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Single range aided navigation and source localization: Observability and filter design

Pedro Batista *, Carlos Silvestre, Paulo Oliveira

Instituto Superior Técnico, Institute for Systems and Robotics, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

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ABSTRACT

This paper addresses the problems of navigation and source localization by mobile agents based on the range to a single source, in addition to relative velocity readings. The contribution of the paper is two-fold: (i) necessary and sufficient conditions on the observability of the nonlinear system are derived, which are useful for trajectory planning and motion control of the agent; and (ii) a nonlinear system, which given the input and output of the system is regarded as linear time-varying, is proposed and a Kalman filter is applied to successfully estimate the system state. Simulation results are presented in the presence of realistic measurement noise that illustrate the performance achieved with the proposed solution.

1. Introduction

The problem of source localization has been subject of intensive research in recent years. Roughly speaking, an agent has access to measurements from a set of sensors and aims to estimate the position of a source. The sensor suite depends on the environment in which the operation takes place, the mission scenario, cost constraints, etc. A popular choice is the distance to the source, designated as range in what follows. Indeed, if both the source and the agent are equipped with transponders, either acoustic or electromagnetic, depending on the surroundings, the range is readily available. A different solution consists in equipping the source with a pinger and have the internal clocks of the source and the agent synchronized, which in general involves the use of more expensive hardware to guarantee the clock synchronization with the required accuracy. Parallel to the topic of source localization is the field of aided navigation systems based on range sensors. Although there exist alternatives such as the Global Positioning System (GPS) or Ultra-Short Baseline (USBL) positioning systems, particular emphasis has been recently placed on the use of range measurements to a single source as a cost efficient solution to improve the performance of navigation systems. Although from the practical point of view this problem differs completely from source localization, it turns out that the theoretical frameworks are similar. Therefore, both problems are addressed in the paper.

Previous work in the field can be found in [1], where a so-called Synthetic Long Baseline navigation algorithm for underwater vehicles is proposed. The vehicle is assumed to have access to range measurements to a single transponder and, between sampling instants, a high performance dead-reckoning system is used to extrapolate the motion of the vehicle. A discrete-time Kalman filter is applied to a linearized model of the system to obtain the required estimates. In [2] the authors deal with the problem of underwater navigation in the presence of unknown currents based on range measurements to a single beacon. The paper presents an observability analysis based on the linearization of the nonlinear system, which yields local results, and an extended Kalman filter (EKF) is implemented to estimate the state, with no guarantees of global asymptotic stability. More recently, the same problem has been studied in [3,4], where EKFs have been extensively used to solve the navigation problem based on single beacon range measurements. In [5] the study of observability of single transponder underwater navigation was carried out resorting to an algebraic approach and algebraic observers were also proposed. More recently, the problem of cooperative navigation based on range and depth sensing was studied in [6], which relies on a linearized model and the EKF. In [7] preliminary experimental results with single beacon acoustic navigation were presented, where the EKF is employed as the state estimator. The problem of source localization has been addressed in [8], where the authors propose a localization algorithm based on the range to the source (more specifically its square) and the inertial position of the agent,

* Corresponding author.
E-mail addresses: pbatista@isr.ist.utl.pt (P. Batista), cjs@isr.ist.utl.pt (C. Silvestre), pjcro@isr.ist.utl.pt (P. Oliveira).

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which provides the necessary self-awareness of the agent motion. Global exponential stability (GES) is achieved under a persistent excitation condition.

This paper addresses the problems of navigation and source localization based on range measurements to a single source. The contribution is two-fold: (i) the observability of the nonlinear system is analyzed and necessary and sufficient conditions are derived; and (ii) a filter design methodology is proposed to estimate the system state. Central to the observability analysis and the filter design is the derivation of a nonlinear system that, given the system input and output, can be regarded as linear time-varying (LTV), with some similarities with the bilinearization of nonlinear systems, addressed in [9]. This system, obtained through appropriate state augmentation, exhibits the original behavior of the nonlinear system. In addition to range measurements, relative velocity readings are assumed and constant unknown drifts are considered. Preliminary work by the authors on the subject of source localization and vehicle navigation based on single range measurements can be found in [10], where the same setup was studied. This paper extends those results and provides more clear and detailed theorems on the observability of the system.

The paper is organized as follows. The problem addressed in the paper is stated in Section 2. Section 3 refers to the observability analysis and provides also the means for design of a state observer. Simulation results are presented in Section 4 and Section 5 summarizes the main conclusions of the paper.

1.1. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as $\operatorname{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \times \mathbf{y}$ and $\mathbf{x} \cdot \mathbf{y}$ represent the cross and inner products, respectively.

2. Problem statement

Consider an agent moving in a scenario where a fixed source is installed. Let $\{I\}$ denote an inertial reference frame and $\{B\}$ a coordinate frame attached to the agent, denominated in what follows as the body-fixed coordinate frame. The linear motion of the agent is given by

$$\dot{\mathbf{p}}(t) = \mathbf{R}(t)\mathbf{v}(t),$$

where $\mathbf{p}(t) \in \mathbb{R}^3$ denotes the inertial position of the agent, $\mathbf{v}(t) \in \mathbb{R}^3$ is the velocity of the agent relative to $\{I\}$ and expressed in body-fixed coordinates, and $\mathbf{R}(t) \in SO(3)$ is the rotation matrix from $\{B\}$ to $\{I\}$, which satisfies $\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}(\boldsymbol{\omega}(t))$, where $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is the angular velocity of $\{B\}$, expressed in body-fixed coordinates, and $\mathbf{S}(\boldsymbol{\omega})$ is the skew-symmetric matrix such that $\mathbf{S}(\boldsymbol{\omega})\mathbf{x}$ is the cross product $\boldsymbol{\omega} \times \mathbf{x}$. Let $\mathbf{s} \in \mathbb{R}^3$ denote the inertial position of the source. Then, the range to the source is given by $r(t) = \|\mathbf{r}(t)\|$, where $\mathbf{r}(t) := \mathbf{R}^T(t)[\mathbf{s} - \mathbf{p}(t)]$ is the location of the source relative to the agent, expressed in body-fixed coordinates, precisely the quantity that the agent aims to estimate.

