DIFFUSION-BASED TRAJECTORY OBSERVERS WITH VARIANCE CONSTRAINTS

A. Alcocer [*,1] **J. Jouffroy** [**,***] **P. Oliveira** [*]
A. Pascoal [*]

[*] *Institute for Systems and Robotics*
Instituto Superior Técnico
Av. Rovisco Pais, 1100 Lisbon, Portugal
e-mail: {alexblau,pjcro,antonio}@isr.ist.utl.pt
[**] *Mads Clausen Institute, University of Southern Denmark, DK-6400 Sønderborg, Denmark*
e-mail: jerome@mci.sdu.dk
[***] *Center for Ships and Ocean Structures*
Norwegian University for Science and Technology
NO-7491 Trondheim, Norway
e-mail: jouffroy@marin.ntnu.no

Abstract: Diffusion-based trajectory observers have been recently proposed as a simple and efficient framework to solve diverse smoothing problems in underwater navigation. For instance, to obtain estimates of the trajectories of an underwater vehicle given position fixes from an acoustic positioning system and velocity measurements from a DVL. The observers are conceptually simple and can easily deal with the problems brought about by the occurrence of asynchronous measurements and dropouts. In its original formulation, the trajectory observers depend on a user-defined constant gain that controls the level of smoothing and is determined by resorting to trial and error. This paper presents a methodology to choose the observer gain by taking into account a priori information on the variance of the position measurement errors. Experimental results with data from an acoustic positioning system are presented to illustrate the performance of the derived observers.

1. INTRODUCTION

In many applications, one is interested in determining a smooth trajectory that fits a set of sparse and noisy measurements in a fixed time interval. For instance, when post-processing underwater navigation data obtained with an acoustic positioning system and a Doppler Velocity Log (DVL) (Jouffroy and Opderbecke, 2007), (Kinsey *et al.*, 2006), (Ferrini *et al.*, 2007). Although off-line navigation is of crucial practical importance for the correction of geo-referenced scientific data, it has received considerably less attention than real-time navigation (Alcocer *et al.*, 2007), (Kinsey and Whitcomb, 2004), (Whitcomb *et al.*, 1999).

Diffusion-based trajectory observers have been recently proposed as a simple and efficient frame-

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work to solve diverse smoothing problems in underwater navigation (Jouffroy and Opderbecke, 2004), (Jouffroy and Opderbecke, 2007). These observers are conceptually simple, and can cope naturally with asynchronous measurements and dropouts (temporally loss of measurements). From a conceptual point of view, diffusion-based trajectory observers have strong links with several other disciplines such as de-noising and snakes in image processing (Rudin *et al.*, 1992), (Cohen and Cohen, 1993), (Xu and Prince, 1998), and nonparametric smoothing splines (Eubank, 1999).

In the original formulation, the objective is to minimize an energy-like functional that penalizes the derivatives of the trajectory and a function of the distances between points along the trajectory and a set of available measurements. The trajectory is modeled as a curve $s \mapsto X(s) \in \mathbb{R}^n$, $s \in [s_b, s_e]$ where $n \in \{2, 3\}$ is the dimension of the trajectory ambient space, and $[s_b, s_e] \subset \mathbb{R}$ is the time interval of interest. Suppose there are N_m position measurements $X_m(\tau_i) \in \mathbb{R}^n$ corresponding to times $\tau_i \in \mathcal{I} = \{\tau_1, \ldots, \tau_{N_m}\}$. Consider an energy-like functional of the form

$$E = \int_{s_b}^{s_e} \alpha_1 \|\nabla X(s)\|^2 + \alpha_2 \|\nabla^2 X(s)\|^2 + K\gamma(s)\|X(s) - X_m(s)\|^2 \mathrm{d}s \quad (1)$$

where $\alpha_1, \alpha_2 > 0$ are user-defined parameters, ∇^i stands for the i'th order derivative with respect to the trajectory time s, and K is a user-defined gain. The factor $\gamma(s)$ is introduced to capture the fact that position measurements are available only at some discrete instants of time and is defined as

