NONLINEAR OBSERVERS FOR TRACKING AND NAVIGATION OF MARINE SYSTEMS: DESIGN BASED ON MOTION LINEARIZATION

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Abstract: In this paper nonlinear estimation problems commonly found in marine systems are addressed, supported on a methodology recently introduced for the design of exponentially stable nonlinear observers for non-autonomous time-varying nonlinear systems (Oliveira, 2007). An extra degree of freedom is exploited in a new synthesis methodology to compute the nonlinear estimator gains, based on the solution of a set of linear equations related to the underlying Lyapunov equation. A nonlinear tracker and a complementary navigation system in the plane illustrate the design procedure and allow the performance assessment obtained with the proposed method

Keywords: Marine Vehicles, Navigation and Tracking Systems, Nonlinear Observers

1. INTRODUCTION

In the design of guidance, mission, process control, and network control systems accurate estimates of the variables describing the state of the dynamic system under consideration are of utmost importance. Indirect measurements from a subset of variables are usually available from a sensor package, given the technical and economical constrains for the specific problem at hand. Estimates on all the relevant variables can be obtained by formulating and solving an estimation problem.

For the class of systems described by nonlinear differential equations, the solution for the optimal estimation problem is not known in general. The Extended Kalman filter has been the most commonly used synthesis methodology, still based on an approximated stochastic characterization of the disturbances affecting the nonlinear systems. However, the stability of the resulting estimator is not guaranteed, there are no performance bounds easily computable, and the convergence rates are not known in general. Moreover, when large mismatch occurs in the initial state estimates or in the presence of relevant unmodelled dynamics, the stability is often hard to be reinforced, see the classical work (Gelb, 1975), and the references therein.

During the last decades a number of alternative methodologies have been proposed, with focus on the local or regional stability, resorting to deterministic models of the nonlinear systems and the disturbances present (Khalil, 2000). Early approaches were proposed by Thau (Thau, 1973) and by Kou in (Kou *et al.*, 1975), where for nonlinear systems verifying a very demanding Lipschitz condition, a solution for the positive definite matrices present in the Lyapunov equation could be deduced. Methodologies for the design of local nonlinear observers were proposed in (Kazantzis and Kravaris, 1998; Krener and Xiao, 2001), based on a change of coordinates, that must verify a set of first order differential equations. Other methods to obtain local observers can also be found in (Conte *et al.*, 1999), borrowing tools from differential geometry, in the case where local weak observability is verified (Hermann and Krener, 1977). New directions of research for observer design were introduced in (Nijmeijer, 1999) featuring local or regional stability,

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namely sliding modes, high-gain observers, and differential geometry based methods.

This paper departs from these approaches and illustrates a nonlinear observer design method based on the linearization of the nonlinear dynamics and measurement equations recently introduced (Oliveira, 2007). The advantage is that, under the assumptions to be detailed next, to compute the unknown **nonlinear** observer gains only a linear equation must be solved, thus avoiding more difficult and restrictive mathematical tools. Motion linearization was already used in (Huang *et al.*, 2003), denominated as trajectory linearization, where given a nominal trajectory, the observer gains are function of the unknown nonlinear system states.

2. MOTION LINEARIZATION OF NONLINEAR DYNAMIC SYSTEMS

Consider the class of finite-dimension nonlinear time-varying systems with dynamics described by a set of n differential equations, uniformly continuous in the time variable t, written in compact form as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \tag{1}$$

where $\mathbf{x} \in \mathcal{R}^n$ is the system state and $\mathbf{u} \in \mathcal{R}^m$, assumed known, are the system inputs represented by bounded, piecewise continuous functions. Moreover, both quantities are represented as column vectors. The system state is not available, however there are p (usually less than n) nonlinear indirect observations available, given by

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, t), \tag{2}$$

where $\mathbf{y} \in \mathcal{R}^p$, is also a column vector. The vector fields $\mathbf{f}(.)$ and $\mathbf{h}(.)$ are assumed to be at least locally Lipschitz and of class $\mathcal{C}^2$. The deterministic nonlinear estimation problem central to this work can be stated as follows:

Proposition 1. Design an observer to provide causal estimates on the system state $\hat{\mathbf{x}} \in \mathcal{R}^n$, given the available measurements $\mathbf{y}$ in (2), such that the estimation error $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ decays exponentially to zero, at least in a region near the motion of the system (1).