Evidently, the signal $r(t)$ does not suffice to estimate the position of the source without some knowledge about the motion of the agent itself. In this paper it is assumed that the agent is moving in the presence of a fluid with velocity relative to the fluid $\mathbf{v}_r(t) \in \mathbb{R}^3$, expressed in body-fixed coordinates and available to the agent. Further assume that the fluid has a constant unknown velocity in inertial coordinates, which when expressed in body-fixed coordinates is denoted as $\mathbf{v}_f(t)$. Considering $\mathbf{r}(t)$ and $\mathbf{v}_f(t)$ as the system states, it is straightforward to show that the system dynamics can be written as

$$\begin{cases} \dot{\mathbf{r}}(t) = -\mathbf{v}_r(t) - \mathbf{v}_f(t) - \mathbf{S}(\boldsymbol{\omega}(t))\mathbf{r}(t) \\ \dot{\mathbf{v}}_f(t) = -\mathbf{S}(\boldsymbol{\omega}(t))\mathbf{v}_f(t) \\ r(t) = \|\mathbf{r}(t)\|, \end{cases} \tag{1}$$

which are nonlinear due to the output, $r(t)$. In (1) $\mathbf{v}_r(t)$ is interpreted as a deterministic system input and $\boldsymbol{\omega}(t)$ is a known bounded smooth function of t. The problem of source localization addressed in the paper is that of estimating $\mathbf{r}(t)$ from the knowledge of $r(t)$, $\mathbf{v}_r(t)$, and $\boldsymbol{\omega}(t)$. For navigation purposes, the position of the agent is readily obtained from

$$\mathbf{p}(t) = \mathbf{s} - \mathbf{R}(t)\mathbf{r}(t). \tag{2}$$

This paper focuses on the estimation of linear motion quantities. Therefore, the angular velocity is assumed to be available from an Inertial Measurement Unit (IMU) or an Attitude and Heading Reference System (AHRS). There are nowadays available a number of attitude estimation solutions that can compensate for rate gyro bias, see [11] and the references therein.

3. Observability analysis

While the observability of linear systems is nowadays fairly well understood, the observability of nonlinear systems is still an open field of research, as evidenced (and in spite of) the large number of recent publication on the subject, see [12–15] and the references therein. This section provides an analysis of the observability of the nonlinear time-varying system (1). First, a state transformation that preserves observability properties is applied that renders the system dynamics time invariant. Afterward, a nonlinear system, that can be regarded as LTV, is derived through state augmentation. Linear system theory is employed and sufficient conditions for the observability of the LTV system are derived. These results are extended to the original nonlinear time-varying system and necessary conditions are also derived. Finally, discussion is provided on the design of a state observer for the nonlinear system based on the results previously presented.

3.1. Coordinate transformation

Let $\mathbf{T}(t) := \operatorname{diag}(\mathbf{R}(t), \mathbf{R}(t)) \in \mathbb{R}^{6\times 6}$ and consider the state transformation

$$\begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} := \mathbf{T}(t) \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}_f(t) \end{bmatrix}, \tag{3}$$

which is a Lyapunov state transformation previously proposed by the authors [16]. The new system dynamics are given by

$$\begin{cases} \dot{\mathbf{x}}_1(t) = -\mathbf{x}_2(t) + \mathbf{u}(t) \\ \dot{\mathbf{x}}_2(t) = \mathbf{0} \\ y(t) = \|\mathbf{x}_1(t)\|, \end{cases} \tag{4}$$

where $\mathbf{u}(t) := -\mathbf{R}(t)\mathbf{v}_r(t)$. Notice that, as (3) is a Lyapunov state transformation, all observability properties are preserved [17]. The purpose of this state transformation is to simplify the system dynamics which, although nonlinear due to the system output, are no longer time-varying. Nevertheless, the ensuing analysis could have been carried out for the original nonlinear time-varying system dynamics, at the expense of additional computations.

Remark 1. Notice that (4) is the standard form for navigation purposes. However, in the problem formulation, (1) was preferred as it facilitates the interpretation of sensor noise and reduces its impact from back-and-forth transformations of sensor measurements. For example, relative velocity measurements, as provided by Doppler Velocity Logs, are expressed in body-fixed coordinates, as in (1). For large vectors, even small errors on the attitude would lead to large errors of the rotated vectors. This issue is further discussed in [16].

3.2. State augmentation

Define three additional scalar state variables as $x_3(t) := y(t)$, $x_4(t) := \mathbf{x}_1(t) \cdot \mathbf{x}_2(t)$, and $x_5(t) := \|\mathbf{x}_2(t)\|^2$, and denote by $\mathbf{x}(t) = \left[\mathbf{x}_1{}^T(t)\, \mathbf{x}_2{}^T(t)\, x_3(t)\, x_4(t)\, x_5(t)\right]^T \in \mathbb{R}^n, n = 9$, the augmented state. It is easy to verify that the dynamics of the augmented system can be written as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ y(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{5}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \dfrac{1}{y(t)}\mathbf{u}^T(t) & \mathbf{0} & 0 & -\dfrac{1}{y(t)} & 0 \\ \mathbf{0} & \mathbf{u}^T(t) & 0 & 0 & -1 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \end{bmatrix}, \tag{6}$$

$\mathbf{B} = [\mathbf{I}\,\mathbf{0}\,0\,0\,0]^T$, and $\mathbf{C} = [\mathbf{0}\,\mathbf{0}\,1\,0\,0]$. The dynamic system (5) can be regarded as a linear time-varying system, even though the system matrix $\mathbf{A}(t)$ depends explicitly on the system input and output, as evidenced by (6). Nevertheless, this is not a problem from the theoretical point of view as both the input and output of the system are known continuous bounded signals and it just suggests, in this case, that the observability of (5) may be connected with the evolution of the system input, output, or both, which is not common and does not happen when this matrix does not depend on the system input or output. The underlying reason for this is that the system is, in fact, nonlinear, even though it admits a linear interpretation for observability purposes. Before presenting the analysis of the observability of the nonlinear system, the following lemma is introduced.

Lemma 1. *Consider the nonlinear system*

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathcal{A}\left(t, \mathbf{u}(t), \mathbf{y}(t)\right)\mathbf{x}(t) + \mathcal{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathcal{C}(t)\mathbf{x}(t). \end{cases} \tag{7}$$

If the observability Gramian $\mathcal{W}\left(t_0, t_f\right)$ associated with the pair $\left(\mathcal{A}\left(t, \mathbf{u}(t), \mathbf{y}(t)\right), \mathcal{C}(t)\right)$ on $\mathcal{I} = \left[t_0, t_f\right]$ is invertible then the nonlinear system (7) is observable in the sense that, given the system input $\{\mathbf{u}(t), t \in \mathcal{I}\}$ and the system output $\{\mathbf{y}(t), t \in \mathcal{I}\}$, the initial condition $\mathbf{x}(t_0)$ is uniquely defined.

