-1}(\sigma)\hat{\mathbf{C}}(\sigma)\tilde{\boldsymbol{\Phi}}(\sigma, t_1)\Big]\, d\sigma \succeq \mathbf{0}, \\ \boldsymbol{\Gamma}_2(t_1, t_2) \;=\; &\int_{t_1}^{t_2} \Big[\boldsymbol{\Phi}^\mathsf{T}(\sigma, t_1)\tilde{\mathbf{C}}^\mathsf{T}(\sigma)\mathbf{R}^{-1}(\sigma)\mathbf{C}(\sigma)\boldsymbol{\Phi}(\sigma, t_1) \\ &+ \boldsymbol{\Phi}^\mathsf{T}(\sigma, t_1)\mathbf{C}^\mathsf{T}(\sigma)\mathbf{R}^{-1}(\sigma)\tilde{\mathbf{C}}(\sigma)\boldsymbol{\Phi}(\sigma, t_1)\Big]\, d\sigma,\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\Gamma}_3(t_1, t_2) = &\int_{t_1}^{t_2} \Big[\tilde{\boldsymbol{\Phi}}^\mathsf{T}(\sigma, t_1)\hat{\mathbf{C}}^\mathsf{T}(\sigma)\mathbf{R}^{-1}(\sigma)\hat{\mathbf{C}}(\sigma)\boldsymbol{\Phi}(\sigma, t_1) \\ &+ \boldsymbol{\Phi}^\mathsf{T}(\sigma, t_1)\hat{\mathbf{C}}^\mathsf{T}(\sigma)\mathbf{R}^{-1}(\sigma)\hat{\mathbf{C}}(\sigma)\tilde{\boldsymbol{\Phi}}(\sigma, t_1)\Big]\, d\sigma.\end{aligned}$$

The norm of $\mathbf{\Gamma}_2(t_1, t_2)$ satisfies

$$\begin{aligned}\|\mathbf{\Gamma}_2(t_1,t_2)\| &= \left\| \int_{t_1}^{t_2} \Big[\mathbf{\Phi}^{\mathsf{T}}(\sigma,t_1)\tilde{\mathbf{C}}^{\mathsf{T}}(\sigma)\mathbf{R}^{-1}(\sigma)\mathbf{C}(\sigma)\mathbf{\Phi}(\sigma,t_1) \right. \\ &\quad \left. + \mathbf{\Phi}^{\mathsf{T}}(\sigma,t_1)\mathbf{C}^{\mathsf{T}}(\sigma)\mathbf{R}^{-1}(\sigma)\tilde{\mathbf{C}}(\sigma)\mathbf{\Phi}(\sigma,t_1)\Big]\, d\sigma \right\| \\ &\le \int_{t_1}^{t_2} \Big\| \mathbf{\Phi}^{\mathsf{T}}(\sigma,t_1)\tilde{\mathbf{C}}^{\mathsf{T}}(\sigma)\mathbf{R}^{-1}(\sigma)\mathbf{C}(\sigma)\mathbf{\Phi}(\sigma,t_1) \\ &\quad + \mathbf{\Phi}^{\mathsf{T}}(\sigma,t_1)\mathbf{C}^{\mathsf{T}}(\sigma)\mathbf{R}^{-1}(\sigma)\tilde{\mathbf{C}}(\sigma)\mathbf{\Phi}(\sigma,t_1)\Big\|\, d\sigma \\ &\le 2\int_{t_1}^{t_2} \|\mathbf{\Phi}(\sigma,t_1)\|^2 \|\tilde{\mathbf{C}}(\sigma)\|\, \|\mathbf{C}(\sigma)\|\, \|\mathbf{R}^{-1}(\sigma)\|\, d\sigma.\end{aligned}$$