$$\gamma(s) = \sum_{\tau_i \in \mathcal{I}} \delta(s - \tau_i), \quad (2)$$

where $\delta(\cdot)$ stands for the Dirac delta function. For simplicity of presentation, in this paper we will focus on energy functionals of the type (1). This corresponds to the *acoustic data smoothing* problem in (Jouffroy and Opderbecke, 2007) where it is assumed that no velocity measurements are available. Including continuous velocity measurements can be easily done with minor modifications in the derivation that follows by considering terms of the form $\|\nabla X(s) - V_m(s)\|^2$ in the energy functional.

The minimizing trajectory must satisfy the Euler Lagrange equation

$$-\alpha_1 \nabla^2 X(s) + \alpha_2 \nabla^4 X(s) + K\gamma(s)(X(s) - X_m(s)) = 0 \quad (3)$$

and can be computed using a dynamical system, called a trajectory observer, that follows the negative gradient-like flow and is defined as follows:

$$\frac{\partial}{\partial t} X(s,t) = \alpha_1 \nabla^2 X(s,t) - \alpha_2 \nabla^4 X(s,t) - K\gamma(s)(X(s,t) - X_m(s)). \quad (4)$$

Note that we are now considering a continuous of trajectories $X(s,t)$ where s is the *trajectory* time, and t is the *improvement* time. The observer is started at some initial trajectory $X(s,0)$, and the desired solution is the limit of $X(s,t)$ as $t \to \infty$. The question of whether the previous flow converges and has a unique stable equilibrium trajectory is of crucial importance. In (Jouffroy and Nguyen, 2004) the stability proof for a similar flow based on Lyapunov analysis is presented. In (Jouffroy and Opderbecke, 2004) the convergence of a finite difference implementation of the observer is guaranteed given that the observer gain K is a positive constant.

In practice, the choice of the value of the observer gain is often left to the user and is done through trial and error. In many situations, however, there is some a-priori knowledge on the variance of the measurement errors that could be used in order to chose K. In this paper we propose a methodology to determine K based on minimizing an energy like function subject to a variance constraint that can for all purposes be viewed as an observer "tuning knob". The observer gain is viewed as a Lagrange multiplier, as inspired by (Rudin *et al.*, 1992), and determined by solving a system of equations involving that constraint.

2. TRAJECTORY OBSERVERS WITH VARIANCE CONSTRAINTS

Suppose the variance of the position measurement errors σ_X^2 is known. Consider the minimization of the cost function

$$E = \int_{s_b}^{s_e} \alpha_1 \|\nabla X(s)\|^2 + \alpha_2 \|\nabla^2 X(s)\|^2 \mathrm{d}s \quad (5)$$

subject to the constraint

$$\frac{1}{N_m} \int_{s_b}^{s_e} \gamma(s)\|X(s) - X_m(s)\|^2 \mathrm{d}s = \sigma_X^2. \quad (6)$$

The minimizing trajectory must now satisfy the Euler Lagrange equations (3) and (6). In order to determine the value of K that satisfies both equations, we can multiply (3) on the left by $\gamma(s)(X(s) - X_m(s))^T$ and integrate in the interval $[s_b, s_e]$ to obtain

$$K = \frac{1}{\sigma_X^2 N_m} \int_{s_b}^{s_e} \gamma(s)(X(s) - X_m(s))^T (\alpha_2 \nabla^4 X(s) - \alpha_1 \nabla^2 X(s)) \mathrm{d}s. \quad (7)$$

Injecting the derived gain in the trajectory observer equations (4) can be interpreted as a gradient projection method for the constrained minimization problem (5)-(6).

2.1 Numerical Implementation

The observer derived above has the form of a nonlinear partial differential equation that is not easy to solve in practice. Next, we formulate a finite difference approximation to the observer that can be easily implemented.