Proof. First, notice that, given the system input $\{\mathbf{u}(t), t \in \mathcal{I}\}$ and the system output $\{\mathbf{y}(t), t \in \mathcal{I}\}$, it is possible to compute the transition matrix associated with the system matrix $\mathcal{A}\left(t, \mathbf{u}(t), \mathbf{y}(t)\right)$

$$\begin{aligned} \boldsymbol{\phi}(t, t_0) = \mathbf{I} &+ \int_{t_0}^{t} \mathcal{A}\left(\sigma_1, \mathbf{u}(\sigma_1), \mathbf{y}(\sigma_1)\right) \mathrm{d}\sigma_1 \\ &+ \int_{t_0}^{t} \mathcal{A}\left(\sigma_1, \mathbf{u}(\sigma_1), \mathbf{y}(\sigma_1)\right) \\ &\quad \int_{t_0}^{\sigma_1} \mathcal{A}\left(\sigma_2, \mathbf{u}(\sigma_2), \mathbf{y}(\sigma_2)\right) \mathrm{d}\sigma_2 \mathrm{d}\sigma_1 + \cdots \end{aligned}$$

on $\mathcal{I}$, which clearly satisfies $\boldsymbol{\phi}(t_0, t_0) = \mathbf{I}$ and

$$\frac{\partial \boldsymbol{\phi}(t, t_0)}{\partial t} = \mathcal{A}\left(t, \mathbf{u}(t), \mathbf{y}(t)\right) \boldsymbol{\phi}(t, t_0).$$

Therefore, it is also possible to compute the observability Gramian

$$\mathcal{W}\left(t_0, t_f\right) = \int_{t_0}^{t_f} \boldsymbol{\phi}^T(t, t_0)\, \mathcal{C}^T(t) \mathcal{C}(t) \boldsymbol{\phi}(t, t_0)\, \mathrm{d}t.$$

Now, notice that it is possible to write the evolution of the state, given the system input and output (which allow to compute the transition matrix), as

$$\mathbf{x}(t) = \boldsymbol{\phi}(t, t_0)\, \mathbf{x}_0 + \int_{t_0}^{t} \boldsymbol{\phi}(t, \tau)\, \mathcal{B}(\tau)\, \mathbf{u}(\tau)\, \mathrm{d}\tau, \tag{8}$$

where $\mathbf{x}_0$ is the initial condition. This is easily verified as with $t = t_0$ in (8) gives $\mathbf{x}(t_0) = \mathbf{x}_0$ and taking the time derivative of (8) gives $\dot{\mathbf{x}}(t) = \mathcal{A}\left(t, \mathbf{u}(t), \mathbf{y}(t)\right)\mathbf{x}(t) + \mathcal{B}(t)\mathbf{u}(t)$. The remainder of the proof follows as in classic theory. The output of the system can be written, from (8), as

$$\mathbf{y}(t) = \mathcal{C}(t)\boldsymbol{\phi}(t, t_0)\, \mathbf{x}_0 + \mathcal{C}(t) \int_{t_0}^{t} \boldsymbol{\phi}(t, \tau)\, \mathcal{B}(\tau)\, \mathbf{u}(\tau)\, \mathrm{d}\tau. \tag{9}$$

Multiplying (9) on both sides by $\boldsymbol{\phi}^T(t, t_0)\, \mathcal{C}^T(t)$ and integrating on $\mathcal{I}$ yields

$$\begin{aligned} \mathcal{W}\left(t_0, t_f\right)\mathbf{x}_0 = &\int_{t_0}^{t_f} \boldsymbol{\phi}^T(t, t_0)\, \mathcal{C}^T(t)\mathbf{y}(t)\mathrm{d}t \\ &- \int_{t_0}^{t_f} \boldsymbol{\phi}^T(t, t_0)\, \mathcal{C}^T(t)\mathcal{C}(t) \\ &\quad \int_{t_0}^{t} \boldsymbol{\phi}(t, \tau)\, \mathcal{B}(\tau)\, \mathbf{u}(\tau)\, \mathrm{d}\tau \mathrm{d}t. \end{aligned} \tag{10}$$

All quantities in (10) but the initial condition are known given the system input and output and therefore it corresponds to a linear algebraic equation on $\mathbf{x}_0$. If the observability Gramian $\mathcal{W}\left(t_0, t_f\right)$ is invertible, then $\mathbf{x}_0$ is uniquely defined, which concludes the proof. □

Remark 2. Notice that, even though the evolution of the state given in (8) seems to resemble the response of a linear system, the response of the system does not correspond to the superposition of the free response (due to the initial conditions) and the forced response (due to system input). This is so because the transition matrix in (8) depends explicitly on the system input. For observability purposes this is not a problem because both the input and output are available.

The observability analysis of (5) will follow using Lemma 1, in which (5) is regarded as a LTV system. Before that, notice that there is nothing in (5) imposing

$$\begin{cases} y(t) = \|\mathbf{x}_1(t)\| \\ x_4(t) = \mathbf{x}_1(t) \cdot \mathbf{x}_2(t) \\ x_5(t) = \|\mathbf{x}_2(t)\|^2. \end{cases} \tag{11}$$

Although these restrictions could be easily imposed including artificial outputs, e.g., $y_2(t) = x_4(t) - \mathbf{x}_1(t) \cdot \mathbf{x}_2(t) = 0$, this form is preferred since it allows to apply Lemma 1. However, care must be taken when extrapolating conclusions from the observability of (5) to the observability of (4) or (1). Finally, notice that (6) is only well defined for $y(t) \neq 0$. This is a mild assumption as, for $y(t) = 0$, the location of the agent or vehicle coincides with the source location, which is impossible in practice. It is also required that the range in bounded. Therefore, the following assumption is introduced.

Assumption 1. There exist scalars $Y_m > 0$ and $Y_M > 0$ such that $Y_m \leq y(t) \leq Y_M$ for all $t \geq t_0$.

It is important to remark that the values of the bounds are not required for the filter design and are only used for theoretical purposes.

3.3. Observability of the LTV system

The following theorem establishes a sufficient condition on the observability of the nonlinear system (5), which is here regarded as LTV.

Theorem 1. *Let*

$$\mathbf{u}^{[1]}(t,t_0) = \left[u_1^{[1]}(t,t_0)\ u_2^{[1]}(t,t_0)\ u_3^{[1]}(t,t_0)\right]^T := \int_{t_0}^{t} \mathbf{u}(\sigma)\,\mathrm{d}\sigma.$$

If the set of functions

$$\mathcal{F} = \Big\{t-t_0, (t-t_0)^2, u_1^{[1]}(t,t_0), u_2^{[1]}(t,t_0), u_3^{[1]}(t,t_0), \\ (t-t_0)u_1^{[1]}(t,t_0), (t-t_0)u_2^{[1]}(t,t_0), (t-t_0)u_3^{[1]}(t,t_0)\Big\}$$

is linearly independent on $\mathcal{I} := [t_0, t_f]$, $t_0 < t_f$, then the LTV system (5) is observable on $\mathcal{I}$ in the sense that, given the system input $\mathbf{u}(t)$, $t \in \mathcal{I}$, and the system output $\mathbf{y}(t)$, $t \in \mathcal{I}$, the initial condition is uniquely defined.