Motivated by the optimal solution obtained in state estimation problems associated with linear systems (Gelb, 1975), the structure proposed for the causal nonlinear observers also replicates the system dynamics and corrects the state estimates with a nonlinear term, given the output measurement error $\mathbf{y} - \mathbf{h}(\hat{\mathbf{x}}, t)$ resulting in

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}, \mathbf{t}) + \mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, \mathbf{t}) \left(\mathbf{y} - \mathbf{h}(\hat{\mathbf{x}}, \mathbf{t})\right), \tag{3}$$

where $\mathbf{K} \in \mathcal{R}^{n\times p}$ is a **nonlinear** observer gain matrix. This technique is denominated as *output injection* and is used in almost all existing linear and nonlinear observer design structures. It is important to remark that in this work the gain $\mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t)$ is a nonlinear time-varying matrix function of the state estimates $\hat{\mathbf{x}}$ and eventually of the known inputs $\mathbf{u}$ and system measurements $\mathbf{y}$, but not function of the true unknown system state $\mathbf{x}$ (as was the case in (Huang *et al.*, 2003)).

Consider that each nonlinear function in the system dynamics (1) and in the measurement equation (2) is approximated by the Taylor series expansion, given the information available from the observer $\hat{\mathbf{x}}$, i.e.

$$f_i(\mathbf{x}, \mathbf{u}, t) \simeq f_i(\hat{\mathbf{x}}, \mathbf{u}, t) + \nabla f_i(\hat{\mathbf{x}}, \mathbf{u}, t)\tilde{\mathbf{x}},$$

$$h_i(\mathbf{x}, t) \simeq h_i(\hat{\mathbf{x}}, t) + \nabla h_i(\hat{\mathbf{x}}, t)\tilde{\mathbf{x}},$$

respectively, assuming that higher order terms are negligible. It is extremely important to remark that in this case the approximation is not considered just in the vicinity of an equilibrium point of the original system (usually the origin), see (Khalil, 2000; Slotine and Li, 1991) and (Huang *et al.*, 2003) to clarify this perspective for the motion / trajectory linearization, applied in this work to nonlinear state estimation. See also the survey paper (Guardabassi and Savaresi, 2001) to clarify the distinction relative to extended-linearization and pseudo-linearization techniques.

The error dynamics $\dot{\tilde{\mathbf{x}}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}$ can then be written as

$$\dot{\tilde{\mathbf{x}}} = \left(\nabla\mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}, \mathbf{t}) - \mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, \mathbf{t})\nabla\mathbf{h}(\hat{\mathbf{x}}, \mathbf{t})\right)\tilde{\mathbf{x}}, \tag{4}$$

where each line of the Jacobian matrices $\nabla\mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}, t) \in \mathcal{R}^{n\times n}$ and $\nabla\mathbf{h}(\hat{\mathbf{x}}, t) \in \mathcal{R}^{p\times n}$ are the gradients previously introduced. Note that in these new coordinates, the estimation error dynamics is described by a non-autonomous, time-varying linear system. Moreover, the proposed methodology can be applied in the state estimation of unstable or non-minimum phase systems that can exhibit multiple isolated equilibria, finite-escape time, limit cycles, or chaotic behavior.

The main theorem for the proposed estimator design method can be presented with a proof that is a straightforward application of the Lyapunov's second method.

Theorem 1. (Oliveira, 2007) The finite-dimension, non-autonomous linear time-varying system (4) expressing the estimation error dynamics, locally (weakly) observable in a region near the origin, has an exponentially stable equilibrium point if for any constant positive definite symmetric matrix $\mathbf{Q} \in \mathcal{R}^{n\times n}$ there exists a constant positive definite symmetric matrix $\mathbf{P} \in \mathcal{R}^{n\times n}$ verifying the Lyapunov equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}. \tag{5}$$

where $\mathbf{A} = \mathbf{A}(\hat{\mathbf{x}}, \mathbf{u}, t) = (\nabla\mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}, t) - \mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t)\nabla\mathbf{h}(\hat{\mathbf{x}}, t))$.

This result generalizes the method proposed in (Kou *et al.*, 1975), where a constant gain K was used in the estimator for autonomous nonlinear systems. It is important to remark that the choice of $\mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t)$ makes the Lyapunov function to verify a Lyapunov equation with a structure equal to those associated with linear time-invariant dynamic systems. Moreover, the unknown nonlinear gains will be function of the observer state thus corresponding to an extra degree of freedom. Only an algebraic equation related with (5), i.e. linear in unknown gains, remains to be solved. Furthermore, as the derivative of $\mathbf{A}$ is not required to be evaluated for the proof, the time-varying dependencies of the gains along the system trajectories do not have to be taken into consideration and do not bring further complexity to the problem at hand.