The differential equation that defines the observer is well suited to estimate the trajectory at all times s in the closed interval $[s_b, s_e]$. In practice, one is often interested in estimating the trajectory only at a set of uniformly distributed instants of time $\{s_1, s_2, \dots, s_N\}$, where $s_1 = s_b$, $s_N = s_e$, and $\Delta s = s_{i+1} - s_i$, that is, in estimating a discretized version of the trajectory defined as

$$\mathbf{X} = \begin{bmatrix} X(s_1)^T \\ X(s_2)^T \\ \vdots \\ X(s_N)^T \end{bmatrix} \in \mathbb{R}^{N \times n}. \tag{8}$$

The position measurements can also be put in this form, assuming that $\mathcal{I} \subseteq \{s_1, s_2, \dots, s_N\}$, by defining a matrix $\mathbf{X}_m \in \mathbb{R}^{N \times n}$ with row i given by $X_m(s_i)^T$ whenever there is a measurement at time s_i and a row of zeros otherwise. Let vec() denote the operator that stacks the columns of a matrix from left to right. Define the vector variables $\mathbf{x} = \mathrm{vec}(\mathbf{X}) \in \mathbb{R}^{Nn}$, and $\mathbf{x}_m = \mathrm{vec}(\mathbf{X}_m) \in \mathbb{R}^{Nn}$. The finite difference implementation of the trajectory observer (4) can now be written as

$$\begin{cases} \dot{\mathbf{x}} &= f(\mathbf{x}) = \mathbf{A}\mathbf{x} - K\mathbf{\Gamma}(\mathbf{x} - \mathbf{x}_m) + \mathbf{b} \\ K &= \dfrac{1}{\sigma_X^2 N_m}(\mathbf{x} - \mathbf{x}_m)^T \mathbf{\Gamma}(\mathbf{A}\mathbf{x} + \mathbf{b}) \end{cases} \tag{9}$$

where

$$\mathbf{A} = I_n \otimes (\alpha_1 \mathbf{L}_1 - \alpha_2 \mathbf{L}_2) \in \mathbb{R}^{Nn \times Nn}, \tag{10}$$

$$\mathbf{\Gamma} = I_n \otimes \begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_N \end{bmatrix} \in \mathbb{R}^{Nn \times Nn}, \tag{11}$$

$$\gamma_i = \int_{s_b}^{s_e} \gamma(s_i) \mathrm{d}s = \begin{cases} 1 & \text{if } s_i \in \mathcal{I}, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

I_n is the $n \times n$ identity matrix, and $\otimes$ denotes the Kronecker product of matrices. The matrices $\mathbf{L}_1, \mathbf{L}_2$, defined as

$$\mathbf{L}_1 = \frac{1}{\Delta s^2} \begin{bmatrix} -2 & 1 & 0 & \cdots \\ 1 & -2 & 1 & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \cdots & 0 & 1 & -2 \end{bmatrix} \in \mathbb{R}^{N \times N}, \tag{13}$$

$$\mathbf{L}_2 = \frac{1}{\Delta s^4} \begin{bmatrix} 6 & -4 & 1 & 0 & \cdots \\ -4 & 6 & -4 & 1 & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ & 1 & -4 & 6 & -4 \\ \cdots & 0 & 1 & -4 & 6 \end{bmatrix} \in \mathbb{R}^{N \times N}, \tag{14}$$

are the centered finite difference approximations with Dirichlet boundary conditions of ∇^2 and ∇^4, respectively. Vector $\mathbf{b} = \mathrm{vec}(\mathbf{B}) \in \mathbb{R}^{Nn}$ is obtained from a constant matrix $\mathbf{B} \in \mathbb{R}^{N \times n}$ containing the Dirichlet boundary conditions. More specifically, assuming that $X(s,t) = \mathbf{x}_b, \forall s \leq s_b$ and $X(s,t) = \mathbf{x}_e, \forall s \geq s_e$, then $\mathbf{B} = \alpha_1 \mathbf{B}_1 - \alpha_2 \mathbf{B}_2$ where

$$\mathbf{B}_1 = \frac{1}{\Delta s^2} \begin{bmatrix} \mathbf{x}_b & 0 & \dots & 0 & \mathbf{x}_e \end{bmatrix}^T, \tag{15}$$

$$\mathbf{B}_2 = \frac{1}{\Delta s^4} \begin{bmatrix} -3\mathbf{x}_b & \mathbf{x}_b & 0 & \dots & 0 & \mathbf{x}_e & -3\mathbf{x}_e \end{bmatrix}^T. \tag{16}$$

Note that this implementation differs from the original formulation, that used complex numbers (Jouffroy and Opderbecke, 2007), and allows to deal easily with both 2D and 3D trajectories.