Then, using the bounds in (2), (4), (6), and (7), it follows that there is a positive scalar constant α_{18} such that

$$\|\mathbf{\Gamma}_2(t, t+\delta_1)\| \le \alpha_{18} e^{-\lambda_3(t-t_0)}$$

for all $t \ge t_0$, which in turn implies that

$$\mathbf{d}^{\mathsf{T}}\mathbf{\Gamma}_2(t,t+\delta_1)\mathbf{d} \le \alpha_{18} e^{-\lambda_3(t-t_0)} \tag{16}$$

for all $t \ge t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. Similar steps can be carried out for $\mathbf{\Gamma}_3(t_1, t_2)$ using Proposition 1 and the bounds in (2), (4), (6), and (7), and it follows that there exists a positive scalar constant α_{19} such that

$$\mathbf{d}^{\mathsf{T}}\mathbf{\Gamma}_3(t,t+\delta_1)\mathbf{d} \le \alpha_{19} e^{-\lambda_7(t-t_0)} \tag{17}$$

for all $t \ge t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. Thus, (15), (16), and (17) imply that

$$\mathbf{d}^{\mathsf{T}}\hat{\mathcal{W}}_O(t,t+\delta_1)\mathbf{d} \ge \gamma_1 - \alpha_{18}e^{-\lambda_3(t-t_0)} - \alpha_{19}e^{-\lambda_7(t-t_0)}$$

for all $t \ge t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. Then, for any $0 < \gamma_3 < \gamma_1$, there exists a $t_1 \ge t_0$ such that

$$\mathbf{d}^{\mathsf{T}}\hat{\mathcal{W}}_O(t,t+\delta_1)\mathbf{d} \ge \gamma_3$$

for all $t \ge t_1$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. In turn, this implies that there exists a positive scalar constant $\delta_3 \ge \delta_1$ such that

$$\mathbf{d}^{\mathsf{T}}\hat{\mathcal{W}}_O(t,t+\delta_3)\mathbf{d} \ge \gamma_3$$

for all $t \ge t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. As the perturbed observability Gramian is a definite integral of bounded functions of time, it also follows that there is a $\gamma_4 > 0$ such that

$$\mathbf{d}^{\mathsf{T}}\hat{\mathcal{W}}_O(t,t+\delta_3)\mathbf{d} \le \gamma_4$$

for all $t \ge t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$. Thus, the perturbed pair $(\hat{\mathbf{A}}, \hat{\mathbf{C}})$ is UCO. □

The following result establishes uniform complete observability of the perturbed pair $(\hat{\mathbf{A}}, \hat{\mathbf{C}})$.

Lemma 2. *Suppose that the pair $(\mathbf{A}, \mathbf{D})$ of the nominal system (1) is UCC. Then, the perturbed pair $(\hat{\mathbf{A}}, \hat{\mathbf{D}})$ is UCC.*

Proof. The proof for this result is very similar to the proof for Lemma 1, as the same steps can be followed using the controllability Gramian instead of the observability Gramian. □

These results will be useful to study the convergence of the perturbed Riccati equation in (8), as they allow to establish bounds for $\hat{\mathbf{P}}$. To be more specific, if the LTV system (1) is UCO and UCC, then there exist positive scalar constants δ_4, α_{20}, and α_{21} such that

$$\begin{cases} \alpha_{20}^{-1} \le \|\mathbf{P}\| \le \alpha_{20} \\ \alpha_{21}^{-1} \le \|\hat{\mathbf{P}}\| \le \alpha_{21} \end{cases}$$

for all $t \ge t_0 + \delta_4$, see [13].

4. Stability of the Riccati equation

This section focuses on showing that the solution $\hat{\mathbf{P}}$ of the Riccati equation in (8) converges to the nominal solution of (3) and thus that the gain $\hat{\mathbf{K}}$ of the PKF also converges to the nominal Kalman gain $\mathbf{K}$.

Define the deviation of $\hat{\mathbf{P}}$ from $\mathbf{P}$ as

$$\tilde{\mathbf{P}} := \mathbf{P} - \hat{\mathbf{P}}. \tag{18}$$

Taking the time derivative of (18) and using (3), (5), and (8) yields

$$\begin{aligned}\dot{\tilde{\mathbf{P}}} &= \dot{\mathbf{P}} - \dot{\hat{\mathbf{P}}} \\ &= \mathbf{AP} + \mathbf{PA}^{\mathsf{T}} + \mathbf{DQD}^{\mathsf{T}} - \mathbf{PC}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{CP} \\ &\quad - \hat{\mathbf{A}}\hat{\mathbf{P}} - \hat{\mathbf{P}}\hat{\mathbf{A}}^{\mathsf{T}} - \hat{\mathbf{D}}\mathbf{Q}\hat{\mathbf{D}}^{\mathsf{T}} + \hat{\mathbf{P}}\hat{\mathbf{C}}^{\mathsf{T}}\mathbf{R}^{-1}\hat{\mathbf{C}}\hat{\mathbf{P}} \\ &= \mathbf{F}(\tilde{\mathbf{P}}) + \mathbf{G}(\tilde{\mathbf{C}}) + \mathbf{H}(\tilde{\mathbf{A}}) + \mathbf{J}(\tilde{\mathbf{D}}),\end{aligned} \tag{19}$$

