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# FORWARD KINEMATIC MODELING OF CONSTANT CURVATURE CONTINUUM ROBOTS USING DUAL QUATERNIONS

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**Summary:** *Different approaches have been applied to derive the kinematics of continuum robots with the assumption of piecewise constant curvature. Despite all these approaches produce identical results and can be reduced to the homogeneous transformation matrix, a particular approach could result to be preferable among the others, with respect to the actual numerical application, thanks to the characteristics of its mathematical formulation. In this paper the above mentioned kinematics is approached through the use of dual quaternions. The resulting formulation offers remarkable characteristics of compactness and numerical efficiency compared with those of the homogeneous transformation matrix.*

## 1. INTRODUCTION

A continuum robot can modify the shape of its elastic structure to grasp an object wrapping around it [1], to move with dexterity in un-structured environments (e.g. nuclear decontamination [2], search and rescue [3]), taking advantage of its compliance to interact safely with the environment (e.g. medical applications [4][5]) and for locomotion [6][7][8].

Piecewise constant curvature approach allowed researchers to apply the mathematical tools widely used to model rigid-links robots (such as Denavit-Hartenberg parameters and Euler-Lagrange equations) on continuum robots. Despite the use of these well-established methods leads continuum robots to inherit experience from a wide literature of applications, the high complexity of the resulting models (due to the continuum nature of these robots) represents a significant issue to their actual numerical implementation [9]. Prior to the introduction of approximations to simplify the model, the possibility of numerical simplification offered by the use of different mathematical tools has to be investigated. In literature, several approaches to obtain the homogeneous transformation with the constant curvature hypothesis are present. These approaches, in particular Denavit-Hartenberg parameters [10], Frenet-Serret frames [10], Chirikjian and Burdick integral formulation [11], and exponential coordinates [13][14], when placed in a common coordinate frame, produce identical results for forward kinematics [12]. With the exception of Chirikjian and Burdick integral formulation [11], the results obtained with these approaches are affected by singularity when the circular arc approaches the zero curvature configuration [12].

In this paper the forward kinematic modelling of a continuum backbone with constant curvature is approached with the use of dual quaternions (DQ) in order to obtain a more compact and efficient kinematic relationship with respect to the homogeneous transformation matrix.

Dual quaternions were introduced by William K. Clifford in 1873 [15] and they are the application of dual numbers theory to quaternions. From quaternions, dual quaternions inherit some important properties. Among all, two are the most interesting. First, dual quaternions are one of the most compact and efficient form for representing rigid transforms [16] [17]. A transform with DQ is described by only 8 variables, instead of the 4x4 matrices used by the majority of the other methods. The composition of more transforms can be achieved through the DQ product that requires less mathematical operations than the matrix product [17]. Second, dual quaternions provide a non-singular representation of rotations. Quaternions, in fact, are frequently adopted to avoid "Gimbal lock" singularity. This singularity occurs when two of the three axes describing the spatial orientation of an object (e.g. Pitch-Roll-Yaw) have the same direction. In Robotics a similar problem, called "wrist singularity", occurs when two links, because of a particular configuration of the robot, rotate around the same axis. In this condition, the rotation of the two links produces the same effect (and consequently one degree of freedom is lost), and movements with high velocity may occur in correspondence of infinitesimal changes of orientation [16]. Dual quaternions are instead not affected by this singularity [16] [17] [18].

The paper is organized as follows. The application of dual quaternions to the forward kinematics of the constant curvature backbone is presented in paragraph 4. In paragraphs 2and 3the paper provides a brief overview on the piecewise constant curvature approach [12] and on the dual quaternions' mathematical definitions to represent rigid transformations[17] respectively. The resulting dual quaternion formulation is compared with the matrix formulation in terms of theoretical numerical weight in paragraph 5.

## 2. PIECEWISE CONSTANT CURVATURE APPROACH

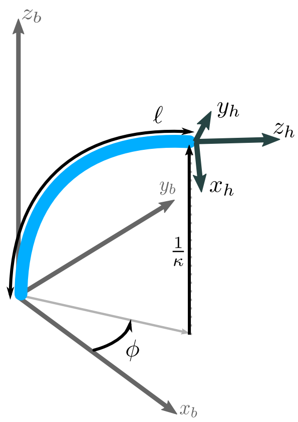


Figure 1: Backbone as a circular arc with variables elongation ($\ell$), curvature ($\kappa$) and orientation ($\phi$); head-frame and base-frame coordinate systems of the backbone.