The initial trajectory estimate $\mathbf{x}(0)$ is an important parameter that needs to be carefully chosen. As shown in the appendix, under some assumptions, the observer is only locally asymptotically stable. One must be aware that there are some initial trajectories for which the observer diverges. A natural choice is to start with a linear interpolation of the available measurements. Another possibility is to chose a constant positive gain K and use the closed form solution to $f(\mathbf{x}) = 0$ given by

$$\mathbf{x} = -(\mathbf{A} - K\mathbf{\Gamma})^{-1}(K\Gamma\mathbf{x}_m + \mathbf{b}). \tag{17}$$

where, provided that $\mathbf{A}$ is negative definite (see Lemma 2 in the appendix) and $K\mathbf{\Gamma}$ is positive semidefinite, matrix $\mathbf{A} - K\mathbf{\Gamma}$ is always invertible.

Velocity measurements, as those provided by a DVL unit, can be easily introduced in the previous formulation with minor modifications. For instance, one way of doing so is by assuming continuous noise-free velocity measurements V_m and considering the energy functional (1) with the first term replaced by $\alpha_1 \|\nabla X(s) - V_m(s)\|^2$ as in (Jouffroy and Opderbecke, 2007). The observer equations (9) remain unchanged, except for vector $\mathbf{b}$ that must now be defined as $\mathbf{b} = \mathrm{vec}(\mathbf{B}) - \alpha_1 \mathrm{vec}(\mathbf{W})$, with $\mathbf{W}$ representing the finite difference approximation of ∇V_m.

2.2 Tuning the trajectory observer

The free parameters in the observer are α_1, α_2 and σ_X. The first two, are also encountered in the original trajectory observers, and weigh the relative penalization of velocities and accelerations. In (Cohen and Cohen, 1993) it is recommended to have parameters of the order Δs^2 for α_1 and of the order Δs^4 for α_2. The introduction of σ_X is the main contribution of the paper. It provides a simple and intuitive way of tuning the observer. Unlike the original formulation, where the observer gain K had no physical interpretation, the new parameter σ_X is the assumed standard deviation of the position measurement errors. It might be known a priori, from sensor specifications, or roughly estimated from the available data (for instance using a moving average filter). The important fact is that it is possible to specify a priori what will be the size of the mismatch between the available position measurements and the estimated trajectory.

3. EXPERIMENTAL RESULTS

The trajectory observer derived was applied to the post-processing of experimental data from sea trials in Sines, Portugal, in June 2004. Position measurements were generated by trilateration of data coming from an underwater acoustic positioning system (Alcocer *et al.*, 2007) while manoeuvring an acoustic emitter from a surface ship. The selected trajectory corresponds to $100s$ of data, and position measurements were available every second. The trajectory was discretized in $N = 100$ elements with $\Delta s = 1$s. The observer parameters were set to $\alpha_1 = \alpha_2 = 1$, and $\sigma_X \in \{2, 3, 4\}$m. The observer was initialized at a linear interpolation of the available measurements. The results are shown in Figures 1, 2, and 3. The final trajectory estimates (the steady state solution of the observer equations) are shown in Figure 1, together with the actual position measurements. The bigger the standard deviation σ_X, the smoother the resulting trajectory is, and less weight is given to the measurements. Figure 2 shows the time evolution of the trajectory estimate, which provides a graphical intuition on how the observer behaves. Figure 3 shows the evolution of the resulting observer gain.