3. ALGEBRAIC SYNTHESIS METHOD

Under the assumption that the systems at hand are at least locally observable and given the theoretical results presented in the previous section, the following new design methodology is suggested to compute the unknown gains:

(1) Write the Lyapunov equation (5), for constant symmetric matrices $\mathbf{P} > 0$ and $\mathbf{Q} = \mathbf{I}_{n\times n}$ (the optimal decay rate case) and identify the number of gains to be determined;
(2) If some elements on the Lyapunov equation are impossible to be verified, e.g. that do not have any gain associated, use the degrees of freedom in $\mathbf{P}$ and $\mathbf{Q}$ to make all entries to be possible and return Step 1). Otherwise go to step 3);

(3) Organize the set of $n(n+1)/2$ linear algebraic equations (in the unknown elements of $\mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t)$), given the symmetric properties of the matrices involved as

$$\mathbf{U}_p(\hat{\mathbf{x}}, \mathbf{u}, t)\mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t) = \mathbf{W}_p(\hat{\mathbf{x}}, \mathbf{u}, t);$$

(4) Compute $r = rank(\mathbf{U}_p)$ and choose a subset of unknowns equal to r. To match this quantity with the number of gains to be determined problem dependent simplifications could be required.

(5) Re-write the equation previously introduced in step 3), setting the parameters in the $\mathbf{P} > 0$ and $\mathbf{Q}$ matrices, such that the equation

$$\mathbf{U}(\hat{\mathbf{x}}, \mathbf{u}, t)\mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t) = \mathbf{W}(\hat{\mathbf{x}}, \mathbf{u}, t); \qquad (6)$$

results. Solve this equation, i.e. compute

$$\mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t) = \mathbf{U}(\hat{\mathbf{x}}, \mathbf{u}, t)^{-1}\mathbf{W}(\hat{\mathbf{x}}, \mathbf{u}, t).$$

(6) Choose the remaining free elements of the $\mathbf{P}$ matrix such a specific decay rate is obtained, preserving the symmetry and the positive definiteness and substitute those values in $\mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t)$.

In this method the **nonlinear** observer gains are computed explicitly without requiring the solution of differential equations, or the need to resort to scheduling techniques (for a set of gains tuned in a number of operating points). They are just the explicit solution of a set of symbolic equations (6), linear in the unknowns. In the cases where $\dot{V}$ can only be guaranteed to be semi-definite negative, i.e corresponding to $\mathbf{Q} \geq 0$, by the design method previously introduced, the stability of the observer can be studied as an immediate application of the Barbalat Lemma (Khalil, 2000; Slotine and Li, 1991), even in this case where we are in presence of a linear time-varying system, as stated in the following theorem:

Theorem 2. For the class of time-varying systems considered in theorem 1, when the solution for the observer gains is only possible for $\mathbf{Q}$ semi-definite positive, if the gain matrix $\mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t)$ is bounded then the error dynamics of (3) is asymptotically stable.

The proof of this theorem consist on the application of the Barbalat Lemma (see (Oliveira, 2007) for details).

4. DESIGN EXAMPLES

Some examples will be considered next, with the purposes of: i) illustrate the proposed methods; ii) allow the assessment on the performance of the observers obtained; iii) show that even in the cases of unstable, multiple equilibria, or in the presence of other characteristic phenomena found in nonlinear systems, the observers obtained have the properties enumerated above; and iv) address some typical examples found in autonomous robotics allowing an assessment on the results obtained.

4.1 Finite-escape Time Nonlinear System

As a first example a scalar system with nonlinear dynamics and linear observations was selected, given by

$$\begin{cases} \dot{x} = -x^2 \\ y = x \end{cases},$$

respectively. In the case where the initial state is $x(0) = -1$ (considered next) this system is unstable, with finite-escape time, verifying the solution $x(t) = 1/(t-1)$. The

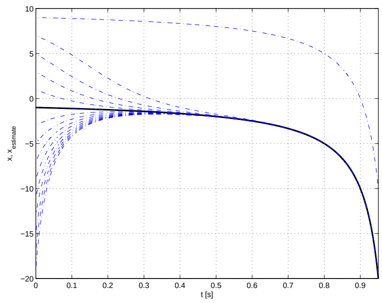

Fig. 1. System motion trajectory (in black) and estimates for a set of different initial conditions.