With piecewise constant curvature approach, an ideal backbone line that bends always like a circular arc is used to describe the displacement of each section of the continuum robot in space. The kinematics of each section is consequently defined in two steps. The first step is the specific mapping "$f_{\text{specific}}$", that defines the relationship between the actuators' state $\boldsymbol{q} = [q_1, .., q_n]$ (where "n" is the number of actuators of the section), and the configuration state $\boldsymbol{\kappa} = [\kappa, \ell, \phi]$, where $\kappa$ is the curvature, $\ell$ the length and $\phi$ the orientation of the circular arc of the backbone. Each robot has its "specific" actuation characteristics, and consequently its own specific mapping.

$$
{}^{ib}\boldsymbol{H_{ih}} = \begin{bmatrix} \cos(\phi)[\cos(\kappa\ell)-1]+1 & \sin(\phi)\cos(\phi)[\cos(\kappa\ell)-1] & \cos(\phi)\sin(\kappa\ell) & \cos(\phi)[1-\cos(\kappa\ell)]1/\kappa \\ \sin(\phi)\cos(\phi)[\cos(\kappa\ell)-1] & \cos(\phi)^2[1-\cos(\kappa\ell)]+\cos(\kappa\ell) & \sin(\phi)\sin(\kappa\ell) & \sin(\phi)[1-\cos(\kappa\ell)]1/\kappa \\ -\cos(\phi)\sin(\kappa\ell) & -\sin(\phi)\sin(\kappa\ell) & \cos(\kappa\ell) & \sin(\kappa\ell)1/\kappa \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)
$$

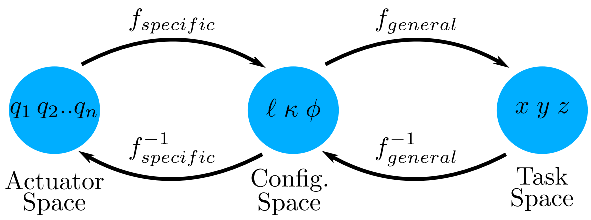


Figure 2: Backbone kinematics: specific and general mappings

The second step is the general mapping "$f_{\text{general}}$", which provides the position and orientation $\boldsymbol{x} = [x, y, z, \phi_x, \phi_y, \phi_z]$ of each point of the backbone in the space. Figure 2 resumes the specific and general mapping for the backbone. Describing the kinematics of an ideal line, this mapping finds general application to all continuum robots modelled with the piecewise constant curvature approach. It consists in the definition of the homogeneous transformation matrix ${}^{ib}\boldsymbol{H}_{ih}(\boldsymbol{\kappa_i})$ in (1). Through this matrix, it is possible to obtain the change of reference frame of a vector from head-frame coordinates $\boldsymbol{x}^{(ih)}$ to base-frame coordinates $\boldsymbol{x}^{(ib)}$ of the $i^{th}$-section as

$$
\boldsymbol{x}^{(ib)} = {}^{ib}\boldsymbol{H}_{ih}\, \boldsymbol{x}^{(ih)} \quad (2)
$$

With more serially-linked sections, the base-frame of the $i^{th}$-section corresponds to the head-frame of the (i−1)$^{th}$-section. Therefore the transformation of the reference coordinate system from $i^{th}$-section's head-frame into $j^{th}$-section's base-frame is

$$
\boldsymbol{x}^{(jb)} = {}^{jb}\boldsymbol{H}_{ih}\, \boldsymbol{x}^{(ih)} \quad (3)
$$

where

$$
{}^{jb}\boldsymbol{H}_{ih} = \prod_{k=j}^{i} {}^{kb}\boldsymbol{H}_{kh} \quad (4)
$$

.

## 3. DUAL QUATERNIONS TO REPRESENT RIGID TRANSFORMATIONS

One dual quaternion $\widehat{q}$ is composed by two quaternions

$$\widehat{q} = q + \varepsilon q_\varepsilon \ ,$$

where $q$ represents the real part quaternion, $q_\varepsilon$ the dual part quaternion and $\varepsilon$ the dual factor. Therefore it consist in the 8-dimensional vectors

$$\begin{aligned} \widehat{q} &= s + xi + yj + zk + \varepsilon(s_\varepsilon + x_\varepsilon i + y_\varepsilon j + z_\varepsilon k) && (5)\\ \widehat{q} &= (s, x, y, z, s_\varepsilon, x_\varepsilon, y_\varepsilon, z_\varepsilon) = (s, \overline{m}, s_\varepsilon, \overline{m_\varepsilon}). && (6) \end{aligned}$$