in which

$$\mathbf{F}(\tilde{\mathbf{P}}) = (\mathbf{A} - \mathbf{PC}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C})\tilde{\mathbf{P}} + \tilde{\mathbf{P}}(\mathbf{A} - \mathbf{PC}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C})^{\mathsf{T}} + \tilde{\mathbf{P}}\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C}\tilde{\mathbf{P}},$$

$$\mathbf{G}(\tilde{\mathbf{C}}) = \hat{\mathbf{P}}\left[\tilde{\mathbf{C}}^{\mathsf{T}}\mathbf{R}^{-1}\tilde{\mathbf{C}} - \tilde{\mathbf{C}}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C} - \mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\tilde{\mathbf{C}}\right]\hat{\mathbf{P}},$$

$$\mathbf{H}(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}\hat{\mathbf{P}} + \hat{\mathbf{P}}\tilde{\mathbf{A}}^{\mathsf{T}},$$

and

$$\mathbf{J}(\tilde{\mathbf{D}}) = \tilde{\mathbf{D}}\mathbf{Q}\mathbf{D}^{\mathsf{T}} + \mathbf{D}\mathbf{Q}\tilde{\mathbf{D}}^{\mathsf{T}} - \tilde{\mathbf{D}}\mathbf{Q}\tilde{\mathbf{D}}^{\mathsf{T}}.$$

The following result establishes a sufficient condition for the exponential convergence of the dynamics of (18).

Lemma 3. *Suppose that the LTV system (1) is UCO and UCC and verifies the bounds in (2) and (6). Then, $\tilde{\mathbf{P}}$ converges exponentially fast to the origin, in the sense that, for any given initial condition $\tilde{\mathbf{P}}(t_0)$, it is possible to choose positive scalar constants α and λ such that $\|\tilde{\mathbf{P}}\| \le \alpha e^{-\lambda(t-t_0)}$ for all $t \ge t_0$.*

Proof. First, note that when there are no perturbations in $\hat{\mathbf{A}}$, $\hat{\mathbf{C}}$, and $\hat{\mathbf{D}}$, the dynamics of $\tilde{\mathbf{P}}$ follow

$$\dot{\tilde{\mathbf{P}}} = \mathbf{F}(\tilde{\mathbf{P}}), \tag{20}$$

whose origin is globally exponentially stable (GES), see [14]. Defining

$$\begin{cases} \tilde{\mathbf{P}}_v = \mathrm{vec}(\tilde{\mathbf{P}}) \\ \mathbf{F}_v(\tilde{\mathbf{P}}) = \mathrm{vec}(\mathbf{F}(\tilde{\mathbf{P}})) \\ \mathbf{G}_v(\tilde{\mathbf{C}}) = \mathrm{vec}(\mathbf{G}(\tilde{\mathbf{C}})) \\ \mathbf{H}_v(\tilde{\mathbf{A}}) = \mathrm{vec}(\mathbf{H}(\tilde{\mathbf{A}})) \\ \mathbf{J}_v(\tilde{\mathbf{D}}) = \mathrm{vec}(\mathbf{J}(\tilde{\mathbf{D}})), \end{cases}$$

the systems (19) and (20) can be represented respectively as

$$\dot{\tilde{\mathbf{P}}}_v = \mathbf{F}_v(\tilde{\mathbf{P}}) + \mathbf{G}_v(\tilde{\mathbf{C}}) + \mathbf{H}_v(\tilde{\mathbf{A}}) + \mathbf{J}_v(\tilde{\mathbf{D}}) \tag{21}$$