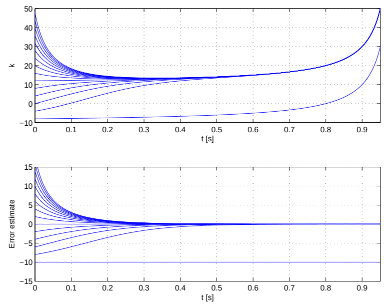

Fig. 2. Observer gains (upper plot) for a set of different initial conditions and estimates errors (lower plot).

gradients of the dynamics and measurement functions are $\nabla f(\hat{x}) = -2\hat{x}$, and $\nabla h(\hat{x}) = 1$, respectively. The steps corresponding to the algebraic synthesis method, introduced in the section 3, will be detailed to provide a clear view to the reader on its application:

Step 1 The equation (5) should be computed for a generic value of p and $q = 1$ resulting in this case in

$$2(-2\hat{x} - k)p = -1.$$

Step 2 Does not apply, continue to next step;

Step 3 From the above equation it is immediate to identify $U_p(\hat{x}, u, t) = U_p(\hat{x}) = -2p$ and $W_p(\hat{x}, u, t) = W_p(\hat{x}) = -1 + 4\hat{x}p$, respectively.

Step 4 The inversion of $U_p(\hat{x})$ is possible for non-null values of p, which goes accordingly with the sufficient conditions in theorem 1.

Step 5 In this case $U = U_p(\hat{x}) = -2p$ and $W = W_p(\hat{x}) = -1 + 4\hat{x}p$ and the observer gain is

$$k = \frac{1}{2p} - 2\hat{x}.$$

To finish the algebraic synthesis method it remains to choose the value of $p > 0$ (as stated in **Step 6**), which will have impact on the decay rate. Note that in this case the gain k obtained makes the value of

$$a = \nabla f - k\nabla h = 1/2p, \qquad (7)$$

thus the assumption for the validity of theorem 1 is verified.

See in Figure 1 the system evolution (in black) and the evolution on the state estimates, for a set of initial conditions in the interval $\hat{x}(0) = [-20, 9]$, for $p = 0.05$. Figure 2 depicts the evolution of the gains and the error estimates for the same set of initial conditions. In order to get a better understanding of the behavior associated with the state starting in $\hat{x}(0) = 9$ and where the estimation

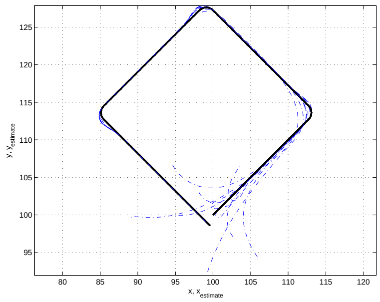

Fig. 3. 2D target trajectory (in black) and estimates for a set of different initial conditions.

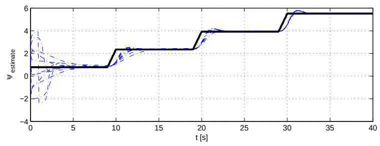

Fig. 4. Nominal target attitude (in black) and estimates.

is maintained constant (upper part of the Figure 1), lets study the equilibria ($\tilde{x}(t) = 0$) of the error dynamics associated with the observer proposed, i.e.

$$0 = -x^2 - \hat{x}^2 - \frac{1}{2p}(x - \hat{x}) + 2x\hat{x},$$

that has the solutions $x = \hat{x}$ and $x - \hat{x} = -1/2p$. Thus for the initial condition $x(0) = -1$ and $p = 0.05$ the choice of $\hat{x}(0) = 9$ corresponds to a second equilibrium (unstable). For initial state estimates greater than 9 the estimator obtained is unstable as can be concluded from the change of signal of $\dot{V} = -\tilde{x}^2(\tilde{x} + 1/2p)$. Reducing the value of $p > 0$ the stability region can be arbitrarily enlarged with a compromise on the decay rate, thus a regional exponential asymptotically stable observer is obtained.