It is important to note that before using the derived observer, one should be careful in removing the outliers from the available measurements. Even if the smoothing process will somehow minimize their influence, the observer is not meant to deal with outliers. In (Vike and Jouffroy, 2005) an integrated outlier rejection and smoothing scheme is presented.

4. CONCLUSIONS AND FUTURE WORK

This paper proposed an extension to diffusion-based trajectory observers that includes variance constraints on the position measurements. The derived observer can be easily implemented and tuned using physical meaningful parameters. Experimental results with data gathered from an acoustic positioning system were presented to illustrate the observer performance. Convergence analysis showed that under some conditions the observer is asymptotically stable.

There are many issues that could be further addressed. First, some improvement over the use of

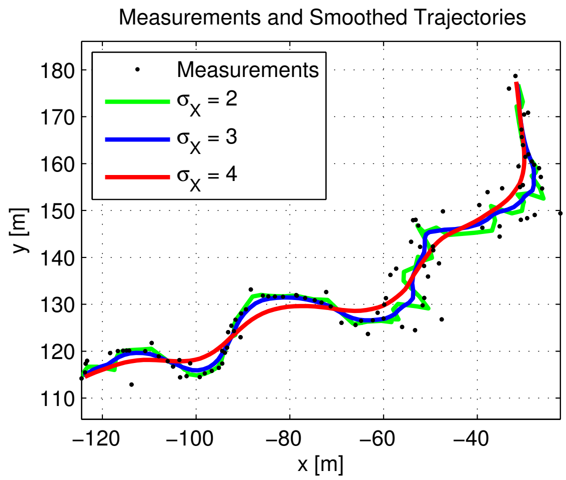

Fig. 1. Position measurements, and final estimated trajectory using different σ_X values.

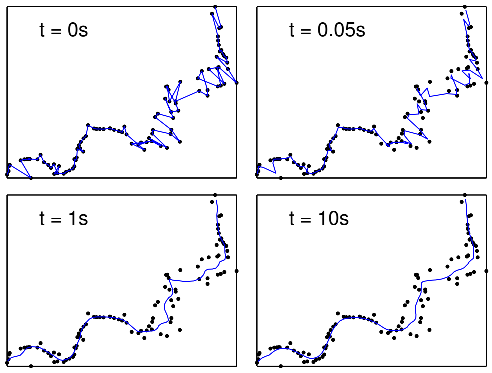

Fig. 2. Position measurements and diffusion trajectory estimate at different improvement times t corresponding to $\sigma_X = 3m$.

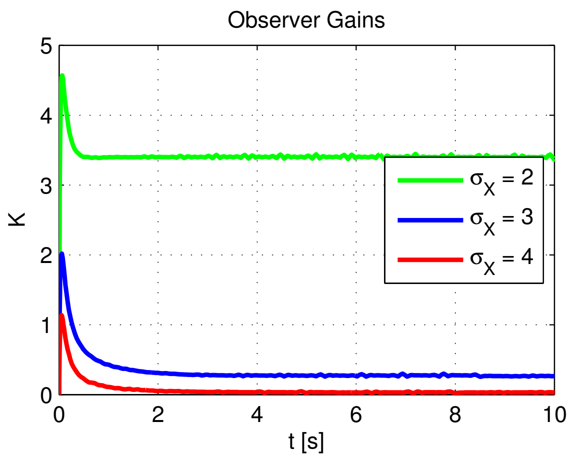

Fig. 3. Evolution of the observer gain K with the improvement time t for different values of σ_X.

Dirac delta functions to weight the position measurements might be done. Instead, one could use Gaussian functions whose variance accounts for the small uncertainty present in the time tags of the position measurements. Second, the equality constrained minimization problem (6) may not seem natural; it was simply chosen so as to yield a simple solution in the framework of diffusion-based trajectory observers. By considering inequality constraints on the variance, the discrete version of problem could have been formulated as a Quadratically Constrained Quadratic Problem (QCQP). Considering constraints on the variance of the velocity measurements is another direction of further research.