and

$$\dot{\tilde{\mathbf{P}}}_v = \mathbf{F}_v(\tilde{\mathbf{P}}). \tag{22}$$

Then, as the origin of (22) is GES, there exist positive constants c_1, c_2, c_3, and c_4 and a Lyapunov function $V : [0,\infty[\times\mathbb{R}^{n^2} \to \mathbb{R}$ that satisfy

$$\begin{cases} c_1\|\tilde{\mathbf{P}}_v\|^2 \le V(t,\tilde{\mathbf{P}}_v) \le c_2\|\tilde{\mathbf{P}}_v\|^2 \\ \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial \tilde{\mathbf{P}}_v}\mathbf{F}_v(\tilde{\mathbf{P}}) \le -c_3\|\tilde{\mathbf{P}}_v\|^2 \\ \left\|\dfrac{\partial V}{\partial \tilde{\mathbf{P}}_v}\right\| \le c_4\|\tilde{\mathbf{P}}_v\| \end{cases}$$

for all $t \geq t_0$, see [11, Theorem 4.14]. On the other hand, there exists a $t_1 \geq t_0$ for which the term $\mathbf{G}(\tilde{\mathbf{C}})$ can be bounded as follows:

$$\begin{aligned}\|\mathbf{G}(\tilde{\mathbf{C}})\| &= \left\|\hat{\mathbf{P}}\left[\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\tilde{\mathbf{C}} - \tilde{\mathbf{C}}^T\mathbf{R}^{-1}\mathbf{C} - \mathbf{C}^T\mathbf{R}^{-1}\tilde{\mathbf{C}}\right]\hat{\mathbf{P}}\right\| \\ &\leq \|\hat{\mathbf{P}}\|^2\|\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\tilde{\mathbf{C}} - \tilde{\mathbf{C}}^T\mathbf{R}^{-1}\mathbf{C} - \mathbf{C}^T\mathbf{R}^{-1}\tilde{\mathbf{C}}\| \\ &\leq \alpha_{21}^2\left(\|\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\tilde{\mathbf{C}}\| + 2\|\tilde{\mathbf{C}}^T\mathbf{R}^{-1}\mathbf{C}\|\right) \\ &\leq \alpha_7\alpha_{21}^2\|\tilde{\mathbf{C}}\|^2 + 2\alpha_5\alpha_7\alpha_{21}^2\|\tilde{\mathbf{C}}\| \end{aligned} \tag{23}$$

for all $t \geq t_1$. Now, define $\tilde{\mathbf{C}}_v := \mathrm{vec}(\tilde{\mathbf{C}})$ and note that

$$\begin{cases}\|\tilde{\mathbf{C}}\| \leq \|\tilde{\mathbf{C}}\|_F = \|\tilde{\mathbf{C}}_v\| \\ \|\mathbf{G}_v(\tilde{\mathbf{C}})\| = \|\mathbf{G}(\tilde{\mathbf{C}})\|_F \leq \sqrt{n}\|\mathbf{G}(\tilde{\mathbf{C}})\|.\end{cases} \tag{24}$$

Using (24) in (23), it is clear that there exist positive scalar constants α_{22} and α_{23} such that

$$\|\mathbf{G}_v(\tilde{\mathbf{C}})\| \leq \alpha_{22}\|\tilde{\mathbf{C}}_v\| + \alpha_{23}\|\tilde{\mathbf{C}}_v\|^2$$

for all $t \geq t_1$. Defining $\tilde{\mathbf{A}}_v := \mathrm{vec}(\tilde{\mathbf{A}})$ and proceeding in the same manner, it is straightforward to show that there exists a positive scalar constant α_{24} such that

$$\|\mathbf{H}_v(\tilde{\mathbf{A}})\| \leq \alpha_{24}\|\tilde{\mathbf{A}}_v\|$$

for all $t \geq t_1$. Finally, defining $\tilde{\mathbf{D}}_v := \mathrm{vec}(\tilde{\mathbf{D}})$ it can be shown that there exist positive scalar constants α_{25} and α_{26} such that

$$\|\mathbf{J}_v(\tilde{\mathbf{D}})\| \leq \alpha_{25}\|\tilde{\mathbf{D}}_v\| + \alpha_{26}\|\tilde{\mathbf{D}}_v\|^2$$

for all $t \geq t_0$. Then, the derivative of V along the trajectories of (21) follows