Both vectors and transformations can be expressed as dual quaternions.
A vector $\boldsymbol{v}$ with components $(v_x, v_y, v_z)$ can be expressed as

$$\widehat{q}_v = (1, 0, 0, 0, 0, v_x, v_y, v_z). \tag{7}$$

A pure rotation of an angle $\theta$ around an axis, defined by the unit-vector $(v_{Rx}, v_{Ry}, v_{Rz})$, is described by the real part as

$$\begin{aligned} \widehat{q}_r =& q_r + \varepsilon 0\\ \widehat{q}_r =& \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)[v_{Rx}, v_{Ry}, v_{Rz}], 0, 0, 0, 0\right) \end{aligned} \tag{8}$$

A pure translation represented by a vector $(T = \ [t_x, t_y, t_z])$, is described by the dual part, with an identity as real part

$$\begin{aligned} \widehat{q}_t =& 1 + \varepsilon \frac{T}{2}\\ \widehat{q}_t =& \left(1, 0, 0, 0, 0, \frac{t_x}{2}, \frac{t_y}{2}, \frac{t_z}{2}\right) \ . \end{aligned} \tag{9}$$

A rotation and a translation can be combined to define a unique transformation, that is a unique DQ:

$$\widehat{q}_{tr} = \widehat{q}_r \,\widehat{\otimes}\, \widehat{q}_t \text{ if translation is applied first} \tag{10}$$

$$\widehat{q}_{rt} = \widehat{q}_t \,\widehat{\otimes}\, \widehat{q}_r \text{ if rotation is applied first} \tag{11}$$

where "$\widehat{\otimes}$" is the dual quaternion product. The dual quaternion product between two dual quaternions $\widehat{q}_{v1}$ and $\widehat{q}_{v2}$ is defined as

$$\widehat{q}_{v1} \,\widehat{\otimes}\, \widehat{q}_{v2} \ = \ q_{v1} \otimes q_{v2} + \varepsilon(q_{v1} \otimes q_{v2\varepsilon} + q_{v1\varepsilon} \otimes q_{v2}), \tag{12}$$

where "$\otimes$" is the quaternion product defined as

$$q_1 \otimes q_2 = s_1 s_2 - m_1 \cdot m_2,\ s_1\, m_2 + s_2\, m_1 + m_1 \wedge m_2 \tag{13}$$

It is possible to apply the generic transformation $\widehat{q}$ to the generic vector in DQ form $\widehat{q}_{v1}$ as

$$\widehat{q}_{v2} \ = \ \widehat{q}\,\widehat{\otimes}\,\widehat{q}_{v1}\,\widehat{\otimes}\,\widehat{q}^{*} \tag{14}$$

where $\widehat{q}^{*}$ is the dual conjugate of $\widehat{q}$, defined as

$$\widehat{q}^{*} = [s, -m, -s_{\varepsilon}, m_{\varepsilon}] \ . \tag{15}$$

The generic transformation represented by the dual quaternion

$$\widehat{q}_{tr} = \left( r_w, r_x, r_y, r_z, 0, \frac{t_x}{2}, \frac{t_y}{2}, \frac{t_z}{2} \right) \tag{16}$$

can be expressed in matrix form as

$$\boldsymbol{H_{tr}} \ = \begin{bmatrix} r_w^2 + r_x^2 - r_y^2 - r_z^2 & 2\,r_x\,r_y - 2\,r_w\,r_z & 2\,r_x\,r_z + 2\,r_w\,r_y & t_x \\ 2\,r_x\,r_y + 2\,r_w\,r_z & r_w^2 - r_x^2 + r_y^2 - r_z^2 & 2\,r_x\,r_z - 2\,r_w\,r_y & t_y \\ 2\,r_x\,r_z - 2\,r_w\,r_y & 2\,r_y\,r_z + 2\,r_w\,r_x & r_w^2 - r_x^2 - r_y^2 + r_z^2 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

## 4. DUAL QUATERNIONS - APPLICATION TO CONSTANT CURVATURE MODEL

The convention used for the base-frame is the same used by Webster and Jones [12]: the positive z-axis ($z_b$) is tangent to the backbone of the section at its base. If the orientation $\boldsymbol{\phi}$ is equal to $0$, not null curvature defines bending about +y-axis ($y_b$) so that the backbone touches the x-axis ($x_b$) after tracing out an angle of $\pi$ radians [12].
The domain of the variables is chosen as :

$$\begin{aligned} &\boldsymbol{\kappa} \geq 0 \\ &\boldsymbol{\phi} \in (-\pi, \pi] \\ &s \in (0, \ell) \end{aligned}$$

The head-frame is also chosen to be tangent to the backbone of the section at its tip. Moreover the projections of its x-axis ($x_h$) and y-axis ($y_h$) on $x_b$-$y_b$-plane are always parallel to $x_b$ and $y_b$ axes respectively.

It is possible to obtain the transformation between base-frame and head-frame through $3$ transformations. In the following description of the three transformations, each DQ will have the superscript "(i)" to indicate the reference frame-i in which it is defined.