Proof. Let $\mathcal{W}(t_0, t_f)$ denote the observability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C})$ and suppose that $\mathbf{d} \in \mathbb{R}^n$ is a unit vector. Then, it is easy to show that

$$\mathbf{d}^T \mathcal{W}(t_0,t)\,\mathbf{d} = \int_{t_0}^{t} \left[\boldsymbol{\phi}_3(\sigma,t_0)\cdot\mathbf{d}\right]^2 \mathrm{d}\sigma, \tag{12}$$

where

$$\boldsymbol{\phi}_3(t,t_0) = \begin{bmatrix} \int_{t_0}^{t} \frac{\mathbf{u}(\sigma)}{y(\sigma)}\mathrm{d}\sigma \\ \int_{t_0}^{t} \frac{-(\sigma-t_0)\mathbf{u}(\sigma) - \left[\mathbf{u}^{[1]}(\sigma,t_0)\right]}{y(\sigma)}\mathrm{d}\sigma \\ 1 \\ -\int_{t_0}^{t}\frac{1}{y(\sigma)}\mathrm{d}\sigma \\ \int_{t_0}^{t}\frac{\sigma-t_0}{y(\sigma)}\mathrm{d}\sigma \end{bmatrix}^T, \tag{13}$$

see Appendix A for the proof. Suppose that the observability Gramian $\mathcal{W}(t_0, t_f)$ associated with the pair $(\mathbf{A}(t), \mathbf{C})$ is not positive definite. Then, there exists $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$, such that $\mathbf{d}^T \mathcal{W}(t_0, t)\,\mathbf{d} = 0$ for all $t \in \mathcal{I}$, or, equivalently,

$$\int_{t_0}^{t} \left[\boldsymbol{\phi}_3(\sigma,t_0)\cdot\mathbf{d}\right]^2 \mathrm{d}\sigma = 0$$

for all $t \in \mathcal{I}$. But that implies that

$$\boldsymbol{\phi}_3(t,t_0)\cdot\mathbf{d} = 0 \tag{14}$$

for all $t \in \mathcal{I}$. Let $\mathbf{d} = \left[\mathbf{d}_1^T\ \mathbf{d}_2^T\ d_3\ d_4\ d_5\right]^T \in \mathbb{R}^n$. Notice that $\boldsymbol{\phi}_3(t_0, t_0)\cdot\mathbf{d} = d_3$ and therefore, for (14) to hold, it must be $d_3 = 0$. On the other hand, from (14) it also follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}_3(t,t_0)\cdot\mathbf{d} = 0$$

for all $t \in \mathcal{I}$, which implies that

$$\mathbf{u}(t)\cdot\mathbf{d}_1 - \left[(t-t_0)\mathbf{u}(t) + \mathbf{u}^{[1]}(t,t_0)\right]\cdot\mathbf{d}_2 - d_4 \\ + (t-t_0)d_5 = 0 \tag{15}$$

for all $t \in \mathcal{I}$. Integrating both sides of (15) gives

$$\mathbf{u}^{[1]}(t,t_0)\cdot\mathbf{d}_1 - (t-t_0)\mathbf{u}^{[1]}(t,t_0)\cdot\mathbf{d}_2 - (t-t_0)d_4 \\ + \frac{(t-t_0)^2}{2}d_5 = 0,$$

which means that the set of functions $\mathcal{F}$ is not linearly independent on $\mathcal{I}$. Therefore, if the set of functions $\mathcal{F}$ is linearly independent on $\mathcal{I}$, then the observability Gramian $\mathcal{W}(t_0, t_f)$ associated with the pair $(\mathbf{A}(t), \mathbf{C})$ on $\mathcal{I}$ is positive definite. The conclusion of the theorem follows from Lemma 1. □

The linear independence condition on the set of functions $\mathcal{F}$ provides little insight on the motion required by the vehicle so that observability is attained. This is established in the following theorem.

Theorem 2. *The LTV system (5) is observable on $[t_0, t_f]$, $t_0 < t_f$, as established in Theorem 1, if the set of functions*

$$\mathcal{F}^* = \{t-t_0, (t-t_0)^2, p_1(t)-p_1(t_0), p_2(t)-p_2(t_0), \\ p_3(t)-p_3(t_0), (t-t_0)[p_1(t)-p_1(t_0)], \\ (t-t_0)[p_2(t)-p_2(t_0)], (t-t_0)[p_3(t)-p_3(t_0)]\}$$

is linearly independent on $[t_0, t_f]$, where

$$\mathbf{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}$$

is the inertial position of the vehicle.

Proof. As it has been shown, in Theorem 1, that the LTV system (5) is observable on $[t_0, t_f]$ if the set of functions $\mathcal{F}$ is linearly independent on that interval, the proof of the theorem follows by establishing that the set of functions $\mathcal{F}$ is linearly independent on $[t_0, t_f]$ if and only if so is the set of functions $\mathcal{F}^*$. From (4) it is possible to write

$$\mathbf{u}^{[1]}(t,t_0) = \mathbf{x}_1(t) - \mathbf{x}_1(t_0) + (t-t_0)\mathbf{x}_2(t_0). \tag{16}$$

Using (3) it is possible to rewrite (16) as

$$\mathbf{u}^{[1]}(t,t_0) = \mathbf{R}(t)\mathbf{r}(t) - \mathbf{R}(t_0)\mathbf{r}(t_0) + (t-t_0)\mathbf{x}_2(t_0). \tag{17}$$

Finally, from (2), it is possible to rewrite (17) as

$$\mathbf{u}^{[1]}(t,t_0) = -\mathbf{p}(t) + \mathbf{p}(t_0) + (t-t_0)\mathbf{x}_2(t_0). \tag{18}$$

Substituting (18) in the set of functions $\mathcal{F}$, it becomes obvious that the set of functions $\mathcal{F}$ is linearly independent on $[t_0, t_f]$ if and only if so is the set of functions $\mathcal{F}^*$, which concludes the proof. □

Notice that in system (5) the additional state constraints (11) on the state variables were discarded. Therefore, it is not possible to conclude about the observability of the nonlinear system (4) from the observability of (5) without further discussion, as the initial condition of (5) could mismatch the initial condition of (4). The following section addresses this issue and provides also a constructive result on the design of state observers for the nonlinear system.

3.4. Observability of the nonlinear system

The observability of the nonlinear system (1) is discussed in this section. Even though all the results derived in this section concern the observability of (4), these also apply to the observability of the original system (1) as they are related by the Lyapunov transformation (3).

The definition of observability for nonlinear systems does not imply that every admissible input distinguishes points of the state space, although that is true for linear systems [18]. Nevertheless, that will be implied in the following result, which provides a sufficient observability condition and a practical result on the design of state observers for the nonlinear system (4).