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \tilde{\mathbf{P}}_v}\left[\mathbf{F}_v(\tilde{\mathbf{P}}) + \mathbf{G}_v(\tilde{\mathbf{C}}) + \mathbf{H}_v(\tilde{\mathbf{A}}) + \mathbf{J}_v(\tilde{\mathbf{D}})\right] \\ &\leq -c_3\|\tilde{\mathbf{P}}_v\|^2 + c_4\|\tilde{\mathbf{P}}_v\|\left(\alpha_{22}\|\tilde{\mathbf{C}}_v\| + \alpha_{23}\|\tilde{\mathbf{C}}_v\|^2\right. \\ &\quad \left. + \alpha_{24}\|\tilde{\mathbf{A}}_v\| + \alpha_{25}\|\tilde{\mathbf{D}}_v\| + \alpha_{26}\|\tilde{\mathbf{D}}_v\|^2\right).\end{aligned}$$

Choosing any θ such that $0 < \theta < 1$, it is straightforward to show that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \tilde{\mathbf{P}}_v}\left[\mathbf{F}_v(\tilde{\mathbf{P}}) + \mathbf{G}_v(\tilde{\mathbf{C}}) + \mathbf{H}_v(\tilde{\mathbf{A}}) + \mathbf{J}_v(\tilde{\mathbf{D}})\right] \leq -c_3(1-\theta)\|\tilde{\mathbf{P}}_v\|^2$$

for all

$$\begin{aligned}\|\tilde{\mathbf{P}}_v\| \geq \frac{c_4}{c_3\theta}&\left(\alpha_{22}\|\tilde{\mathbf{C}}_v\| + \alpha_{23}\|\tilde{\mathbf{C}}_v\|^2 + \alpha_{24}\|\tilde{\mathbf{A}}_v\|\right. \\ &\left. + \alpha_{25}\|\tilde{\mathbf{D}}_v\| + \alpha_{26}\|\tilde{\mathbf{D}}_v\|^2\right),\end{aligned}$$

and thus (21) is input-to-state stable (ISS) with $\tilde{\mathbf{A}}_v$, $\tilde{\mathbf{C}}_v$, and $\tilde{\mathbf{D}}_v$ as inputs on the interval $[t_1, +\infty[$ [11, Theorem 4.19]. As (6) implies that $\tilde{\mathbf{A}}_v$, $\tilde{\mathbf{C}}_v$, and $\tilde{\mathbf{D}}_v$ decay exponentially with time, it follows that $\tilde{\mathbf{P}}$ also does decay exponentially fast for $t \geq t_1$ [11, Lemma 4.7]. Then, since neither $\mathbf{P}$ nor $\hat{\mathbf{P}}$ (and, consequently, $\tilde{\mathbf{P}}$) grow unbounded between t_0 and t_1, see [15], it follows that there exist positive scalar constants α and λ such that $\|\tilde{\mathbf{P}}\| \leq \alpha e^{-\lambda(t-t_0)}$, which concludes the proof. □

5. Stability of the perturbed Kalman filter

Following the previous results, it is now possible to show that the PKF (8) is indeed a suitable state observer for the LTV system (1).

Theorem 1. *Suppose that the LTV system (1) is UCO and UCC and that the bounds in (2), (4), and (6) hold for all $t \geq t_0$ Then, the PKF (8) is a state observer for the LTV system (1) with exponentially convergent error dynamics in the sense that, for any given initial condition $\tilde{\boldsymbol{x}}(t_0) := \boldsymbol{x}(t_0) - \hat{\boldsymbol{x}}(t_0)$, it is possible to choose positive scalar constants α and λ such that the estimation error $\tilde{\boldsymbol{x}} := \boldsymbol{x} - \hat{\boldsymbol{x}}$ follows $\|\tilde{\boldsymbol{x}}\| \leq \alpha e^{-\lambda(t-t_0)}$ for all $t \geq t_0$.*

Proof. First, note that as a consequence of Lemma 3 the deviation of $\hat{\mathbf{K}}$ from the nominal Kalman gain $\mathbf{K}$, defined as $\tilde{\mathbf{K}} := \mathbf{K} - \hat{\mathbf{K}}$, converges exponentially fast to zero. Using (1), (5), and (8), it can be shown that the time derivative of $\tilde{\boldsymbol{x}}$ follows