Appendix A. CONVERGENCE ANALYSIS

The important question of whether the derived observer converges warrants careful analysis. Note that the observer equations are nonlinear due the gain K which is a quadratic function of the state. We start by characterizing the equilibrium points of the observer and then show that, under some conditions, the desired points are asymptotically stable.

The observer equations can be written in compact form as

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}) = \boldsymbol{\xi} - K\mathbf{e} \\ K = c\mathbf{e}^T\boldsymbol{\xi}, \end{cases} \tag{A.1}$$

where

$$\mathbf{e} = \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}_m) \in \mathbb{R}^{Nn\times 1} \tag{A.2}$$

$$\boldsymbol{\xi} = \mathbf{A}\mathbf{x} + \mathbf{b} \in \mathbb{R}^{Nn\times 1} \tag{A.3}$$

$$c = \frac{1}{n_X\sigma_X^2} \in \mathbb{R}. \tag{A.4}$$

The equilibrium trajectories are those satisfying $f(\mathbf{x}) = 0$. From (A.1), and since K is a scalar, it can be seen that for this to happen vectors $\boldsymbol{\xi}$ and $\mathbf{e}$ must be aligned. That is, their inner product must be $\boldsymbol{\xi}^T\mathbf{e} = \pm\|\boldsymbol{\xi}\|\|\mathbf{e}\|$. The set of equilibrium trajectories can then be characterized as $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\begin{aligned} \Omega_1 &= \{\boldsymbol{\xi} = 0\}, \\ \Omega_2 &= \{\boldsymbol{\xi}^T\mathbf{e} = +\|\boldsymbol{\xi}\|\|\mathbf{e}\| > 0,\ c\|\mathbf{e}\|^2 = 1\}, \\ \Omega_3 &= \{\boldsymbol{\xi}^T\mathbf{e} = -\|\boldsymbol{\xi}\|\|\mathbf{e}\| < 0,\ c\|\mathbf{e}\|^2 = 1\}. \end{aligned}$$

The first set Ω_1 consists only of the solution $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$, which in general does not satisfy the variance constraint and can be therefore considered as degenerate. The following proposition gives sufficient conditions that ensure that the sets Ω_2 and Ω_3 are composed of isolated asymptotically stable and unstable equilibrium points, respectively.

Proposition 1. Define the constants

$$\begin{aligned} \beta_2 &= \sqrt{c}(\underline{\sigma}(\mathbf{A})\|\mathbf{x}_m\| - \|\mathbf{b}\|) - \overline{\sigma}(\mathbf{A}) \\ \beta_3 &= \sqrt{c}(\underline{\sigma}(\mathbf{A})\|\mathbf{x}_m\| - \|\mathbf{b}\|) - 2\overline{\sigma}(\mathbf{A}) - \underline{\sigma}(\mathbf{A}) \end{aligned}$$

where $\underline{\sigma}(\mathbf{A}), \overline{\sigma}(\mathbf{A})$ denote the minimum and maximum singular values of matrix $\mathbf{A}$, respectively. Then,

(1) If $\beta_2 > 0$, the equilibrium points in Ω_2 are isolated and (locally) asymptotically stable.
(2) If $\beta_3 > 0$, the equilibrium points in Ω_3 are isolated and unstable.

Matrix $\mathbf{A}$ has an important role in determining the observer properties. The following result will be useful in proving the previous proposition:

Lemma 2. Matrix $\mathbf{A}$ defined in (10) is negative definite, and in particular invertible.