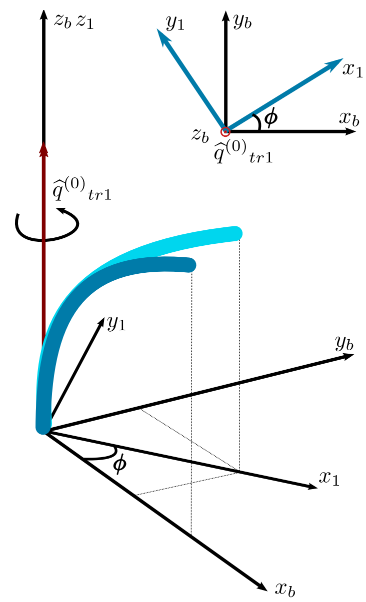


Figure 3: Transformation I - Rotation of base-frame around $z_b$ of the angle $\phi$

### 4.1 Transformation I

As shown in figure 3, frame-1 is chosen so that its origin is coincident with the origin of the base-frame, its z-axis $z_1$ is coincident with $z_b$, and its axes $x_1$ and $y_1$ are rotated of the angle $\phi$ around $z_b$. Using (8), the transformation that describes the rotation between the two frames is therefore

$$\widehat{q}^{(b)}{}_{tr1} = \left[cos\left(\frac{\phi}{2}\right), 0, 0, sin\left(\frac{\phi}{2}\right), 0, 0, 0, 0\right] \tag{18}$$

### 4.2 Transformation II

Frame-2 is placed so that it has its origin coincident with the origin of the head-frame, the axis $z_2$ tangent to the backbone at its tip, and the axis $y_2$ parallel to the axis $y_1$. In figure 4 it can be noted that with this choice the transformation that makes frame-1 to be coincident to frame-2 can be defined as a roto-translation around the axis parallel to $y_1$ and passing through the center of the curvature's arc. The position of this point varies with the curvature: it translates on the $x_1$ axis with a distance from the origin of $1/\kappa$, that is the curvature's radius. The dual quaternion corresponding to this transformation can be calculated as the product between the dual quaternion $\widehat{q}^{(1)}{}_{t2}$, that represents the translation along $x_1$ with distance $r = 1/\kappa$, and the

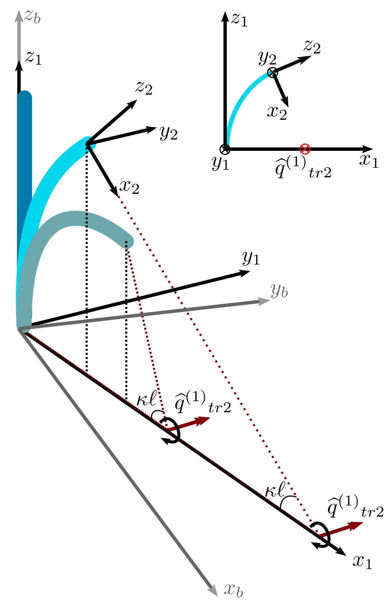


Figure 4: Transformation II - Roto-translation of frame-1 around the axis parallel to $y_1$ and passing through the center of the curvature's arc

dual quaternion $\widehat{q}^{(1)}{}_{r2}$, that represents the rotation around the axis $y_1$:

$$\widehat{q}^{(1)}{}_{tr2} = \widehat{q}^{(1)}{}_{t2} \,\widehat{\otimes}\, \widehat{q}^{(1)}{}_{r2} \,\widehat{\otimes}\, \widehat{q}^{(1)*}{}_{t2}; \tag{19}$$

$$\widehat{q}^{(1)}{}_{tr2} = \left(1 + \frac{\varepsilon}{2} d_2\right) \,\widehat{\otimes}\, \left(R_2 + \frac{\varepsilon}{2} 0\right) \,\widehat{\otimes}\, \left(1 - \frac{\varepsilon}{2} d_2\right) \tag{20}$$

$$\text{with } d_2 = [0,\ r,\ 0,\ 0] \tag{21}$$

$$\text{and } R_2 = \left[\cos\left(\frac{\kappa\ell}{2}\right),\ 0,\ \sin\left(\frac{\kappa\ell}{2}\right),\ 0\right]. \tag{22}$$

The calculation of the application of a pure translation to a pure rotation can be simplified [17], so that (20) becomes

$$\widehat{q}^{(1)}{}_{tr2} = R_2 - \frac{\varepsilon}{2}(R_2 \otimes d_2 - d_2 \otimes R_2) \tag{23}$$

that results in (see Appendix A)

$$\widehat{q}^{(1)}{}_{tr2} = \left[\cos\left(\frac{\kappa\ell}{2}\right), 0, \sin\left(\frac{\kappa\ell}{2}\right), 0, 0, 0, 0, r\sin\left(\frac{\kappa\ell}{2}\right)\right]. \tag{24}$$