Theorem 3. *Suppose that the set of functions $\mathcal{F}^*$ is linearly independent on $[t_0, t_f]$. Then,*

1. *the nonlinear system (4) is observable on $[t_0, t_f]$ in the sense that, given the system input $\mathbf{u}(t)$, $t \in [t_0, t_f]$ and the system output $\mathbf{y}(t)$, $t \in [t_0, t_f]$, the initial condition is uniquely defined;*

2. *a state observer with globally asymptotically stable error dynamics for the LTV system (5) is also a state observer for the nonlinear system (4), with globally asymptotically stable error dynamics.*

Proof. Let $\left[\mathbf{x}_1{}^T(t_0)\ \mathbf{x}_2{}^T(t_0)\right]^T$ be the initial state of the nonlinear system (4). Then, the output at time t is given by $y(t) = \sqrt{\|\mathbf{x}_1(t)\|^2}$, where

$$\begin{aligned}\|\mathbf{x}_1(t)\|^2 &= \left\|\mathbf{x}_1(t_0) - (t-t_0)\,\mathbf{x}_2(t_0) + \mathbf{u}^{[1]}(t,t_0)\right\|^2 \\ &= \|\mathbf{x}_1(t_0)\|^2 + \|\mathbf{x}_2(t_0)\|^2 (t-t_0)^2 + \left\|\mathbf{u}^{[1]}(t,t_0)\right\|^2 \\ &\quad - 2(t-t_0)\,\mathbf{x}_1(t_0)\cdot\mathbf{x}_2(t_0) + 2\mathbf{x}_1(t_0)\cdot\mathbf{u}^{[1]}(t,t_0) \\ &\quad - 2(t-t_0)\,\mathbf{x}_2(t_0)\cdot\mathbf{u}^{[1]}(t,t_0).\end{aligned} \tag{19}$$

Assuming that the set of functions $\mathcal{F}^*$ is linearly independent on $\left[t_0, t_f\right]$, then it follows, from Theorem 2, that the LTV system (5) is observable on $\left[t_0, t_f\right]$. Thus, given $\left\{u(t) : t \in \left[t_0, t_f\right]\right\}$ and $\left\{y(t) : t \in \left[t_0, t_f\right]\right\}$, the initial state of (5) is uniquely determined. Let $\left[\mathbf{z}_1{}^T(t_0)\ \mathbf{z}_2{}^T(t_0)\ z_3(t_0)\ z_4(t_0)\ z_5(t_0)\right]^T$ be the initial state of the linear system (5). Then, the square of the output satisfies

$$\begin{aligned}y^2(t) &= 2\mathbf{z}_1(t_0)\cdot\mathbf{u}^{[1]}(t,t_0) - 2(t-t_0)\,\mathbf{z}_2(t_0)\cdot\mathbf{u}^{[1]}(t,t_0) \\ &\quad + [z_3(t_0)]^2 - 2(t-t_0)\,z_4(t_0) + (t-t_0)^2 z_5(t_0) \\ &\quad + \left\|\mathbf{u}^{[1]}(t,t_0)\right\|^2.\end{aligned} \tag{20}$$

From the comparison between (19) and (20) it follows that

$$\begin{aligned}&2\left[\mathbf{x}_1(t_0) - \mathbf{z}_1(t_0)\right]\cdot\mathbf{u}^{[1]}(t,t_0) \\ &\quad - 2\left[\mathbf{x}_2(t_0) - \mathbf{z}_2(t_0)\right]\cdot(t-t_0)\,\mathbf{u}^{[1]}(t,t_0) \\ &\quad + \left[\|\mathbf{x}_1(t_0)\| - z_3(t_0)\right] \\ &\quad - 2\left[\mathbf{x}_1(t_0)\cdot\mathbf{x}_2(t_0) - z_4(t_0)\right](t-t_0) \\ &\quad + \left[\|\mathbf{x}_2(t_0)\|^2 - z_5(t_0)\right](t-t_0)^2 = 0\end{aligned} \tag{21}$$

for all $t \in \left[t_0, t_f\right]$. Notice that $z_3(t_0) = \|\mathbf{x}_1(t_0)\|$. Since it is assumed that the set of functions $\mathcal{F}^*$ is linearly independent, it is easy to see that the only solution of (21) is

$$\begin{bmatrix}\mathbf{x}_1(t_0) - \mathbf{z}_1(t_0) \\ \mathbf{x}_2(t_0) - \mathbf{z}_2(t_0) \\ \|\mathbf{x}_1(t_0)\| - z_3(t_0) \\ \mathbf{x}_1(t_0)\cdot\mathbf{x}_2(t_0) - z_4(t_0) \\ \|\mathbf{x}_2(t_0)\|^2 - z_5(t_0)\end{bmatrix} = \mathbf{0}.$$

This concludes the proof of the first part of the theorem, as the initial state of the linear system (5), which is uniquely determined under the conditions of the theorem, matches the initial state of the nonlinear system (4), which is also, consequently, uniquely determined under the conditions of the theorem. The second part of the theorem follows from this fact. Indeed, the estimation error of an observer for the linear system (5) with globally asymptotically stable error dynamics converges to zero, which means that its estimates asymptotically approach the true state. But as the true state of the linear system (5) matches that of the nonlinear system (4), that means that the observer for the linear system is also an observer for the nonlinear system, with globally asymptotically stable error dynamics. □

The application of Theorem 3 is two-fold: (i) on one hand, it provides a constructive result on the design of a state observer for the nonlinear system (5); and (ii) on the other hand, it provides conditions directly related to the trajectory of the agent so that the state is observable. This allows for trajectory planning and motion control of the agent so that observability is attained.

Theorem 3 does not imply that the estimates of a state observer for the LTV system (5) fulfill the constraints (11) at all time. Indeed, with the proposed solution, the trajectories of the estimates do not belong, in general, to the space of plausible solutions, i.e., the space of the solutions that satisfy the additional constraints. However, as the estimation error converges to zero, the trajectories of the state estimates converge to the set of plausible trajectories or, in other words, those constraints are verified asymptotically. As, in general, the same happens for all observers, i.e., the error converges to zero asymptotically, this approach seems effective.

Theorem 3 introduces a sufficient observability condition for the nonlinear system (4) that requires the linear independence of 8 functions, which may be considered, at first glance, a conservative result attending to the fact that the original nonlinear system only has 6 states. Nevertheless, it is shown in what follows that this linear independence condition is not that conservative. The following proposition establishes a lower bound on the number and nature of functions necessarily required to be linearly independent.