Proof If the matrix $\mathbf{C} = (\alpha_1\mathbf{L}_1 - \alpha_2\mathbf{L}_2)$ is negative definite the result follows since $\mathbf{A} = I_n \otimes \mathbf{C}$ is symmetric and with the same eigenvalues as $\mathbf{C}$ (with n times its multiplicities). Matrix $\mathbf{L}_1$ defined in (13) is negative definite (Ames, 1977, p.57). It follows that its square $\mathbf{L}_1^2$ is positive definite. Moreover, it is easy to see that matrix $\mathbf{L}_2$ defined in (14) is also positive definite since it can be expressed as

$$\mathbf{L}_2 = \mathbf{L}_1^2 + \mathbf{e}_1\mathbf{e}_1^T + \mathbf{e}_N\mathbf{e}_N^T \succeq \mathbf{L}_1^2$$

where $\mathbf{e}_i \in \mathbb{R}^N$ has a 1 on its i'th entry and zeros elsewhere, and where given two symmetric matrices $\mathbf{A}, \mathbf{B}$, the expression $\mathbf{A} \succeq \mathbf{B}$ denotes that the difference $\mathbf{A} - \mathbf{B}$ is positive semidefinite. Because $\alpha_1, \alpha_2 > 0$, $\mathbf{C}$ is the sum of two negative definite matrices and is itself negative definite. The result follows immediately. □

Note that if instead of Dirichlet (fixed extreme positions) one had considered Neumann boundary conditions (fixed extreme velocities), the corresponding finite difference approximation matrix of ∇^2, $\mathbf{L}_1$ would be only negative semidefinite, and the result would not be valid.

Proof (Proposition 1) After some computations, the Jacobian of the system can be found to be:

$$\frac{\partial f}{\partial \mathbf{x}} = (I - c\mathbf{e}\mathbf{e}^T)\mathbf{A} - c\,\mathbf{e}\boldsymbol{\xi}^T\boldsymbol{\Gamma} - c\,\mathbf{e}^T\boldsymbol{\xi}I \tag{A.5}$$

When evaluated in Ω_2,

$$\frac{\partial f}{\partial \mathbf{x}}|_{\Omega_2} = (I - c\mathbf{e}\mathbf{e}^T)\mathbf{A} - c\sqrt{c}\|\boldsymbol{\xi}\|\mathbf{e}\mathbf{e}^T - \sqrt{c}\|\boldsymbol{\xi}\|I$$

since $\boldsymbol{\Gamma}\mathbf{e} = \mathbf{e}$, $\boldsymbol{\Gamma}\boldsymbol{\xi} = K\boldsymbol{\Gamma}\mathbf{e} = \boldsymbol{\xi}$, and $\sqrt{c}\|\mathbf{e}\| = 1$. Moreover, when restricted to Ω_2 the following bounds apply

$$\begin{aligned} \sqrt{c}/c = \|\mathbf{e}\| &= \|\boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}_m)\| \geq \|\boldsymbol{\Gamma}\mathbf{x}_m\| - \|\boldsymbol{\Gamma}\mathbf{x}\| \\ &= \|\mathbf{x}_m\| - \|\boldsymbol{\Gamma}\mathbf{x}\| \Longrightarrow \\ \|\mathbf{x}\| &\geq \|\boldsymbol{\Gamma}\mathbf{x}\| \geq \|\mathbf{x}_m\| - \sqrt{c}/c \end{aligned} \tag{A.6}$$

$$\begin{aligned} \|\boldsymbol{\xi}\| &= \|\mathbf{A}\mathbf{x} + \mathbf{b}\| \geq \|\mathbf{A}\mathbf{x}\| - \|\mathbf{b}\| \\ &\geq \underline{\sigma}(\mathbf{A})\|\mathbf{x}\| - \|\mathbf{b}\| \\ &\geq \underline{\sigma}(\mathbf{A})(\|\mathbf{x}_m\| - \sqrt{c}/c) - \|\mathbf{b}\| \end{aligned} \tag{A.7}$$

$$
\begin{aligned}
c\mathbf{e}\mathbf{e}^T\mathbf{A} + c\mathbf{A}\mathbf{e}\mathbf{e}^T &\preceq \|c\mathbf{e}\mathbf{e}^T\mathbf{A} + c\mathbf{A}\mathbf{e}\mathbf{e}^T\|I \\
&\preceq \|c\mathbf{e}\mathbf{e}^T\mathbf{A}\|I + \|c\mathbf{A}\mathbf{e}\mathbf{e}^T\|I \\
&= 2\|c\mathbf{e}\mathbf{e}^T\mathbf{A}\|I \\
&\preceq 2\|c\mathbf{e}\mathbf{e}^T\|\|\mathbf{A}\|I \\
&= 2\overline{\sigma}(\mathbf{A})I
\end{aligned} \tag{A.8}
$$