### 4.3 Transformation III

In order to align the frame-2 to the head-frame, transformation III consists in the rotation around $z_2$ of an angle $-\phi$ as shown in figure 5. This corresponds to rotate around the vector

$\boldsymbol{v}_3{}^{(2)}$, that is defined with respect to the frame-2, with components

$$\left[v^{(2)}{}_{x3}, v^{(2)}{}_{y3}, v^{(2)}{}_{z3}\right] = [0, 0, 1]\,.$$

The corresponding DQ for this rotation is therefore

$$\widehat{q}^{(2)}{}_{tr3} = \left[cos\left(\frac{-\phi}{2}\right), sin\left(\frac{-\phi}{2}\right)[0,0,1], 0,0,0,0\right]\ . \tag{25}$$

### 4.4 Composition of the transformations

It is possible to apply the transformations to the generic DQ vector $\widehat{q}^{(\mathrm{h})}{}_v = (1,0,0,0,0,v^{(\mathrm{h})}{}_x, v^{(\mathrm{h})}{}_y, v^{(\mathrm{h})}{}_z)$ expressed with head-frame coordinates, to find its counterpart in base-frame coordinates as

$$\widehat{q}^{(\mathrm{b})}{}_v = {}^{ib}\widehat{q}_{ih} \widehat{\otimes}\ \widehat{q}^{(\mathrm{h})}{}_v \ \widehat{\otimes}\ {}^{ib}\widehat{q}^{*}_{ih} \tag{26}$$

with:

$${}^{ib}\widehat{q}_{ih} = \widehat{q}^{(\mathrm{b})}{}_{tr1}\ \widehat{\otimes}\ \widehat{q}^{(1)}{}_{tr2} \widehat{\otimes}\ \widehat{q}^{(2)}{}_{tr3} \tag{27}$$

The transformation expressed by the dual quaternion ${}^{ib}\widehat{q}_{ih}$ is equivalent to the homogeneous transformation presented in (1). It is important to remark that $\widehat{q}^{(2)}{}_{tr3}$ (25) results to be the dual conjugate of $\widehat{q}^{(\mathrm{b})}{}_{tr1}$ (18). From this property and from the equation (14) we can conclude that the DQ products in (27) is exactly the application of the transformation-1 to the transformation-2. This demonstrates that the homogeneous transformation is actually the roto-translation of the base-frame around an axis that lies on the plane $x_b - y_b$ and that passes through the center of curvature's arc of the backbone, as shown in Figure 6. The center of curvature's arc translates on the plane $x_b - y_b$ with the curvature's radius $(r = {}^1\!/\!_k)$ as distance from the origin of the base-frame axes, and with the angle $\phi$ as angular position in the plane.

Computing the DQ products, (27) results in: (see Appendix B)

$$\begin{aligned} {}^{ib}\widehat{q}_{ih} \ =& \\ &\Big[\cos\left(\frac{\kappa\ell}{2}\right), -\sin\left(\frac{\kappa\ell}{2}\right)\sin(\phi), \sin\left(\frac{\kappa\ell}{2}\right)\cos(\phi),\ 0, \\ &-\ r\ \sin\left(\frac{\kappa\ell}{2}\right)\sin(\phi),\ \ 0,\ \ 0,\ \ r\ \sin\left(\frac{\kappa\ell}{2}\right)\Big] \end{aligned} \tag{28}$$

Compared with the homogeneous transformation (1), this is a much more compact representation of the same information. The singularity in the translational terms is still present: if $\kappa \to 0$, the curvature's radius $r \to \infty$.

In some cases, the head-frame of the section is assumed to be oriented like the frame-2. Consequently, the transformation between base and head frames is the composition of only transformation I and II, that results in (see Appendix B):

$$\widehat{q}_{1-2} = \Big[ \cos\left(\frac{\kappa\ell}{2}\right)\cos\left(\frac{\phi}{2}\right),\ -\sin\left(\frac{\kappa\ell}{2}\right)\sin\left(\frac{\phi}{2}\right), \\ \sin\left(\frac{\kappa\ell}{2}\right)\cos\left(\frac{\phi}{2}\right),\quad \cos\left(\frac{\kappa\ell}{2}\right)\sin\left(\frac{\phi}{2}\right), \\ -r\sin\left(\frac{\kappa\ell}{2}\right)\sin\left(\frac{\phi}{2}\right),\ 0,\ 0,\ r\sin\left(\frac{\kappa\ell}{2}\right)\cos\left(\frac{\phi}{2}\right) \Big] \tag{29}$$