Proposition 1. *If the nonlinear system (4) is observable on $\left[t_0, t_f\right]$, $t_0 < t_f$, then the set of functions*

$$\begin{aligned}\mathcal{F}_r = \Big\{&u_1^{[1]}(t,t_0),\, u_2^{[1]}(t,t_0),\, u_3^{[1]}(t,t_0), \\ &(t-t_0)\,u_1^{[1]}(t,t_0),\, (t-t_0)\,u_2^{[1]}(t,t_0),\, (t-t_0)\,u_3^{[1]}(t,t_0)\Big\}\end{aligned}$$

is linearly independent on $\left[t_0, t_f\right]$.

Proof. Suppose that the set of functions $\mathcal{F}_r$ is linearly dependent. Then, it is clear that there exists a nonzero vector $\mathbf{c} = \left[\mathbf{c}_1^T\quad \mathbf{c}_2^T\right]^T \in \mathbb{R}^6$ such that

$$\mathbf{c}\cdot\begin{bmatrix}\mathbf{u}^{[1]}(t,t_0) \\ (t-t_0)\,\mathbf{u}^{[1]}(t,t_0)\end{bmatrix} = \mathbf{0}$$

for all $t \in \left[t_0, t_f\right]$. Let $y^a(t)$ denote the output of (4) with initial condition $\mathbf{x}_1(t_0) = -\mathbf{c}_1$, $\mathbf{x}_2(t_0) = \mathbf{c}_2$, and $y^b(t)$ denote the output of (4) with initial condition $\mathbf{x}_1(t_0) = \mathbf{c}_1$, $\mathbf{x}_2(t_0) = -\mathbf{c}_2$. Then, it is straightforward to show that $y^a(t) = y^b(t)$ for all $t \in \left[t_0, t_f\right]$. Thus, if the set of functions $\mathcal{F}_r$ is not linearly independent, then there exist, at least, two states that are indistinguishable. Therefore, the system is not observable, which concludes the proof. □

Remark 3. The observability conditions that are derived in the paper are useful for predicting trajectories that are not observable. Examples of such trajectories are straight lines that pass through the origin, as previously evidenced in [2]. In fact, this is true for all straight line trajectories. Let

$$\mathbf{p}(t) = \mathbf{p}(t_0) + \gamma(t)\mathbf{d}_v, \quad \gamma(t) \in \mathbb{R}.$$

It is straightforward to show, from (18), that

$$\mathbf{u}^{[1]}(t,t_0) = (t-t_0)\,\mathbf{x}_2(t_0) - \gamma(t)\mathbf{d}_v.$$

Clearly, in this case, the set of functions $\mathcal{F}_r$ is not linearly independent, as

$$\begin{bmatrix}\mathbf{c}^\perp \\ \mathbf{c}^\perp\end{bmatrix}\cdot\begin{bmatrix}\mathbf{u}^{[1]}(t,t_0) \\ (t-t_0)\,\mathbf{u}^{[1]}(t,t_0)\end{bmatrix} = \mathbf{0},$$

where $\mathbf{c}^\perp$ is a unit vector orthogonal to $\mathbf{x}_2(t_0)$ and $\mathbf{d}_v$. From Proposition 1 it immediately follows that the nonlinear system (4) is not observable.

3.5. Filter design

As a result of Theorem 3, a filtering solution for the nonlinear system (1) is simply obtained with the design of a Kalman filter for the augmented LTV system (5) transformed to the original coordinate space. The design is trivial and therefore it is omitted. It is important to stress that the proposed solution is not an EKF, which would not offer GAS guarantees, and no approximate linearizations are carried out. Instead, the solution is a standard Kalman filter for an augmented LTV system, which was shown in Theorem 3 to suffice to estimate the state of the original nonlinear system.

In order to guarantee that the Kalman filter has globally asymptotically stable error dynamics, stronger forms of observability are required, in particular uniform complete observability, see [19,20]. The following proposition [21, Proposition 4.2] is useful in what follows.

Proposition 2. *Let $\mathbf{f}(t)\colon \left[t_0, t_f\right] \subset \mathbb{R} \to \mathbb{R}^n$ be a continuous and i-times continuously differentiable function on $\mathcal{I} := \left[t_0, t_f\right]$, $T := t_f - t_0 > 0$, and such that*

$$\mathbf{f}\left(t_0\right) = \dot{\mathbf{f}}\left(t_0\right) = \cdots = \mathbf{f}^{(i-1)}\left(t_0\right) = \mathbf{0}.$$

Further assume that

$$\max_{t\in\mathcal{I}} \left\|\mathbf{f}^{(i+1)}(t)\right\| \leq C.$$

If

$$\underset{\substack{\alpha>0\\ t_1\in\mathcal{I}}}{\exists} : \left\|\mathbf{f}^{(i)}\left(t_1\right)\right\| \geq \alpha,$$

then

$$\underset{\substack{0<\delta\leq T\\ \beta>0}}{\exists} : \left\|\mathbf{f}\left(t_0+\delta\right)\right\| \geq \beta.$$

The following theorem addresses the issue of uniform complete observability.

Theorem 4. *If there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$ and $\mathbf{c} \in \mathbb{R}^{n-1}$, $\|\mathbf{c}\| = 1$, it is possible to choose $t^* \in [t, t+\delta]$ such that*

$$\Big|\mathbf{u}^{[1]}\left(t^*, t\right)\cdot\mathbf{c}_1 + \left(t^*-t\right)\mathbf{u}^{[1]}\left(t^*, t\right)\cdot\mathbf{c}_2 + \left(t^*-t\right)c_4 + \left(t^*-t\right)^2 c_5\Big| \geq \alpha,$$

where

$$\mathbf{c} = \begin{bmatrix}\mathbf{c}_1\\ \mathbf{c}_2\\ c_4\\ c_5\end{bmatrix}, \quad \mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^3,\ c_4, c_5 \in \mathbb{R},$$

then the pair $(\mathbf{A}(t), \mathbf{C})$ is uniformly completely observable.

Proof. That there exists a positive constant c_M such that

$$\mathbf{d}^T \mathcal{W}\left(t, t+\delta\right)\mathbf{d} \leq c_M$$

for all $t \geq t_0$ and $\mathbf{d} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$, is trivially concluded as the integrand of the observability Gramian is a continuous bounded function on all intervals $[t, t+\delta]$, $t \geq t_0$, uniformly in t. Suppose now that

$$\mathbf{d} = \begin{bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\\ d_3\\ d_4\\ d_5\end{bmatrix} \in \mathbb{R}^n, \quad \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^3,\ d_3, d_4, d_5 \in \mathbb{R},$$

with $d_3 \neq 0$. Then, notice that

$$\left|\boldsymbol{\phi}_3\left(t, t\right)\cdot\mathbf{d}\right| = \left|d_3\right| > 0$$

for all $t \geq t_0$. From Proposition 2 it immediately follows that there exists a positive constant c_{m1} such that