4.2 2D Tracker

In this subsection a target tracker in 2D with range and bearing measurements will be considered. A nonlinear observer will be synthesized following the design methodology previously introduced. Consider the nonlinear system described by

$$\begin{cases} \dot{x} &= cos(\psi)u \\ \dot{y} &= sin(\psi)u \\ \dot{\psi} &= r \end{cases}$$

where x, y are the linear coordinates of the target with an unknown linear velocity u and ψ is the heading relative to the inertial frame of reference. The target rotation rate r is piecewise constant but not accessible by the estimator (i.e. the underlying system is time-varying) thus considered null in the observer design. The range and bearing measured by the sensor (assumed at the origin) will be given by $d = \sqrt{x^2 + y^2}$ and $\theta = atant(\frac{y}{x})$, respectively. The Jacobians are given by

$$\nabla f(\hat{\mathbf{x}}, \mathbf{u}) = \begin{bmatrix} 0 & 0 & -sin(\hat{\psi})u \\ 0 & 0 & cos(\hat{\psi})u \\ 0 & 0 & 0 \end{bmatrix},$$

$$\nabla h(\hat{\mathbf{x}}) = \begin{bmatrix} \dfrac{\hat{x}}{\sqrt{\hat{x}^2 + \hat{y}^2}} & \dfrac{\hat{y}}{\sqrt{\hat{x}^2 + \hat{y}^2}} & 0 \\ -\dfrac{\hat{y}}{\hat{x}^2 + \hat{y}^2} & \dfrac{\hat{x}}{\hat{x}^2 + \hat{y}^2} & 0 \end{bmatrix},$$

respectively. The system is observable for all values such that $\hat{x} \neq 0 \vee \hat{y} \neq 0$.

Step 1 The equation (5) should be computed for a generic value of $\mathbf{P} \in \mathcal{R}^{3\times3}$ and $\mathbf{Q} = I_{3\times3}$. The algebraic computations are obvious but due to its length they are omitted here. The reader is referred to the online technical report (Oliveira, 2007) and the companion Matlab script.
Step 2 The element $(3, 3)$ of the resulting equation is

$$-2sin(\hat{\psi})up_{13} + 2cos(\hat{\psi})up_{23} = -1.$$

As there is no gain such that this relation could be verified for generic input $u(t)$, p_{13}, p_{23}, and q_{33} must all be set to 0 and Step 1 should be repeated.
Step 1 (bis) The equation (5) is recomputed and no inconsistent entries exist.
Step 3 From that same equation, after selecting the 6 possible equations in this symmetric matrix, it is immediate to identify $U_p(\hat{x}, u, t) = U_p(\hat{x})$ and $W_p(\hat{x}, u, t) = W_p(\hat{x})$ as

$$U_p(\hat{x}) = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & 0 & 0 \\ u_{21} & u_{22} & u_{23} & u_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & u_{35} & u_{36} \\ u_{41} & u_{42} & u_{43} & u_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & u_{55} & u_{56} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $W(\hat{x})^T = [-1 \quad 0 \quad w_3 \quad -1 \quad w_5 \quad 0]^T$, respectively, where $u_{11} = -4p_{11}\hat{x}$, $u_{12} = 2p_{11}\hat{y}/A^2$, $u_{13} = -4p_{12}\hat{x}$, $u_{14} = 2p_{12}\hat{y}/A^2$, $u_{21} = -2p_{12}\hat{x} - 2p_{11}\hat{y}$, $u_{22} = -(p_{11}\hat{x} - p_{12}\hat{y})/A^2$, $u_{23} = -2p_{22}\hat{x} - 2p_{12}\hat{y}$, $u_{24} = -(p_{12}\hat{x} - p_{22}\hat{y})/A^2$, $u_{35} = -2p_{33}\hat{x}$, $u_{36} = p_{33}\hat{y}/A^2$, $u_{41} = -4p_{12}\hat{y}$, $u_{42} = -2p_{12}\hat{x}/A^2$, $u_{43} = -4p_{22}\hat{y}$, $u_{44} = -2p_{22}\hat{x}/A^2$, $u_{55} = -2p_{33}\hat{y}$, $u_{56} = -p_{33}\hat{x}/A^2$, $w_3 = (sin(\hat{\psi})p_{11} - cos(\hat{\psi})p_{12})u$, $w_5 = (sin(\hat{\psi})p_{12} - cos(\hat{\psi})p_{22})u$, and $A = \sqrt{\hat{x}^2 + \hat{y}^2}$.
Step 4 It is obvious that the rank of matrix $U_p(\hat{x})$ can not be 6, it is at maximum 5. So, there are more unknowns than degrees of freedom on the gain vector. A reasonable option to reduce the number of unknowns is, considering the physics of the problem at hand, to consider the gains $k_{12} = k_{21} = k$, which goes accordingly with the sufficient conditions in theorem 1.
Step 5 For the choice introduced in the previous step the following matrices