where we used the fact that, since matrix $\mathbf{A}$ is negative definite, $-\overline{\sigma}(\mathbf{A})I \preceq \mathbf{A} \preceq -\underline{\sigma}(\mathbf{A})I$. The symmetric part of the Jacobian

$$
\mathbf{J}_s = \frac{1}{2}\left(\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial f}{\partial \mathbf{x}}^T\right), \tag{A.9}
$$

when evaluated in Ω_2, satisfies

$$
\begin{aligned}
\mathbf{J}_s|_{\Omega_2} = \mathbf{A} &- \frac{c}{2}(\mathbf{e}\mathbf{e}^T\mathbf{A} + \mathbf{A}\mathbf{e}\mathbf{e}^T) - \sqrt{c}\|\boldsymbol{\xi}\|I \\
&- c\sqrt{c}\|\boldsymbol{\xi}\|\mathbf{e}\mathbf{e}^T \\
&\preceq -\underline{\sigma}(\mathbf{A})I + \frac{1}{2}\|c\mathbf{e}\mathbf{e}^T\mathbf{A} + c\mathbf{A}\mathbf{e}\mathbf{e}^T\|I - \sqrt{c}\|\boldsymbol{\xi}\|I \\
&\preceq -\underline{\sigma}(\mathbf{A})I + \overline{\sigma}(\mathbf{A})I - \sqrt{c}\|\boldsymbol{\xi}\|I \\
&\preceq -\overline{\sigma}(\mathbf{A})I + \sqrt{c}(\underline{\sigma}(\mathbf{A})\|\mathbf{x}_m\| - \|\mathbf{b}\|)I \\
&= -\beta_2 I
\end{aligned} \tag{A.10}
$$

As a result, if $\beta_2 > 0$, $\mathbf{J}_s|_{\Omega_2}$ is negative definite. This implies that the Jacobian has all of its eigenvalues negative and, in particular, it is invertible. By the inverse function theorem, there is a neighborhood D of every point in Ω_2 in which f is bijective, and thus the equilibriums points are isolated. Moreover, the stability of such equilibriums can be analyzed by using a Lyapunov function candidate $V : D \to \mathbb{R}$ given by

$$
V(\mathbf{x}) = \|f(\mathbf{x})\|^2 \tag{A.11}
$$

that is positive definite in D. Its time derivative, given by

$$
\dot{V} = f(\mathbf{x})^T\mathbf{J}_s f(\mathbf{x}) \leq -\beta_2\|f(\mathbf{x})\|^2, \tag{A.12}
$$

is negative definite in D. It follows that the equilibrium points in Ω_2 are isolated and asymptotically stable (Khalil, 2000). In order to prove part (2) of the proposition we follow a similar reasoning. Evaluating the symmetric part of the Jacobian on Ω_3, and using (A.6)-(A.8), we find that

$$
\begin{aligned}
\mathbf{J}_s|_{\Omega_3} = \mathbf{A} &- \frac{c}{2}(\mathbf{e}\mathbf{e}^T\mathbf{A} + \mathbf{A}\mathbf{e}\mathbf{e}^T) + \sqrt{c}\|\boldsymbol{\xi}\|I \\
&+ c\sqrt{c}\|\boldsymbol{\xi}\|\mathbf{e}\mathbf{e}^T \\
&\succeq -2\overline{\sigma}(\mathbf{A})I + \sqrt{c}\|\boldsymbol{\xi}\|I \\
&\succeq \beta_3 I
\end{aligned} \tag{A.13}
$$

Now if $\beta_3 > 0$, then $\mathbf{J}_s|_{\Omega_3}$ is positive definite, and all of its eigenvalues are positive. The equilibrium points in Ω_3 can then be shown to be isolated and unstable. □

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