Considering more sections serially linked, the transformation of reference coordinate system from $i^{th}$-section's head-frame into $j^{th}$-section's base-frame (with j<i) is obtained with

$$\widehat{q}^{(\mathrm{jb})}{}_v = {}^{jb}\widehat{q}_{ih} \mathbin{\widehat{\otimes}} \widehat{q}^{(\mathrm{ih})}{}_v \mathbin{\widehat{\otimes}} {}^{jb}\widehat{q}^{*}_{ih} \tag{30}$$

where

$${}^{jb}\widehat{q}_{ih} = \prod_{k=j}^{i} {}^{kb}\widehat{q}_{kh} \tag{31}$$

and with $\prod$ the DQ-product sequence is intended.

## 5. COMPARISON BETWEEN DQ and HT IN TERMS OF NUMERICAL WEIGHT

In Table 1 dual quaternions and matrix transformations presented in the previous paragraphs are compared in terms of theoretical numerical weight of the calculations. The effort to evaluate the elements of the homogeneous transformation matrix (1) with respect to its DQ counterpart (28) is first evaluated. In case of n-sections serially linked, the product of n-homogeneous transformation is needed. Therefore the second comparison considers the effort to evaluate this product through matrices with (4) and through DQs with (31). The last comparison is made on the effort to apply the generic homogeneous transformation to the generic vector. With both formulations these products could be

## 6. Conclusion

The application of the dual quaternions to the kinematic modelling of the constant curvature backbone has led to positive results with respect the compactness and efficiency of the representation of the transformations. With its five variables among the eight constitutive elements, the dual quaternion (28) is a much more compact representation of the homogeneous transformation with respect to the matrix (1) of twelve variables. Major benefits due this representation are inherited by the differential kinematic problem, where the derivative of the homogeneous transformation with respect to the three configuration space variables are defined by three dual

| Type of calculation | Form | Expression | Add &Sub | Mult | Cos &Sin | Tot | Diff | Storage | Diff |
|---|---|---|---|---|---|---|---|---|---|
| Homogeneous transformation | Matrix | $^{\boldsymbol{ib}}\boldsymbol{H_{ih}}$ | 8 | 17 | 25 | 50 | | 16 | |
| | DQ | $^{ib}\widehat{q}_{ih}$ | 0 | 13 | 8 | 21 | -29 | 8 | $-50\%$ |
| Product of n Homogeneous transformations (with $m = n - 1$) | Matrix | $\prod_{k=1}^{n} {}^{\boldsymbol{kb}}\boldsymbol{H_{kh}}$ | $m * 36$ | $m * 60$ | 0 | $m * 96$ | | $m * 16$ | |
| | DQ | $\prod_{k=1}^{n} {}^{kb}\widehat{q}_{kh}$ | $m * 40$ | $m * 48$ | 0 | $m * 88$ | $-m * 8$ | $m * 8$ | $-50\%$ |
| Transformation applyied to a vector | Matrix | $^{\boldsymbol{jb}}\boldsymbol{H_{ih}} * \boldsymbol{v^{(ih)}}$ | 12 | 16 | 0 | 28 | | 8 | |
| | DQ | $^{jb}\widehat{q}_{ih} \mathbin{\widehat{\otimes}} \widehat{q}_v^{(ih)} \mathbin{\widehat{\otimes}} {}^{jb}\widehat{q}_{ih}^{*}$ | 80 | 96 | 0 | 176 | $+148$ | 16 | $+50\%$ |

Table 1: Theoretical numerical weight of DQ and matrix formulations

quaternions instead of a matrix with dimensions 4x4x3 of much more complex terms. As table 1 shows, improvements on the theoretical numerical weight of the kinematic modelling are related only to the calculation of the homogeneous transformations and the products between them. Therefore, once all the necessary operations between them are computed, the resulting dual quaternions have to be transformed into the matrix form with the relation (17), to avoid the inefficiency of the application of DQtransformations to DQ vectors. Lastly, the spatial representation of the entire kinematic transformation has been identified as consisting of a single transformation around a well-defined axis (Figure 6).