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{f}(\tilde{\boldsymbol{x}}) + \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}, \tilde{\mathbf{C}}, \tilde{\mathbf{K}}) + \boldsymbol{g}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{y}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{K}}), \tag{25}$$

in which

$$\boldsymbol{f}(\tilde{\boldsymbol{x}}) = (\mathbf{A} - \mathbf{K}\mathbf{C})\,\tilde{\boldsymbol{x}}$$

are the error dynamics of the Kalman filter (3),

$$\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}, \tilde{\mathbf{A}}, \tilde{\mathbf{C}}, \tilde{\mathbf{K}}) = \left(\mathbf{K}\tilde{\mathbf{C}} + \tilde{\mathbf{K}}\hat{\mathbf{C}} - \tilde{\mathbf{A}}\right)\tilde{\boldsymbol{x}},$$

and

$$\boldsymbol{g}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{y}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{K}}) = \left(\tilde{\mathbf{K}}\tilde{\mathbf{C}} - \mathbf{K}\tilde{\mathbf{C}} + \tilde{\mathbf{A}}\right)\boldsymbol{x} + \mathbf{B}\tilde{\boldsymbol{u}} + \tilde{\mathbf{B}}\hat{\mathbf{u}} + \hat{\mathbf{K}}\tilde{\boldsymbol{y}}. \tag{26}$$

Now, consider the system

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{f}(\tilde{\boldsymbol{x}}) + \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}, \tilde{\mathbf{A}}, \tilde{\mathbf{C}}, \tilde{\mathbf{K}}). \tag{27}$$

This system can be regarded as a perturbation of the linear system $\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{f}(\tilde{\boldsymbol{x}})$ which, under the assumptions of the theorem, is GES. Then, since the terms that multiply $\tilde{\boldsymbol{x}}$ in $\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}, \tilde{\mathbf{A}}, \tilde{\mathbf{C}}, \tilde{\mathbf{K}})$ converge exponentially fast to zero, there exists a $t_1 \geq t_0$ such that the system (27) is also GES on the interval $[t_1, +\infty[$, see [11, Lemma 9.1].

Taking the norm of both sides of (26) and using the bounds in (2) and (6), it follows that there exist positive scalar constants α_{27}, α_{28}, α_{29}, α_{30}, α_{31}, and α_{32} such that

$$\begin{aligned}\|\boldsymbol{g}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{y}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{K}})\| \leq\ & \alpha_{27}\|\tilde{\mathbf{K}}\|\,\|\tilde{\mathbf{C}}\| + \alpha_{28}\|\tilde{\mathbf{A}}\| \\ & + \alpha_{29}\|\tilde{\mathbf{B}}\| + \alpha_{30}\|\tilde{\mathbf{C}}\| + \alpha_{31}\|\tilde{\boldsymbol{u}}\| + \alpha_{32}\|\tilde{\boldsymbol{y}}\|\end{aligned}$$

for all $t \geq t_0$. Then, using the same method that was used in the proof of Lemma 3, it is straightforward to show that, for all $t \geq t_1$, the system (25) is ISS with $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{y}}$, $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$, and $\tilde{\mathbf{K}}$ as inputs. Then, since all those quantities converge exponentially fast to zero, the error dynamics (25) also converge exponentially fast to the origin for $t \geq t_1$ [11, Lemma 4.7]. Finally, as under the assumptions of the theorem neither $\boldsymbol{x}$ nor $\hat{\boldsymbol{x}}$ can grow unbounded between t_0 and t_1, it follows that there exist positive scalar constants α and λ such that $\|\tilde{\boldsymbol{x}}\| \leq \alpha e^{-\lambda(t-t_0)}$ for all $t \geq t_0$. □

6. Conclusions

This paper detailed the stability analysis of the continuous-time Kalman filter dynamics for linear time-varying systems subject to exponentially decaying perturbations. It was assumed that estimates of the input, output, and matrices of the system are available, but subject to unknown perturbations which decay exponentially with time. It was shown that if the nominal system is uniformly completely observable and uniformly completely controllable and if the state, input, and matrices of the system verify the bounds specified in Section 2, then the Kalman filter built using the perturbed estimates is a suitable state observer for the nominal system, featuring exponentially convergent error dynamics.

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