$$\mathbf{d}^T \mathcal{W}\left(t, t+\delta\right)\mathbf{d} \geq c_{m1}$$

for all $t \geq t_0$ and $d_3 \neq 0$. Suppose now that $d_3 = 0$, i.e.,

$$\mathbf{d} = \begin{bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\\ 0\\ d_4\\ d_5\end{bmatrix} \in \mathbb{R}^n,$$

which is also a unit vector. Under the conditions of the theorem, and resorting to the Lagrange's theorem (mean value theorem), it follows that, for all $t \geq t_0$, it is possible to choose $t_i \in]t, t_*[\subset [t, t+\delta]$ such that, for all $\mathbf{d} = \begin{bmatrix}\mathbf{d}_1^T & \mathbf{d}_2^T & 0 & d_4 & d_5\end{bmatrix} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$, it is true that

$$\left|\mathbf{u}\left(t_i\right)\cdot\mathbf{d}_1 - \left[\left(t_i-t\right)\mathbf{u}\left(t\right) + \mathbf{u}^{[1]}\left(t_i, t\right)\right]\cdot\mathbf{d}_2 - d_4 + \left(t_i-t\right)d_5\right| \geq \frac{\alpha}{\delta}.$$

Under Assumption 1, it follows that, for all $t \geq t_0$, it is possible to choose $t_i \in [t, t+\delta]$ such that, for all $\mathbf{d} = \begin{bmatrix}\mathbf{d}_1^T & \mathbf{d}_2^T & 0 & d_4 & d_5\end{bmatrix} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$, it is true that

$$\frac{\left|\mathbf{u}\left(t_i\right)\cdot\mathbf{d}_1 - \left[\left(t_i-t\right)\mathbf{u}\left(t\right) + \mathbf{u}^{[1]}\left(t_i, t\right)\right]\cdot\mathbf{d}_2 - d_4 + \left(t_i-t\right)d_5\right|}{y\left(t_i\right)} \geq \frac{\alpha}{\delta Y_M}$$

or, equivalently,

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{\phi}_3\left(\tau, t\right)\cdot\mathbf{d}\right|\Bigg|_{\tau=t_i} \geq \frac{\alpha}{\delta Y_M}.$$

But then, applying Proposition 2 twice, it is possible to conclude that there exists a positive constant c_{m2} such that, for all $t \geq t_0$ and $\mathbf{d} = \begin{bmatrix}\mathbf{d}_1^T & \mathbf{d}_2^T & 0 & d_4 & d_5\end{bmatrix} \in \mathbb{R}^n$, $\|\mathbf{d}\| = 1$, it is true that

$$\mathbf{d}^T \mathcal{W}\left(t, t+\delta\right)\mathbf{d} \geq c_{m2}.$$

This concludes the proof, as it is shown that there exists a positive constant c_m such that

$$\mathbf{d}^T \mathcal{W}\left(t, t+\delta\right)\mathbf{d} \geq c_m$$

for all $t \geq t_0$ and $\|\mathbf{d}\| = 1$. □

Notice that the condition for uniform complete observability follows naturally from the observability conditions previously derived, as it corresponds essentially to a persistent excitation condition based on the linear independence of the set of functions previously considered.

4. Simulation results

This sections presents a simulation example in order to illustrate the performance of the proposed solution. The example provided is similar to one examined in [8]. In this simulation the inertial position of the source is given by $\mathbf{s} = [2\ 3\ 1]^T$ (m) and the inertial position of the agent by $\mathbf{p}(t) = [2 + 2\sin(t)\ \ 2\cos(2t)\ \ 2\sin(0.5t)]^T$ (m). Notice that the motion of the agent is such that the sufficient condition of Theorem 3 is satisfied. The range and velocity measurements are assumed perturbed by zero-mean white Gaussian noise, with standard deviations of 0.0316 m and 0.005 m/s, respectively, which

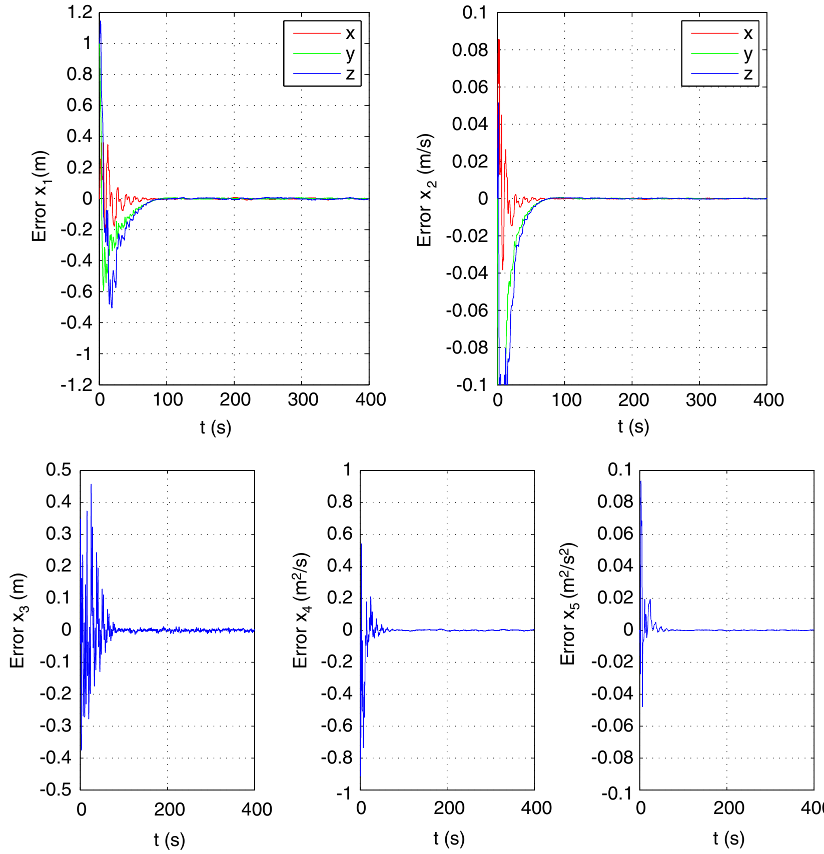

Fig. 1. Initial convergence of the estimation error variables.

given the scale of the problem are reasonable. As the estimation is carried out in body-fixed coordinates, no attitude measurements are required. The Kalman filter parameters were chosen as $\mathbf{Q} = 0.01\mathrm{diag}\,(1, 1, 1, 0.001, 0.001, 0.01, 1, 1, 0.01)$ for the state disturbance covariance matrix and the output noise variance as $R = 1$. The initial convergence of the error variables is depicted in Fig. 1, while the steady-state evolution is shown in detail in Fig. 2. Clearly, the filter is able to estimate the position of the source with great accuracy.