## A CALCULATION OF $\widehat{q}^{(1)}{}_{tr2}$

$$\widehat{q}^{(1)}{}_{tr2} = R_2 - \frac{\varepsilon}{2} \left( R_2 \otimes d_2 - d_2 \otimes R_2 \right) \tag{32}$$

with:

$$d_2 = [0, r, 0, 0]; \tag{33}$$

$$R_2 = \left[ \cos\left(\frac{\kappa\ell}{2}\right), 0, \sin\left(\frac{\kappa\ell}{2}\right), 0 \right]. \tag{34}$$

Evaluating the two quaternion pruducts in (32)

$$\begin{aligned} & R_2 \otimes d_2 = \\ & \left( 0 - 0, \left[ r\cos\left(\frac{\kappa\ell}{2}\right), 0, 0 \right] + \left[ 0, 0, -r\sin\left(\frac{\kappa\ell}{2}\right) \right] \right); \end{aligned} \tag{35}$$

$$\begin{aligned} & d_2 \otimes R_2 = \\ & \left( 0 - 0, \left[ r\cos\left(\frac{\kappa\ell}{2}\right), 0, 0 \right] + \left[ 0, 0, +r\sin\left(\frac{\kappa\ell}{2}\right) \right] \right). \end{aligned} \tag{36}$$

Therefore it results

$$R_2 \otimes d_2 - d_2 \otimes R_2 = \left(0, 0, 0, -2r \sin\left(\frac{\kappa\ell}{2}\right)\right); \tag{37}$$

$$\widehat{q}_{tr2} = \left[\cos\left(\frac{\kappa\ell}{2}\right), 0, \sin\left(\frac{\kappa\ell}{2}\right), 0, 0, 0, 0, r\sin\left(\frac{\kappa\ell}{2}\right)\right]. \tag{38}$$

## B Calculation of ${}^{ib}\widehat{q}_{ih}$

$${}^{ib}\widehat{q}_{ih} = \ \widehat{q}^{(0)}{}_{tr1} \ \widehat{\otimes} \ \widehat{q}^{(1)}{}_{tr2} \widehat{\otimes} \ \widehat{q}^{(2)}{}_{tr3} \,. \tag{39}$$

Computing the first DQ product as:

$$\widehat{q}_{1-2} = \widehat{q}^{(0)}{}_{tr1} \ \widehat{\otimes} \ \widehat{q}^{(1)}{}_{tr2} \tag{40}$$

with:

$$\widehat{q}^{(0)}{}_{tr1} = \left[cos\left(\frac{\phi}{2}\right), 0, 0, sin\left(\frac{\phi}{2}\right), 0, 0, 0, 0\right]; \tag{41}$$

$$\widehat{q}^{(1)}{}_{tr2} = \left[cos\left(\frac{\kappa\ell}{2}\right), 0, sin\left(\frac{\kappa\ell}{2}\right), 0, 0, 0, 0, rsin\left(\frac{\kappa\ell}{2}\right)\right] \tag{42}$$

(40) results in:

$$\begin{aligned}\widehat{q}_{1-2} = \\ \Bigg[ \ & \cos\left(\frac{\kappa\ell}{2}\right)\cos\left(\frac{\phi}{2}\right), -\sin\left(\frac{\kappa\ell}{2}\right)\sin\left(\frac{\phi}{2}\right), \\ & \sin\left(\frac{\kappa\ell}{2}\right)\cos\left(\frac{\phi}{2}\right), \quad \cos\left(\frac{\kappa\ell}{2}\right)\sin\left(\frac{\phi}{2}\right), \\ & -r\ \sin\left(\frac{\kappa\ell}{2}\right)\sin\left(\frac{\phi}{2}\right), \ 0, \ 0, \ r\ \sin\left(\frac{\kappa\ell}{2}\right)\cos\left(\frac{\phi}{2}\right)\Bigg].\end{aligned} \tag{43}$$

Now computing the second DQ product as:

$${}^{ib}\widehat{q}_{ih} = \widehat{q}_{1-2} \ \widehat{\otimes} \ \widehat{q}^{(2)}{}_{tr3} \tag{44}$$

where

$$\widehat{q}^{(2)}{}_{tr3} = \left[cos\left(\frac{-\phi}{2}\right), sin\left(\frac{-\phi}{2}\right)[0, 0, 1], 0, 0, 0, 0\right] \tag{45}$$

(44) results in

$$
\begin{aligned}
{}^{ib}\widehat{q}_{ih}(1) &= \quad \cos\left(\frac{\kappa\ell}{2}\right)\left(\cos^2\left(\frac{\phi}{2}\right) + \sin^2\left(\frac{\phi}{2}\right)\right) && (46)\\
{}^{ib}\widehat{q}_{ih}(2) &= -\sin\left(\frac{\kappa\ell}{2}\right)\left(2\,\cos\left(\frac{\phi}{2}\right)\,\sin\left(\frac{\phi}{2}\right)\right) && (47)
\end{aligned}
$$