$$U(\hat{x}) = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & 0 & 0 \\ u_{21} & u_{22} & u_{23} & u_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & u_{35} & u_{36} \\ u_{41} & u_{42} & u_{43} & u_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & u_{55} & u_{56} \end{bmatrix},$$

and $W(\hat{x})^T = [-1 \;\; 0 \;\; w_3 \;\; -1 \;\; w_5]^T$ results.
Step 6 Setting the free elements available to $p_{11} = p_{22} = p_{33} = 10$ and $p_{12} = 0$, the following gain results

$$\mathbf{K}(\hat{\mathbf{x}}, \mathbf{u}, t) = \begin{bmatrix} \hat{x}/40 & 0 \\ 0 & \hat{x}A^2/20 \\ k_{31} & k_{32} \end{bmatrix},$$

where $k_{31} = -u(t)(\hat{x}sin(\hat{\psi}) - \hat{y}cos(\hat{\psi}))/2A^2$ and $k_{32} = u(t)(\hat{y}sin(\hat{\psi}) + \hat{x}cos(\hat{psi}))$. Note that in this case an anti-symmetric matrix $\mathbf{A} = \nabla\mathbf{f} - \mathbf{K}\nabla\mathbf{h}$ has been implicitly obtained, from the algebraic synthesis method detailed above, i.e.

$$\mathbf{A} = \begin{bmatrix} -1/20 & -\hat{y}/20\hat{x} & -usin(\hat{\psi}) \\ \hat{y}/20\hat{x} & -1/20 & ucos(\hat{\psi}) \\ usin(\hat{\psi}) & -ucos(\hat{\psi}) & 0 \end{bmatrix} \quad (8)$$

so the validity of theorem 1 is verified for non-constant dynamics matrices.

The target trajectory is plot in black in Figure 3 for the initial conditions $x = 100m, y = 100m, \psi = \pi/4rad$ and for estimator initial states randomly selected for 10 different runs. The target velocity is constant along the $40\,s$ of each experiment and equal to $2\ m/s$. In Figure 4 the target heading ψ and the estimates $\hat{\psi}$ are plotted for the same experiments.

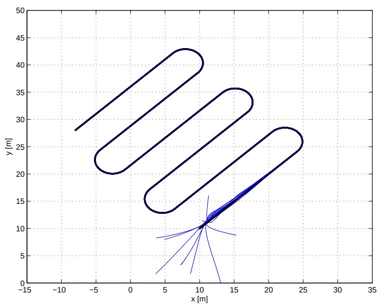

Fig. 5. Robot trajectory in 2D (in black) and estimates for 10 different initial conditions.

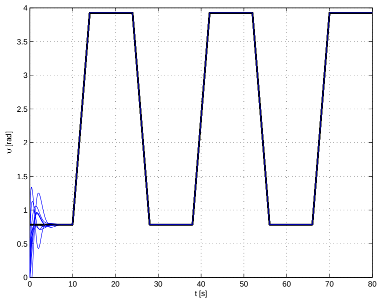

Fig. 6. Yaw angle (in black) and yaw angle estimate, for a set of different initial conditions.

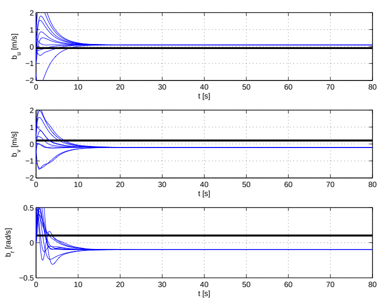

Fig. 7. Doppler log biases b_u and b_v, in the upper pictures, and rate-gyro bias b_r on the lower picture, for a set of different initial conditions.