5. Conclusions

The problems of source localization by mobile agents and vehicle aided navigation based on range measurements to a single source were addressed in this paper. In addition to range readings, the vehicle is assumed to have relative velocity measurements and constant unknown drifts relative to an inertial reference frame are also considered, as happens, for example, with ocean robotic vehicles in the presence of sea currents. The contribution of the paper is two-fold: (i) necessary and sufficient conditions on the observability of the nonlinear system were derived, which are useful for motion planning and control of the agent; and (ii) a nonlinear system that can be regarded as LTV was developed that is appropriate for state estimation of the nonlinear range-based system. To solve the estimation problem a Kalman filter is proposed for the LTV system previously derived and characterized. Since no linearization is considered, the stability of the filter is well characterized from classic Kalman filter theory. Simulation results are presented in the presence of realistic measurement noise that illustrate the performance achieved with the proposed solution. Future work will include the application of the techniques proposed in this paper to similar problems such as localization in sensor networks given distances to anchor nodes and unknown sensor nodes, see e.g. [22–24].

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Appendix. Complements to the proof of Theorem 1

It is shown in this section that (12) is true. By definition, the observability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C})$ in $\left[t_0, t_f\right]$ is given by

$$\mathcal{W}\left(t_0, t_f\right) = \int_{t_0}^{t_f} \boldsymbol{\phi}^T\left(t, t_0\right)\mathbf{C}^T\mathbf{C}\boldsymbol{\phi}\left(t, t_0\right)\mathrm{d}t,$$

where $\boldsymbol{\phi}\left(t, t_0\right)$ denotes the transition matrix associated with $\mathbf{A}(t)$. Therefore, it immediately follows that

$$\begin{aligned}\mathbf{d}^T\mathcal{W}\left(t_0, t\right)\mathbf{d} &= \mathbf{d}^T\int_{t_0}^{t_f} \boldsymbol{\phi}^T\left(t, t_0\right)\mathbf{C}^T\mathbf{C}\boldsymbol{\phi}\left(t, t_0\right)\mathrm{d}t\mathbf{d}\\ &= \int_{t_0}^{t_f}\left[\mathbf{C}\boldsymbol{\phi}\left(t, t_0\right)\mathbf{d}\right]^2\mathrm{d}t.\end{aligned} \tag{22}$$

By definition, the transition matrix $\boldsymbol{\phi}\left(t, t_0\right)$ is given by

$$\begin{aligned}\boldsymbol{\phi}\left(t, t_0\right) &= \mathbf{I} + \int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)\mathrm{d}\sigma_1\\ &\quad + \int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)\int_{t_0}^{\sigma_1}\mathbf{A}\left(\sigma_2\right)\mathrm{d}\sigma_2\mathrm{d}\sigma_1 + \cdots.\end{aligned} \tag{23}$$

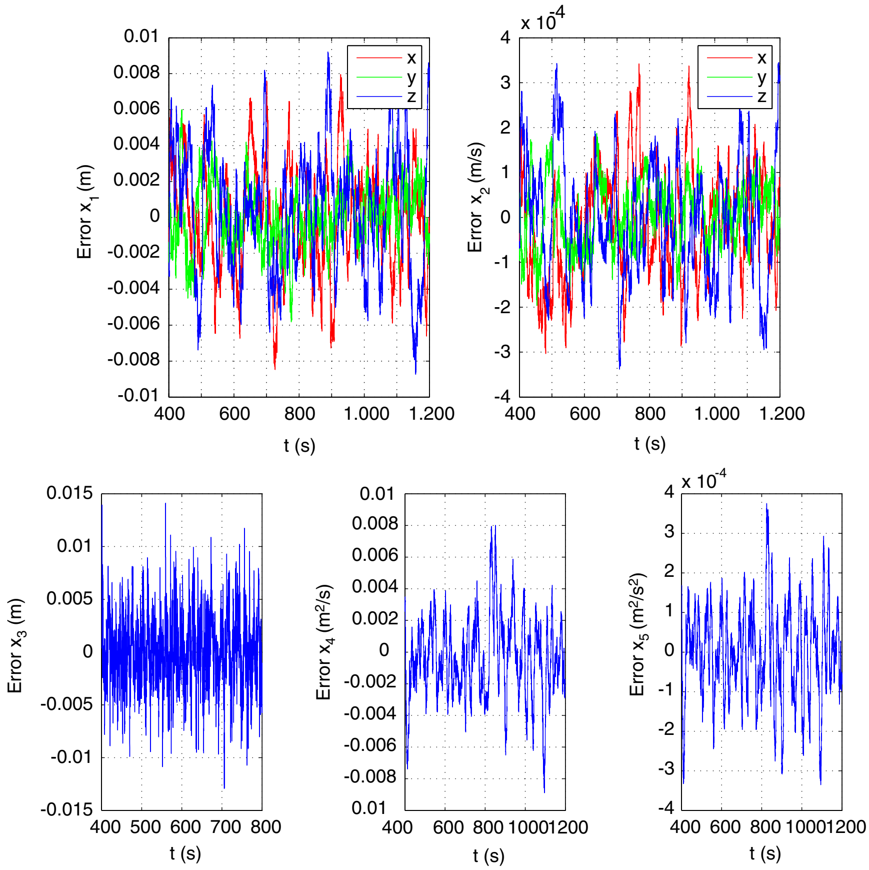

Fig. 2. Steady-state evolution of the estimation error variables.

Computing the first two terms of (23) and noticing that

$$\mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1}\mathbf{A}(\sigma_2)\int_{t_0}^{\sigma_2}\mathbf{A}(\sigma_3)\,\mathrm{d}\sigma_3\mathrm{d}\sigma_2=\mathbf{0}$$

allows to conclude that

$$\boldsymbol{\phi}(t,t_0)=\left[\begin{array}{c|c}\boldsymbol{\phi}_A(t,t_0) & \mathbf{0}\\ \hline \multicolumn{2}{c}{\boldsymbol{\phi}_3^T(t,t_0)}\\ \hline \boldsymbol{\phi}_B(t,t_0) & \boldsymbol{\phi}_C(t,t_0)\end{array}\right], \tag{24}$$

where

$$\boldsymbol{\phi}_A(t,t_0)=\begin{bmatrix}\mathbf{I} & -(t-t_0)\,\mathbf{I}\\ \mathbf{0} & \mathbf{I}\end{bmatrix},$$

$$\boldsymbol{\phi}_B(t,t_0)=\begin{bmatrix}\mathbf{0} & \left[\mathbf{u}^{[1]}(t,t_0)\right]^T\\ \mathbf{0} & \mathbf{0}\end{bmatrix},$$

$$\boldsymbol{\phi}_C(t,t_0)=\begin{bmatrix}0 & 1 & -(t-t_0)\\ 0 & 0 & 1\end{bmatrix},$$

and $\boldsymbol{\phi}_3(t,t_0)$ is given by (13). Substituting (24) in (22) immediately yields (12).

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