$$
\begin{aligned}
{}^{ib}\widehat{q}_{ih}(3) &= \sin\left(\frac{\kappa\ell}{2}\right)\left(1 - 2\,\sin^2\left(\frac{\phi}{2}\right)\right) && (48)\\
{}^{ib}\widehat{q}_{ih}(4) &= \cos\left(\frac{\kappa\ell}{2}\right)\left((1-1)\,\cos\left(\frac{\phi}{2}\right)\,\sin\left(\frac{\phi}{2}\right)\right) && (49)\\
{}^{ib}\widehat{q}_{ih}(5) &= -r\,\sin\left(\frac{\kappa\ell}{2}\right)\left(2\,\cos\left(\frac{\phi}{2}\right)\,\sin\left(\frac{\phi}{2}\right)\right) && (50)\\
{}^{ib}\widehat{q}_{ih}(6) &= 0 && (51)\\
{}^{ib}\widehat{q}_{ih}(7) &= 0 && (52)\\
{}^{ib}\widehat{q}_{ih}(8) &= \quad r\,\sin\left(\frac{\kappa\ell}{2}\right)\left(\cos^2\left(\frac{\phi}{2}\right) + \sin^2\left(\frac{\phi}{2}\right)\right) && (53)
\end{aligned}
$$

Using the trigonometric properties:

$$
\begin{aligned}
1 - 2\sin\frac{\phi}{2}^2 &= \cos\phi && (54)\\
2\sin\frac{\phi}{2}\cos\frac{\phi}{2} &= \sin\phi && (55)
\end{aligned}
$$

(46) to (53) result in

$$
\begin{aligned}
&{}^{ib}\widehat{q}_{ih} = \\
&\Big[\cos\left(\frac{\kappa\ell}{2}\right),\, -\sin\left(\frac{\kappa\ell}{2}\right)\sin(\phi),\, \sin\left(\frac{\kappa\ell}{2}\right)\cos(\phi),\, 0,\\
&-\,r\,\sin\left(\frac{\kappa\ell}{2}\right)\sin(\phi),\;\; 0,\;\; 0,\;\; r\,\sin\left(\frac{\kappa\ell}{2}\right)\Big] && (56)
\end{aligned}
$$

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## C EQUATIONS

A displayed equation is automatically numbered, using Arabic numbers in parentheses. The following example is a single line equation:

$$Ax = b \tag{57}$$

The next example is a multi-line equation:

$$\begin{aligned} Ax &= b \\ Ax &= b \end{aligned} \tag{58}$$

## D TABLES

All tables are automatically numbered consecutively and captioned.

| | | |
|---|---|---|
| C11 | C12 | C13 |
| C21 | C22 | C23 |
| C31 | C32 | C33 |
| C41 | C42 | C43 |
| C51 | C52 | C53 |

Table 2: Example of the construction of a table.

## E FORMAT OF REFERENCES

References should be quoted in the text by numbers [1, 2] and grouped together at the end of the Abstract in numerical order as shown in these instructions. Use the $unsrt$ style either with the $BibTeX$ or the $\backslash bibitem$ format.

## F CONCLUSIONS

We look forward to receive your contributions for this conference.

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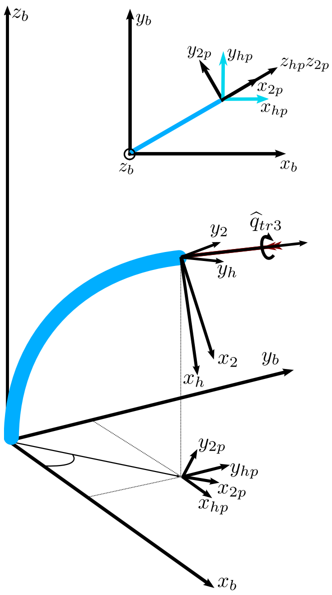


Figure 5: Transformation III - Rotation of frame-2 around $z_2$ of an angle $-\phi$

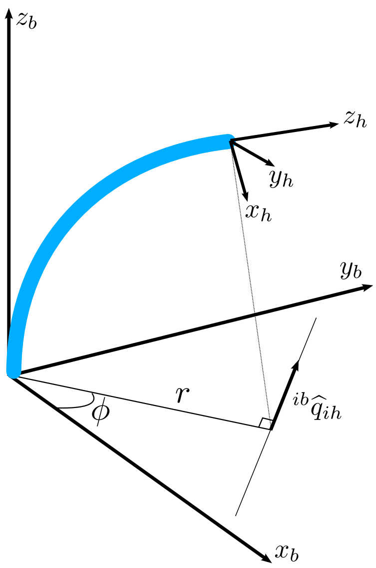


Figure 6: Homogeneous transformation as roto-translation around an axis passing through the center of curvature's radius and lying on the plane $x_b$-$y_b$.