4.3 Nonlinear Complementary Filters for Position and Attitude in 2D

In this section joint design of nonlinear complementary filters for position and attitude estimation of a robotic platform in 2D will be presented. The following sensor package is considered to be available onboard:

- GPS - A Global Positioning System receiver, that based on measurements relative to a constellation of satellites, provides the position coordinates of the robot in 2D, i.e. x and y, respectively.
- Doppler Log - A velocity Doppler Log, that emitting acoustic waves from an array, and based on the Doppler shift experienced by the acoustic waves, provides measurements on the linear velocities u and v, relative to the seafloor, expressed in the body coordinate frame. Unfortunately due to construction and calibration difficulties constant unknown bias terms b_u and b_v, respectively, also in the body frame are present in the measurements.
- Fluxgate - An heading sensor, based on a magnetic fluxgate, provides the yaw heading angle ψ, relative to the north. The rotation matrix from inertial to body frame is

$$\mathcal{R}(\psi) = \begin{bmatrix} cos(\psi) & -sin(\psi) \\ sin(\psi) & cos(\psi) \end{bmatrix}.$$

- Rate Gyro - A rate gyro providing measurements of the yaw rate r, between the inertial and the body frames, is installed onboard. The data is also corrupted by a constant unknown bias b_r.

Consider the nonlinear system resulting from the rigid body kinematics, described by

$$\begin{cases} \dot{x} &= cos(\psi)(u+b_u) - sin(\psi)(v+b_v) \\ \dot{y} &= sin(\psi)(u+b_u) + cos(\psi)(v+b_v) \\ \dot{b_u} &= 0 \\ \dot{b_v} &= 0 \\ \dot{\psi} &= r + b_r \\ \dot{b_r} &= 0 \end{cases}$$

that can be written in the form presented in (1), considering that the linear and angular velocities of the robot will be piecewise constant. The Doppler and the rate-gyro measurements are considered as inputs to the system, following a methodology commonly used in complementary filters. The GPS and the heading sensor measurements will be considered as the observables to the estimation problem at hand $\mathbf{y} = [x \;\; y \;\; \psi]^T$. The Jacobians are given by

$$\nabla f(\hat{x}) = \begin{bmatrix} 0 & 0 & c\hat{\psi} & -s\hat{\psi} & -s\hat{\psi}u_e - c\hat{\psi}v_e & 0 \\ 0 & 0 & s\hat{\psi} & c\hat{\psi} & -s\hat{\psi}u_e - c\hat{\psi}v_e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\nabla h(\hat{x}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

where s and c are the abbreviation for the sine and cosine trignometric functions, respectively, and the corrected estimates for the velocities $u_e = u + \hat{b_u}$ and $v_e = v + \hat{b_v}$ were used to get more compact relations.

Step 1 The Lyapunov equation (5) computed for a generic value of $\mathbf{P} \in \mathcal{R}^{6\times 6}$ and $\mathbf{Q} = I_{6\times 6}$. The algebraic computations are obvious but due to its length they are omitted here. The reader is referred to the online technical report (Oliveira, 2007) and the companion Matlab script.
Step 2 In the resulting matrix equation, let the element (i,j) be expressed as $\mathbf{L}(i,j)$. The following impossible relations are present

$$\begin{cases} \mathbf{L}(3,3) \;:\; 2c(\hat{\psi})p_{13} + 2s(\hat{\psi})p23 = -1 \\ \mathbf{L}(3,4) \;:\; -s(\hat{\psi})p_{13} + c(\hat{\psi})p_{23} + c(\hat{\psi})p_{14} + s(\hat{\psi})p_{24} = 0 \\ \mathbf{L}(3,6) \;:\; p_{35} + c(\hat{\psi})p_{16} + s(\hat{\psi})p_{26} = 0 \\ \mathbf{L}(4,4) \;:\; -2s(\hat{\psi})p_{14} + 2c(\hat{\psi})p_{24} = -1 \\ \mathbf{L}(4,6) \;:\; p_{45} - s(\hat{\psi})p_{16} + c(\hat{\psi})p_{26} = 0 \\ \mathbf{L}(6,6) \;:\; 2p_{56} = -1 \end{cases}$$

As there are no gains such that those relations could be verified for generic state estimates, a possible solution is to set p_{13}, p_{23}, p_{14}, p_{24}, p_{16}, p_{26}, p_{35}, p_{45} to 0 and $p_{56} = -q_{66}/2$. Then, Step 1 should be repeated.
Step 1 (bis) The equation (5) is recomputed and no inconsistent entries exist.
Step 3 In that Lyapunov equation involving a symmetric matrix, 15 possible equations can be identified for the 18 gain unknowns. A reasonable option to reduce the number of unknowns is, considering the physics of the problem at hand, to consider that the gains $k_{13} = k_{23} = 0$, i.e. the errors in attitude $\tilde{\psi}$ do not impact directly in the corrections in the position estimates due to the kinematics based model adopted and also the following relation in the

gains $k_{12} = k_{21}$, i.e the space is homogeneous and equal gains could be considered for both x and y directions for a given position error. Note that this choice goes accordingly with the sufficient conditions present in theorem 1. It is then immediate to identify

$$U_p = \begin{bmatrix}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & d & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & d & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2d & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & d
\end{bmatrix},$$

where $a = -2/5$, $b = -1/4$, $c = 1/2$, and $d = -1$, and $V_p = [-1\ 0\ -1/5c\hat{\psi}\ 1/5s\hat{\psi}\ 1/5s\hat{\psi}u_e + 1/5c\hat{\psi}v_e\ 0\ -1\ -1/5s\hat{\psi}\ -1/5c\hat{\psi}\ -1/5c\hat{\psi}u_e + 1/5s\hat{\psi}v_e\ 0\ 0\ 0\ -1\ -1]^T$.

Step 4 The rank of matrix U_p is 15. There are the same number of unknowns in the gain matrix (15) as the number of independent equations.

Step 5 In the case obtained in this example, given the previous choices it is immediate to identify $U = U_p$ and $W = W_p$, where the arguments were ignored. Finally, setting the remaining degrees of freedom (**Step 6**) in the **P** matrix as $p_{11} = .2$, $p_{22} = .2$, $p_{33} = .25$, $p_{44} = .25$, $p_{55} = 1$, $p_{66} = 1$, $p_{12} = 0$, $p_{15} = 0$, $p_{25} = 0$, $p_{34} = 0$, $p_{36} = 0$, and $p_{46} = 0$ the resulting nonlinear observer gain matrix is obtained according to Step 5 of the algebraic method as

$$\mathbf{K}(\hat{\mathbf{x}}) = \begin{bmatrix}
2.5 & 0 & 0 \\
0 & 2.5 & 0 \\
0.8c\hat{\psi} & 0.8s\hat{\psi} & 0 \\
-0.8s\hat{\psi} & 0.8c\hat{\psi} & 0 \\
-0.26(s\hat{\psi}u_e + c\hat{\psi}v_e) & 0.26(c\hat{\psi}u_e - s\hat{\psi}v_e) & 1.33 \\
-0.13(s\hat{\psi}u_e + c\hat{\psi}v_e) & 0.13(c\hat{\psi}u_e - s\hat{\psi}v_e) & 1.66
\end{bmatrix}.$$

Some remarks are important: i) only a numeric matrix was required to be inverted to tackle the observer gain design problem; ii) the elements $k_{33} = k_{43} = 0$, thus the errors in ψ do not impact directly in corrections in the Doppler bias estimates; and iii) the elements

$$\begin{bmatrix} k_{31} & k_{32} \\ k_{32} & k_{42} \end{bmatrix} = 0.8R^T(\hat{\psi}),$$

thus the position errors in the inertial frame are rotated back to the body frame prior to the Doppler biases estimates corrections.

The robot trajectory for a low-mowing survey trajectory is depicted in Figure 5 with initial conditions $x_0 = 10m, y_0 = 10\ m$, and $\psi_0 = \pi/4\ rad$ and the estimator initial states are randomly selected for 10 different runs in the vicinity of real values. In Figure 6 the heading angle ψ and its estimates are plotted for the same experiments. The biases associated with the Doppler log were set to $b_u = -0.1$ and $b_v = 0.2$, respectively, and the rate-gyro bias was set to $b_r = 0.3$. The results obtained for the estimates of those quantities are depicted in Figure 7, where due to the relations used, the biases estimated are the symmetric of the real ones to cancel out their effects. Moreover, note that as the gain matrix $\mathbf{K}(\hat{\mathbf{x}})$ is bounded and then according to theorem 2, even in the present case where $\mathbf{Q}$ is semi-definite negative, the estimation error dynamics is asymptotically stable.

5. CONCLUSIONS AND FUTURE WORK

In this paper the design of nonlinear observers for marine applications, based on a new methodology, is illustrated. The results obtained for some design examples allow to conclude that the proposed methods can be successfully used in a number of applications in nonlinear estimation.

In the future, variants on the linearization methodology will be exploited and an indepth study on techniques to solve Lyapunov equations will be addressed, to try to find more general solutions to the underlying Lyapunov equation. The derivation a discrete-time version of this method is also foreseen, with clear advantages for real time applications.

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